# Super-symmetrically Combined KdV-CDG Equation: Bilinear Approach

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**Abstract:** A combined KdV-CDG equation is extended and an $N$=1 super-symmetrically combined KdV-CDG equation is established using Hirota's bilinear method. A Bäcklund transformation is obtained.

**Key words:** combined KdV-CDG equation; supersymmetric; Hirota's bilinear

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In recent year, the study of supersymmetric integrable system has been a very interesting subject. The physical interest in the study of these systems was initiated by the seminal paper of Alvarez-Gaume, et al[1] concerning the partition function and super-Virasoro constrains of two dimensional (2D) quantum supergravity. There are a number of well known integrable equations have been generalized into the symmetric context and there also have various methods such as painlevé test[2], prolongation structures[3], Darboux and Bäcklund transformation[4], Hirota bilinear method[4-7] and Hamiltonian formalism[8] etc have been extended to study supersymmetric integrable systems.

On the other hand, the construction of soliton solutions for a given nonlinear evolution equation is an important topic. There have been many notifies about it[9-10]. And It is well known that Hirota's bilinear approach is a very effective method for construction particular solutions for soliton systems[11]. Although Hirota bilinear method is used to study supersymmetric integrable system in [4-6], the bilinear formalism for supersymmetric integrable system is very little investigated.

In this paper, we extend combined KdV-CDG equation and consider the $N$=1 supersymmetric combined KdV-CDG equation using Hirota's bilinear method. As a result, a Bäcklund transformation is obtained.

This paper is organized as follows: In section two, combined KdV-CDG equation

$$u_t + a[u_{2x} + 1/5\alpha u^2]_x + b[1/15\alpha^2 u^3 + \alpha u u_{2x} + u_{4x}]_x = 0 \quad (1)$$

is supersymmetrize and bilinearize. In section three, we construct a Bäcklund transformation for the $N$=1 supersymmetric combined KdV-CDG equation. At last, we summarize our results.

## 1 Supersymmetric combined KdV-CDG equation

The Caudrey-Dodd-Gibbon equation (CDG)[12] is

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given by

$$u_t + 1/5\alpha^2 u^2 u_x + \alpha u_x u_{3x} + u_{5x} = 0\,, \tag{2}$$

with $u(x,t)$ is a sufficiently often differentiable function. The CDG equation is completely integrable and therefore it admits multiple-soliton solutions and infinite number of conserved quantities.

Moreover, the CDG equation (2) possesses the Painleve property[13]. Following the approach in [12], equatin (2) becomes equatin (1)

$$u_t + a[u_{2x} + 1/5\alpha u^2]_x + b[1/15\alpha^2 u^3 + \alpha u u_{2x} + u_{4x}]_x = 0\,. \tag{1}$$

Equation (1) will be reduced to the KdV equation for $b$=0 and for the CDG equation (2) for $a$=0. Eq. (1) has the bilinear form

$$D_x(D_t + aD_X^3 + bD_X^5)\,f\cdot f = 0\,, \tag{3}$$

where the definition of the Hirota's bilinear operators D is given by

$$D_t^m D_x^n f\cdot g = (\partial/\partial t_1 - \partial/\partial t_2)^m(\partial/\partial x_1 - \partial/\partial x_2)^n f(x_1,t_1)g(x_2,t_2)|_{\substack{x_1=x_2=x\\ t_1=t_2=t}}\,.$$

We consider combined KdV-CDG equation Eq. (1) where the subscripts denote partial derivatives. We extend independent variables to obtain the supersymmetric combined KdV-CDG equation. As usual, we choose to extend the variable $x$ to a doublet $(x,\theta)$, where $\theta$ is a Grassmann variable and satisfy $\theta^2 = 0$. Thus, the original independent variables $(x,t)$ is extended to supercase $(x,t,\theta)$. And the associated superderivative

$$D = \partial_\theta + \theta\partial_x. \tag{4}$$

A bosonic superfield is given that

$$F(x,t,\theta) = u(x,t) + \theta\eta(x,t)\,, \tag{5}$$

or a fermionic superfield

$$\Phi(x,t,\theta) = \eta(x,t) + \theta u(x,t). \tag{6}$$

Let us proceed with a direct extension, namely multiplying each term of the Eq. (1) by $\theta$ and rewriting the resultant terms in terms of superfields. For simpilify, we choose $\alpha = 15$.

