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Higher-order Integrality for Weak *A*-harmonic Tensors

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Abstract: The higher integrality for the very weak solutions of the non-homogeneous *A*-harmonic equation $d^*A(x,g+\mathrm{d}u)=d^*h$ is obtained using the result of Riesz transforms and interpolation.

Key words: interpolation; Hodge decomposition; weak *A*-harmonic tensor

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Let $\wedge^l=\wedge^l(R^n)$ denote the linear space of l-covectors in $\boldsymbol{R^n}\geqslant \mathrm{d}$, $l=1,2,\cdots,n$. For $l=0$ we put $\wedge^0(R^n)=R^n$. Also, $\wedge^l=\wedge^l(R^n)=0$ if $l<0$ or $l>n$. This is an inner product space of dimension C_n^l. A differential form of degree l on $\boldsymbol{R^n}$ is simply a function or Schwarz distribution on $\boldsymbol{R^n}$ with values in $\wedge^l$. We shall consider a nonlinear mapping $A:R^n\times\wedge^l(R^n)\to\wedge^l(R^n)$.

Which satisfies the usual measurability conditions (Carathéodory conditions) and for some $1<p<\infty$, the following conditions hold:

(i) the monotonicity inequality

$\langle A(x,\xi)-A(x,\zeta),\xi-\zeta\rangle\geqslant\alpha|\xi-\zeta|^2(|\xi+\zeta|)^{p-2}$.

(ii) the Lipschitz type condition

$|A(x,\xi)-A(x,\zeta)|\leqslant\beta|\xi-\zeta|(|\xi+\zeta|)^{p-2}$.

(iii) the homogeneity condition

$A(x,\lambda\xi)=|\lambda|^{p-2}\lambda A(x,\xi)$.

For almost every $x\in\boldsymbol{R^n},\lambda\in\boldsymbol{R}$ and all $\xi,\zeta\in\wedge^l(R^n)$. The exponent $p>1$ will determine the natural Sobolev class, denoted by $W^p_{d,loc}(R^n,\wedge^{l-1})$, in which to consider the nonhomogeneous *A*-harmonic equation

$$d^*A(x,g+\mathrm{d}u)=d^*h, \tag{1}$$

(see [1] for the definition of $W^p_{d,loc}(R^n,\wedge^{l-1})$ and other Sobolev classes of differential forms).

Definition 1 For $g\in L^p(R^n,\wedge^l)$ and $h\in L^q(R^n,\wedge^l)$ with $1/p+1/q=1$, an *A*-harmonic tensor $u\in W^p_{d,loc}(R^n,\wedge^{l-1})$ is a weak solution of (1) defined as

$$\int_{R^n}\langle A(x,g+\mathrm{d}u),\mathrm{d}\phi\rangle=\int_{R^n}\langle h,\mathrm{d}\phi\rangle, \tag{2}$$

for all $\phi\in C_0^\infty(R^n,\wedge^{l-1})$.

Definition 2 A differential form $u\in W^r_{d,loc}(R^n,\wedge^{l-1})$, $r\geqslant\max\{1,p-1\}$, is called a weak *A*-harmonic tensor if it satisfies equation (1) in the distributional sense for $g\in L^r(R^n,\wedge^l)$ and $h\in L^{r/(p-1)}(R^n,\wedge^l)$, that is, the integral identity (2) holds for all $\phi\in L^{r/(r-p+1)}(R^n,\wedge^{l-1})$.

Iwaniec T and Sbordone C showed in [2] the regularity result for the very weak solutions of the

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equation

$$\mathrm{div}A(x,\nabla u(x))=0. \tag{3}$$

They used the Hodge decomposition to build a test function, and obtained that there exist $r_1=r_1(p,\Omega)$ and $r_2=r_2(p,\Omega)$ satisfying $1<r_1<p<<r_2<\infty$, such that every very weak solution $u\in W^{1,r_1}_{loc}(\Omega)$ of equation (3) belongs to $u\in W^{1,r_2}_{loc}(\Omega)$. Stroffolini B used the similar technique in [1] to obtain the regularity results for the very weak solutions of the homogeneous *A*-harmonic equation

$$d^*A(x,\mathrm{d}u)=0. \tag{4}$$

In the paper [3], Stroffolini Capone C, et al proved the similar results via Riesz transforms and interpolation. For other results on very weak solution, see [4-8]. Now, inspired by the paper [3], we obtain the following regularity theorem.