$$u_t \to \Phi_t,$$

$$u_{2x} + 1/5\alpha u^2 \to D^4\Phi + 3\Phi D\Phi,$$

$$1/15\alpha^2 u^3 + \alpha u u_{2x} + u_{4x} \to 10D\Phi D^4\Phi + 5D^5\Phi\Phi + 15\left(D\Phi\right)^2\Phi + D^8\Phi\,. \tag{7}$$

The $N$=1 supersymmetric combined KdV-CDG equation is

$$\Phi_t + aD^2[D^4\Phi + 3\Phi D\Phi] + bD^2[10D\Phi D^4\Phi + 5D^5\Phi\Phi + 15\left(D\Phi\right)^2\Phi + D^8\Phi] = 0\,. \tag{8}$$

By means of equation (4), this equation in components reads as

$$\eta_t + a[\eta_{2x} + 3u\eta]_x + b[10u\eta_{xx} + 5u_{2x}\eta + 15u^2\eta + \eta_{4x}]_x = 0\,, \tag{9}$$

$$u_t + a[u_{2x} + 3u^2 + 3\eta\eta_x]_x + b[15u^3 + 15uu_{2x} + 10\eta_x\eta_{2x} + 5\eta\eta_{3x} + 30u\eta\eta_x + u_{4x}]_x = 0\,. \tag{10}$$

Eq. (8) is Eq. (1) and Eq. (7) vanish when $\eta = 0$.

We reformulate Eq. (6) into Hirota bilinear form. We make a dependent variable transformation as follows:

$$\eta = 2D^3\ln f(x,t,\theta)\,, \tag{11}$$

then through straightforward manipulations we find that equation (6) is transformed into

$$S_x(D_t + aS_x^6 + bS_x^{10})\,f\cdot f = 0\,, \tag{12}$$

which is equivalent to the form

$$S_x(D_t + aS_x^3 + bS_x^5)\,f\cdot f = 0\,, \tag{13}$$

where we used the Hirota derivative which is defined as

$$S_x D_t^m D_x^n f\cdot g = (D_{\theta_1} - D_{\theta_2})(\partial/\partial t_1 - \partial/\partial t_2)^m(\partial/\partial x_1 - \partial/\partial x_2)^n f(x_1,t_1,\theta_1)g(x_2,t_2,\theta_2)|_{\substack{x_1=x_2=x\\ t_1=t_2=t\\ \theta_1=\theta_2=\theta}}\,.$$

In the sequent sections we mainly study Eq. (13).

## 2 Bäcklund transformation

BT is a useful concept and an effective tool for solution systems as well as a characteristic of integrability. In this section, we derive a bilinear BT for Eq. (13). Our results are summarized in the following.

**Proposition** Suppose that $f$ is a solution of Eq. (13), then g satisfying the following relations

$$S_x D_x f\cdot g - \lambda S_x f\cdot g = 0\,, \tag{14}$$

$$S_x D_x^2 f\cdot g - \lambda^2 S_x f\cdot g = 0\,, \tag{15}$$

$$S_x D_x^3 f\cdot g - \lambda^3 S_x f\cdot g = 0\,, \tag{16}$$

$$[D_t + (3\lambda^2 a + 5\lambda^4 b)D_x - (3\lambda a + 10\lambda^3 b)D_x^2 + (a + 10\lambda^2 b)D_x^3 - 5\lambda b D_x^4 + bD_x^5]\,f\cdot g = 0\,, \tag{17}$$

where $\lambda$ is an arbitrary constant. Then $g$ is a new solution of the equation (11).

**Proof** We consider

$$Q=(S_x(D_t+aD_x^3+bD_x^5)f\cdot f)gg-f\cdot f(S_x(D_t+aD_x^3+bD_x^5)g\cdot g)\,.$$

We will use various bilinear identities which, for convenience, are presented in the appendix.

$$Q\overset{(A1)-(A3)}{=\!=}2S_x[(D_t+aD_x^3+bD_x^5)f\cdot g]f\cdot g+6aS_x(D_x^2f\cdot g)\cdot(D_xg\cdot f)-3a(S_xD_xf\cdot f)\cdot(D_x^2g\cdot g)+3a(S_xD_xg\cdot g)(D_x^2f\cdot f)-10bS_x(D_x^4f\cdot g)\cdot(D_xf\cdot g)+20bS_x(D_x^3f\cdot g)\cdot(D_x^2f\cdot g)-5b(S_xD_xf\cdot f)(D_x^4g\cdot g)+5b(S_xD_x^4g\cdot g)(D_x^4f\cdot f)-10b(S_xD_x^3f\cdot f)(D_x^4g\cdot g)+10b(S_xD_x^3g\cdot g)(D_x^2f\cdot f)$$

$$\overset{(A4),(A6)}{=\!=}2S_x[(D_t+aD_x^3+bD_x^5)f\cdot g]fg+6aD_x(S_xD_xf\cdot g)\cdot(D_xg\cdot f)-5bD_x(S_xD_x^3f\cdot g)\cdot(D_xf\cdot g)-5bD_x^3(S_xD_xf\cdot g)\cdot(D_xf\cdot g)-5bD_x(S_xD_xf\cdot g)\cdot(D_x^3f\cdot g)$$

$$\overset{(14)-(15)}{=\!=}2S_x[(D_t+aD_x^3+bD_x^5)f\cdot g]-(6\lambda a+5b\lambda^3)D_x(S_xf\cdot g)\cdot(D_xf\cdot g)-5b\lambda D_x^3(S_xf\cdot g)\cdot(D_xf\cdot g)-5b\lambda D_x(S_xf\cdot g)\cdot(D_x^3f\cdot g)$$

$$\overset{(A5),(A7),(A8)}{=\!=}2S_x[(D_t+aD_x^3+bD_x^5)f\cdot g]fg-(6\lambda a+5b\lambda^3)D_x(S_xf\cdot g)(D_xf\cdot g)-5b\lambda[S_x(D_x^4f\cdot g)\cdot fg-D_x(S_xD_x^3f\cdot g)\cdot fg-3D_x\cdot(S_xD_xf\cdot g)\cdot(D_x^2f\cdot g)]-5\lambda b[S_x(D_x^4f\cdot g)\cdot fg-D_x^3(S_xD_xf\cdot g)\cdot fg-3D_x(S_xD_x^2f\cdot g)\cdot(D_xf\cdot g)]$$

$$\overset{(14)-(16)}{=\!=}2S_x[(D_t+aD_x^3-5\lambda bD_x^4+bD_x^5)f\cdot g]fg+5\lambda^4bD_x(S_xf\cdot g)\cdot fg+(10\lambda^3b-6\lambda a)D_x(S_xf\cdot g)\cdot(D_xf\cdot g)+15\lambda^2bD_x(S_xf\cdot g)\cdot(S_xf\cdot g)+5\lambda^2bD_x^3(S_xf\cdot g)\cdot fg$$

$$\overset{(A9)-(A11)}{=\!=}2S_x[(D_t+aD_x^3-5\lambda bD_x^4+bD_x^5)f\cdot g]fg+5\lambda^4bD_x(S_xf\cdot g)\cdot fg+(10\lambda^3b-6\lambda a)D_x(S_xf\cdot g)\cdot(D_xf\cdot g)+15b\lambda^2[S_x(D_x^3f\cdot g)\cdot fg-D_x(S_xD_x^2f\cdot g)\cdot fg-2D_x(S_xD_xf\cdot g)\cdot(D_xf\cdot g)+S_x(D_x^2f\cdot g)\cdot(D_xf\cdot g)]+5\lambda^2b\cdot[S_x(D_x^3f\cdot g)\cdot fg-3S_x(D_x^2f\cdot g)\cdot(D_xf\cdot g)]$$

$$\overset{(15),(16)}{=\!=}2S_x[(D_t+(a+10\lambda^2b)D_x^3-5\lambda bD_x^4+bD_x^5)f\cdot g]fg-10\lambda^4bD_x(S_xf\cdot g)\cdot fg-(20\lambda^3b+6\lambda a)D_x(S_xf\cdot g)\cdot(D_xf\cdot g)$$

$$\overset{(A5)}{=\!=}2S_x[(D_t+(a+10\lambda^2b)D_x^3-5\lambda bD_x^4+bD_x^5)f\cdot g]fg-10\lambda^4bD_x(S_xf\cdot g)\cdot fg-(20\lambda^3b+6\lambda a)[S_x(D_x^2f\cdot g)\cdot fg-D_x(S_xD_xf\cdot g)\cdot fg]$$

$$\overset{(14),(A9)}{=\!=}2S_x[[D_t+(3\lambda^2a+5\lambda^4b)D_x-(3\lambda a+10\lambda^3b)D_x^2+(a+10\lambda^2b)D_x^3-5\lambda bD_x^4+bD_x^5]f\cdot g]\cdot fg\overset{(15)}{=\!=}0.$$

This proves our result.

# 3 Conclusion

In this paper, the $N$=1 supersymmetric combined KdV-CDG equation is given out. We studied it within the framework of Hirota's bilinear method. A Bäcklund transformation is obtained. We assure that these are important in discussing the integrability and other properties about this system.