Theorem 1 There exist exponent $\max\{1,p-1\}<p_1=p_1(n,p)<p$ and $p_2=p_2(n,p)>p$, such that for $p_1\leqslant r\leqslant p\leqslant s\leqslant p_2$, $g\in L^s(R^n,\wedge^l)$ and $h\in L^{s/(p-1)}R^n,\wedge^l)$, we have if $u\in W^r_d(R^n,\wedge^{l-1})\cap L^1_{loc}(R^n,\wedge^{l-1})$ is weak *A*-harmonic, then $u\in W^s_d(R^n,\wedge^{l-1})\cap L^1_{loc}(R^n,\wedge^{l-1})$ as well and moreover

$$\int_{\boldsymbol{R}^n}|\mathrm{d}u|^s\leqslant C\int_{\boldsymbol{R}^n}(|g|^s+|h|^{s/(p-1)}), \tag{5}$$

Then, u is an *A*-harmonic tensor in the usual sense.

1 Notations and Preliminary Results

Throughout we use the notation of [9]. For the sake of completeness we list basic notions of exterior calculus. The direct sum $\wedge(R^n)=\oplus_{l=0}^n\wedge^l(R^n)$ is an exterior algebra with respect to the wedge product $\wedge$. We define the Hodge star operator $*:\wedge\to$ by the rule $*1=e_1\wedge e_2\wedge\cdots\wedge e_n$, and $\alpha\wedge*\beta=\beta\wedge*\alpha=\langle\alpha,\beta\rangle(*1)$.

For all $\alpha,\beta\in\wedge.$ the norm of $\alpha\in\wedge.$ is given by the formula $|a|^2=\langle\alpha,\alpha\rangle=*(\alpha\wedge*\alpha)\in\wedge^0=R.$

The Hodge star is an isometric isomorphism on $\wedge$ with $*:\wedge^l\to\wedge^{n-l}$ and $**(-1)^{l(n-l)}:\wedge^l\to\wedge^l$..

A differential *l*-form ω on $\boldsymbol{R^n}$ is a locally integrable function or a Schwarz distribution on $\boldsymbol{R^n}$ with values in $\wedge(R^n)$. We denote the space of differential *l*-form by $D'(\boldsymbol{R^n},\wedge^l)$. If $x_1,x_2,\cdots,x_n$ denote the coordinate functions in $\boldsymbol{R^n}$, then the natural generators for the algebra of differential forms are the differentials $\mathrm{d}x_1,\mathrm{d}x_2,\cdots,\mathrm{d}x_n$. Thus each $\omega:R^n\to\wedge^l(R^n)$ can be written as

$$\omega(x)=\sum_I\omega_I(x)\mathrm{d}x_I=$$
$$\sum\omega_{i_1i_2\cdots i_l}(x)\mathrm{d}x_{i_1}\wedge\mathrm{d}x_{i_2}\wedge\cdots\wedge\mathrm{d}x_{i_l},$$

where $\sum\omega_{i_1i_2\cdots i_l}(x)$ are either functions or distributions. We write $L^p(R^n,\wedge^l)$ for the *l*-forms $\omega(x)$ with $\omega_I\in L^p(\boldsymbol{R^n},\boldsymbol{R^n})$ for all ordered l tuples I. Thus $L^p(R^n,\wedge^l)$ is a Banach space with norm

$$\|\omega\|_p=(\int_{R^n}|\omega(x)|^p\mathrm{d}x)^{1/p}=$$
$$(\int_{R^n}(\sum_I|\omega_I(x)|^2)^{p/2}\mathrm{d}x)^{1/p}.$$

We denote the exterior derivative by

$$d:D'(R^n,\wedge^l)\to D'(R^n,\wedge^{l+1}).$$

For $l=0,1,2,\cdots,n$ by the following conditions:

(i) For $l=0$, $\mathrm{d}f$ is the differential of f. (ii) For $\alpha\in D'(R^n,\wedge^l)$ and $\beta\in D'(R^n,\wedge^k)$, we have $\mathrm{d}(\alpha\wedge\beta)=\mathrm{d}\alpha\wedge\beta+(-1)^l\alpha\wedge\mathrm{d}\beta.$ (iii) $\mathrm{d}(\mathrm{d}\alpha)=0$.

The elements of the kernel of $d:D'(R^n,\wedge^l)\to D'(R^n,\wedge^{l+1})$ are called closed *l*-forms and those in the image of $d:D'(R^n,\wedge^{l-1})\to D'(R^n,\wedge^l)$ are the exact *l*-forms.