# 4 Appendix: Some bilinear identities

In this appendix, we list the relevant bilinear identities, which can be proved directly. Here $a,b,c$ and $d$ are arbitrary even functions of the independent variables $x,t,\theta$.

$$(S_xD_ta\cdot a)b^2-a^2(S_xD_tb\cdot b)=2S_x(D_ta\cdot b)\cdot ab\,, \tag{A1}$$

$$(S_xD_x^3a\cdot a)b^2-a^2(S_xD_x^3b\cdot b)=2S_x[(D_x^3a\cdot b)\cdot ab-3(D_x^2a\cdot b)\cdot(D_xa\cdot b)]-3[(S_xD_xa\cdot a)\cdot(D_x^2b\cdot b)-(S_xD_xb\cdot b)(D_x^2a\cdot a)]\,, \tag{A2}$$

$$(S_xD_x^5a\cdot a)b^2-a^2(S_xD_x^5b\cdot b)=2S_x[(D_x^5a\cdot b)\cdot ab-5(D_x^4a\cdot b)\cdot(D_xa\cdot b)+10(D_x^3a\cdot b)\cdot(D_x^2a\cdot b)]-5[(S_xD_xa\cdot a)(D_x^4b\cdot b)+2(S_xD_x^3a\cdot a)(D_x^2b\cdot b)-2(S_xD_x^3b\cdot b)\cdot(D_x^2a\cdot a)-(S_xD_xb\cdot b)(D_x^4a\cdot a)], \tag{A3}$$

$$(S_xD_xa\cdot a)(D_x^4b\cdot b)+2(S_xD_x^3a\cdot a)(D_x^2b\cdot b)-2(S_xD_x^3b\cdot b)(D_x^2a\cdot a)-(S_xD_xb\cdot b)\cdot(D_x^4a\cdot a)=D_x(S_xD_x^3a\cdot b)\cdot(D_xa\cdot b)+D_x(S_xD_xa\cdot b)(D_x^3a\cdot b)+D_x^3(S_xD_xa\cdot b)\cdot(D_xa\cdot b)-2S_x(D_x^4a\cdot b)\cdot(D_xa\cdot b)+4S_x(D_x^3a\cdot b)\cdot(D_x^2a\cdot b), \tag{A4}$$

$$D_x(S_xa\cdot b)\cdot(D_xa\cdot b)=S_x(D_x^2a\cdot b)\cdot ab-D_x(S_xD_xa\cdot b)\cdot ab, \tag{A5}$$

$$2S_x(D_x^2a\cdot a)\cdot(D_xb\cdot a)-2D_x(S_xD_xa\cdot b)\cdot (D_xb\cdot a)=(S_xD_xa\cdot a)(D_x^2b\cdot b)-(S_xD_xb\cdot b)(D_x^2a\cdot a), \tag{A6}$$

$$D_x^3(S_xa\cdot b)\cdot(D_xa\cdot b)=S_x(D_x^4a\cdot b)\cdot ab-D_x(S_xD_x^3a\cdot b)\cdot ab-3D_x(S_xD_xa\cdot b)\cdot (D_x^2a\cdot b), \tag{A7}$$

$$D_x(S_xa\cdot b)\cdot(D_x^3a\cdot b)=S_x(D_x^4a\cdot b)\cdot ab-D_x^3(S_xD_xa\cdot b)\cdot ab-3D_x(S_xD_x^2a\cdot b)\cdot(D_xa\cdot b), \tag{A8}$$

$$D_x(S_xa\cdot b)\cdot ab=S_x(D_xa\cdot b)\cdot ab\ , \tag{A9}$$

$$D_x(S_xa\cdot b)\cdot(D_x^2a\cdot b)=S_x(D_x^3a\cdot b)\cdot ab+S_x(D_x^2a\cdot b)\cdot(D_xa\cdot b)-D_x(S_xD_x^2a\cdot b)\cdot ab-2D_x(S_xD_xa\cdot b)\cdot(D_xa\cdot b), \tag{A10}$$

$$D_x^3(S_xa\cdot b)\cdot ab=S_x(D_x^3a\cdot b)\cdot ab-3S_x(D_x^2a\cdot b)\cdot(D_xa\cdot b). \tag{A11}$$

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# 超对称复合 KdV-CDG 方程：双线性方法


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**摘要：**延拓了复合 KdV-CDG 方程，并用 Hirota 双线性方法考虑了 $N$=1 超对称复合 KdV-CDG 方程，得到了它的一个 Bäcklund 变换.



**关键词：**复合 KdV-CDG 方程；超对称；Hirota 双线性

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