The formal adjoint operator (the Hodge co-differential) is the operator $d^*:D'(R^n,\wedge^{l+1})\to D'(R^n,\wedge^l)$ given by $d^*=(-1)^{nl+1}*d*$.

The forms in the image of $d^*:D'(R^n,\wedge^{l+1})\to D'(R^n,\wedge^l)$ are called coexact *l*-forms.

We make a brief list of spaces of differential forms: $L^p(R^n,\wedge^l)$ -the space of differential forms with coefficients in $L^p(\boldsymbol{R^n})$; $L^p_1(R^n,\wedge^l)$ -the space of differential forms ω such that $\nabla\omega$ is a regular distribution of class $L^p(R^n,\wedge^l)$; $W^{1,p}(R^n,\wedge^l)$ -the Sobolev space of l-forms defined by $L^p(R^n,\wedge^l)\cap L_1^p(R^n,\wedge^l)$; $W_d^p(R^n,\wedge^l)$ -the space of l-forms ω such that $\mathrm{d}\omega\in L^p(R^n,\wedge^{l+1})$; $W_{d^*}^p(R^n,\wedge^l)$ -the space of l-forms ω such that $d^*\omega\in L^p(R^n,\wedge^{l-1})$. Of special importance is the following Hodge decomposition theorem, see [10].

Theorem 2 (Hodge decomposition) For each $\omega\in L^r(R^n,\wedge^l)$, $1<r<\infty$, there exist differential forms

$\alpha\in\ker d^*\cap L_1^r(R^n,\wedge)$ and $\beta\in\ker d\cap L_1^r(R^n,\wedge)$,

such that

$$\omega=\mathrm{d}\alpha+d^*\beta. \tag{6}$$

The forms $\mathrm{d}\alpha$ and $d^*\beta$ are unique and satisfy the uniform estimate

$$\|\alpha\|_{L_1^r(R^n)}+\|\beta\|_{L_1^r(R^n)}\leqslant C_r(n)\|\omega\|_r. \tag{7}$$

For some constant $C_r(n)$ independent of ω.

The following results about interpolation are coming from [3]. Given $1<p_j<\infty$, for $j=1,2$, we consider the function

$$\varPhi(s)=\varPhi(s;p_1,p_2)=\begin{cases}s^{p_1}, 0\leqslant s\leqslant 1,\\ s^{p_2},\ s\geqslant 1.\end{cases} \tag{8}$$

if $q_j=p_j/(p_j-1)$ is the Hölder conjugate exponent to p_j, we set $\varPsi=\varPhi(t;q_1,q_2)$. Then, for all $s,t\geqslant 0$,

$$st\leqslant\varPhi(s)+\varPsi(t), \tag{9}$$

$$\varPhi(s+t)\leqslant\max\{2^{p_1-1},2^{p_2-1}\}(\varPhi(s)+\varPhi(t)). \tag{10}$$

Due to the inequality (9), we say that $\varPhi$ and $\varPsi$ are conjugate functions to each other.

Now, let (X,μ) be a measure space and E be a separable complex Hilbert space. The Hermitian product will be denote by $\langle,\rangle$ and the induced norm by $|\cdot|$. We shall consider μ-measurable functions from X into E. To simplify notation, if $\varPhi$ is any function on $[0,\infty]$ and $f\in E$, then we will write $\varPhi(f)$ instead of $\varPhi(|f|)$. Also, we define the truncation

$$[f]=\begin{cases}|f|,\ |f|\leqslant 1,\\ 1/|f|,\ |f|>1.\end{cases}$$

Let $T:L^r\to L^r$ be a linear operator, bounded for $r\in[r_1,r_2], 1<r_1<r_2<\infty$. In the paper [3], there are the following results.

Lemma 1 There exists $C=C(r_1,r_2,\|T\|_{r_1},\|T\|_{r_2})>0$, such that, for all $f\in L^{r_1}+L^{r_2}$,

$$\int_X\varPhi(Tf)\mathrm{d}\mu\leqslant C\int_X\varPhi(f)\mathrm{d}\mu, \tag{11}$$

where $\varPhi(s)=\varPhi(s;p_1,p_2)$ is defined in (8), $p_1,p_2\in[r_1,r_2]$.

Lemma 2 Suppose that $r_1\leqslant r/(1+\varepsilon)\leqslant r_2$, and $[f]^\varepsilon f\in L^r(X,E)$ verifies $Tf=0$. Then, there exists $C=C(r_1,r_2,\|T\|_{r_1},\|T\|_{r_2})>0$ such that

$$\int_X|T[f]^\varepsilon f\,\|f|^{r-1}\mathrm{d}\mu\leqslant C\varepsilon\int_X[f]^\varepsilon|f|^r\mathrm{d}\mu. \tag{12}$$

2 The Proof of the Theorem

We shall prove Theorem 1 with $r=p-\varepsilon$, $s=p+\varepsilon$ for $\varepsilon\in[0,1]$ sufficiently small. We begin with an a priori estimate.

Lemma 3 If u is a solution of (1) and $[\mathrm{d}u]^\varepsilon|\mathrm{d}u|^p$ is integrable, then $\mathrm{d}u\in L^s(\boldsymbol{R}^n,\wedge^l)$ and (5) holds.

Proof We first show that (5) holds for $p>2$. Under the hypotheses of Lemma and the Lipschitz type condition, we know that $A(x,g+\mathrm{d}u)\in L^{s/(p-1)}+L^{r/(p-1)}$ and thus the test function of Definition 1 extends to $\mathrm{d}\phi\in L^{s/(s-p+1)}\cap L^{r/(r-p+1)}$. Then in view of the Hodge decomposition (Theorem 2), we decompose

$$[\mathrm{d}u]^\varepsilon\mathrm{d}u=\mathrm{d}\phi+H. \tag{13}$$

Thus, ϕ can be used as a test function. Using (2) and the Lipschitz type condition (ii), we obtain

$$\int_{R^n}\langle A(x,\mathrm{d}u),\mathrm{d}\phi\rangle=\int_{R^n}\langle A(x,\mathrm{d}u),\mathrm{d}\phi\rangle-$$

$$\int_{R^n}\langle A(x,g+\mathrm{d}u),\mathrm{d}\phi\rangle+\int_{R^n}\langle A(x,g+\mathrm{d}u),\mathrm{d}\phi\rangle=\int_{R^n}\langle A(x,\mathrm{d}u),\mathrm{d}\phi\rangle-\int_{R^n}\langle A(x,g+\mathrm{d}u),\mathrm{d}\phi\rangle+\int_{R^n}\langle h,\mathrm{d}\phi\rangle\leqslant C_1\int_{\boldsymbol{R}^n}|g|(|g+\mathrm{d}u|+\mathrm{d}u)^{p-2}\mathrm{d}\phi+\int_{\boldsymbol{R}^n}|h||\mathrm{d}\phi|\leqslant C_2\int_{\boldsymbol{R}^n}(|g||\mathrm{d}u|^{p-2}+|g|^{p-1}+|h|)|\mathrm{d}\phi|. \quad (14)$$

By the Hodge decomposition (13), we find that

$$\int_{R^n}\langle A(x,\mathrm{d}u),\mathrm{d}\phi\rangle=\int_{R^n}\left\langle A(x,\mathrm{d}u),[\mathrm{d}u]^{\varepsilon}\mathrm{d}u-H\right\rangle=\int_{R^n}\left\langle A(x,\mathrm{d}u),[\mathrm{d}u]^{\varepsilon}\mathrm{d}u\right\rangle-\int_{R^n}\langle A(x,\mathrm{d}u),H\rangle\geqslant\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p-\int_{\boldsymbol{R}^n}|H||\mathrm{d}u|^{p-1}. \quad (15)$$

Combing (15) and (14), we have

$$\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p-\int_{\boldsymbol{R}^n}|H||\mathrm{d}u|^{p-1}\leqslant C_2\int_{\boldsymbol{R}^n}(|g||\mathrm{d}u|^{p-2}+|g|^{p-1}+|h|)|\mathrm{d}\phi|. \quad (16)$$

We apply Lemma 2 to the measure space $(\boldsymbol{R}^n,\mathbf{d}\boldsymbol{x})$ and $E=\wedge^l(R^n)$. The operator T is then defined by $T\omega=d^*\beta$ and the operator G is defined by $G\omega=\mathrm{d}\alpha$, where $\omega=\mathrm{d}\alpha+d^*\beta$ (see the Hodge decomposition (6)). In view of (7), T: $L^r(R^n,\wedge)\to L^r(R^n,\wedge)$ and G:$L^r(R^n,\wedge)\to L^r(R^n,\wedge)$ are bounded linear operators for all $1<r<\infty$.

It follows from the uniqueness of the Hodge decomposition that the kernel of T consists of the exact forms from $\mathrm{d}W_d^r(R^n,\wedge^{l-1})$, while the range of T consists of the coexact forms. In view of Lemma 2, we have

$$\int_{\boldsymbol{R}^n}|H||\mathrm{d}u|^{p-1}\leqslant C_3\varepsilon\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p. \quad (17)$$

Therefore, we can estimate the left-hand side of (16) from below

$$\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p-\int_{\boldsymbol{R}^n}|H||\mathrm{d}u|^{p-1}\geqslant(1-C_3\varepsilon)\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p. \quad (18)$$

Now, we estimate each term in the right-hand side of (16). By Young's inequality, we obtain

$$|g||\mathrm{d}u|^{p-2}\leqslant\theta|\mathrm{d}u|^{p-1}+C_\theta|g|^{p-1}.$$

Moreover, in view of (9), we have

$$|\mathrm{d}u|^{p-1}|\mathrm{d}\phi|\leqslant\Phi(|\mathrm{d}u|^{p-1})+\Psi(\mathrm{d}\phi).$$

With $\Phi(t)=\Phi(t;s/(p-1),r/(p-1))$ and $\Psi(t)=\Psi(t;s/(s-p+1),r/(r-p+1))$. Then $\Phi(t^{p-1})=[t]^{\varepsilon}t^p$ and $\Psi([t]^{\varepsilon}t)=[t]^{\varepsilon}t^p$. By the definition of the operator G, the equality (13), and Lemma 1, we find that

$$\int_{\boldsymbol{R}^n}\Psi(\mathrm{d}\phi)=\int_{\boldsymbol{R}^n}\Psi(G([\mathrm{d}u]^{\varepsilon}\mathrm{d}u))\leqslant C_4\int_{\boldsymbol{R}^n}\Psi([\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|)=C_4\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p.$$

$$\int_{\boldsymbol{R}^n}|g||\mathrm{d}u|^{p-2}|\mathrm{d}\phi|\leqslant(1+C_4)\theta\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}\cdot|\mathrm{d}u|^p+C_\theta\int_{\boldsymbol{R}^n}|g|^{p-1}|\mathrm{d}\phi|. \quad (19)$$

By Hölder inequality, and the uniform estimate (7) in the Hodge decomposition, and inequality $([\mathrm{d}u]^{\varepsilon}\mathrm{d}u)^{s/(s-p+1)}\leqslant[\mathrm{d}u]^{\varepsilon}[\mathrm{d}u]^p$, we find that

$$\int_{R^n}|g|^{p-1}\mathrm{d}\phi|\leqslant\|g\|_s^{p-1}\|\mathrm{d}\phi\|_{s/(s-p+1)}\leqslant C_5\|g\|_s^{p-1}\|[\mathrm{d}u]^{\varepsilon}\mathrm{d}u\|_{s/(s-p+1)}\leqslant C_5\|g\|_s^{p-1}\left(\int_{R^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p\right)^{(s-p+1)/s}, \quad (20)$$

$$\int_{R^n}|h||\mathrm{d}\phi|\leqslant\|h\|_{s/(p-1)}\|\mathrm{d}\phi\|_{s/(s-p+1)}\leqslant C_6\|h\|_{s/(p-1)}\left(\int_{R^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p\right)^{(s-p+1)/s}. \quad (21)$$

Combing (16) and (18)~(21), for ε and θ small enough, we deduce

$$\int_{R^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p\leqslant C_7(\|g\|_s^{p-1}+\|h\|_{s/(p-1)})\cdot\left(\int_{R^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p\right)^{(s-p+1)/s}.$$

which implies

$$\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^p\leqslant C_8\int_{\boldsymbol{R}^n}(|g|^s+|h|^{s/(p-1)}). \quad (22)$$

On the other hand, in view of the homogeneity condition (iii), for $\delta>0$, (1) can be written as

$$d^*A(x,(g+\mathrm{d}u)/\delta)=d^*(\delta^{1-p}h).$$

Therefore, using the inequality (22), we have

$$\int_{\boldsymbol{R}^n}(\delta[\mathrm{d}u/\delta])^{\varepsilon}|\mathrm{d}u|^p\leqslant C_9\int_{\boldsymbol{R}^n}(|g|^s+|h|^{s/(p-1)}).$$

Noticing that $\lim_{\delta\to\infty}\delta[\mathrm{d}u/\delta]=|\mathrm{d}u|$ *a.e.*, by Fatou Lemma, we have proved that (5) is true.

Now, we prove that (5) holds for $1<p\leqslant 2$. According to the above argument, we know that

$$\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^{p}-\int_{\boldsymbol{R}^n}|H||\mathrm{d}u|^{p-1}\leqslant C_{10}\int_{\boldsymbol{R}^n}|g|\cdot (|g+\mathrm{d}u|+\mathrm{d}u)^{p-2}\mathrm{d}\phi+\int_{\boldsymbol{R}^n}|h||\mathrm{d}\phi|. \quad (23)$$

In view of $|g|\leqslant|g+\mathrm{d}u|+|\mathrm{d}u|$ and $p-2\leqslant 0$, we obtain

$$(|g+\mathrm{d}u|+|\mathrm{d}u|)^{p-2}\leqslant|g|^{p-2}. \quad (24)$$

Substituting (24) into (23), we find that

$$\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^{p}-\int_{\boldsymbol{R}^n}|H||\mathrm{d}u|^{p-1}\leqslant C_{11}\int_{\boldsymbol{R}^n}(|g|^{p-2}+|h|)|\mathrm{d}\phi|,$$

and The following argument is similar. We have completed the proof of Lemma 3.

Proof of Theorem 1

Let us put $X_1=\{x\in\boldsymbol{R}^n:|\mathbf{d}\boldsymbol{u}|\leqslant 1\}$ and $X_2=\{x\in\boldsymbol{R}^n:|\mathbf{d}\boldsymbol{u}|\geqslant 1\}$, then, $\boldsymbol{R}^n=\boldsymbol{X}_1\cup\boldsymbol{X}_2$. By the hypothese of Theorem 1, we know $\mathrm{d}u\in L^{p-\varepsilon}(R^n,\wedge^l)$, and

$$\int_{\boldsymbol{R}^n}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^{p}=\int_{X_1\cup X_2}[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^{p}= \int_{X_1}|\mathrm{d}u|^{p+\varepsilon}+\int_{X_2}|\mathrm{d}u|^{p-\varepsilon}\leqslant \int_{X_1}|\mathrm{d}u|^{p-\varepsilon}+\int_{X_2}|\mathrm{d}u|^{p-\varepsilon}=\int_{\boldsymbol{R}^n}|\mathrm{d}u|^{p-\varepsilon}.$$

Therefore, $[\mathrm{d}u]^{\varepsilon}|\mathrm{d}u|^{p}\in L^1$. By the Lemma 3, we have $\mathrm{d}u\in L^s(R^n,\wedge^l)$ and (5) holds. This ends the proof of Theorem 1.

References:

[1] Stroffolini B. On weakly *A*-harmonic tensors[J]. Studia Mathematica, 1995, 114:289-301.

[2] Iwaniec T, Sbordone C. Weak minima of variational integrals[J]. J Reine Angew Math, 1994, 454:143-161.

[3] Stroffolini Capone C, Greco L, Iwaniec T. Higher integrability via Riesz transforms and interpolation[J]. Nonlinear Analysis, 2002, 49:513-523.

[4] Lewis J. On very weak solutions of certain elliptic equations[J]. Comm Part Diff Equa, 1993, 18:1515-1537.

[5] Hongya G. Regularity for very weak solutions of *A*-harmonic equation[J]. Acta Math Sin, 2001, 4(44): 605-610.

[6] Hongya G, Yinzhu C. Caccioppoli type inequality for weak solutions of *A*-harmonic equation and its applications[J]. Kyungpook Math J, 2004, 44(3):363-368.

[7] Hongya G, Yuquan Y, Suying X. Uniqueness for very weak solutions of *A*-harmonic equations with very weak boundary values[J]. Acta Math Sci, 2002, 22(1):41-46.

[8] Hongya G, Min W, Hongliang Z. On very weak solutions for obstacle problems of *A*-harmonic equations[J]. J Math Res Exp, 2004, 24(1):159-167.

[9] Iwaniec T, Lutoborski A. Integral estimates for null Lagrangians[J]. Arch Rational Mech Anal, 1993, 125:25-79.

[10] Iwaniec T, Martin G. Quasiregular mappings in even dimensions[J]. Acta Math, 1993, 170:29-81.

弱 *A*-调和张量的高阶可积性

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摘要: 首先, 利用 Riesz 变换和内插理论的结果得到了非齐次 *A*-调和方程 $d^*A(x,g+\mathrm{d}u)=d^*h$ 很弱解的一个先验估计. 然后, 利用这个先验估计得到了该方程很弱解的高阶可积性.

关键词: 内插; Hodge 分解; 弱 *A*-调和张量

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