A Recursive Method for Determining the One-Dimensional Submodules of Laurent-Ore Modules *

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ABSTRACT

We present a method for determining the one-dimensional submodules of a Laurent-Ore module. The method is based on a correspondence between hyperexponential solutions of associated systems and one-dimensional submodules. The hyperexponential solutions are computed recursively by solving a sequence of first-order ordinary matrix equations. As the recursion proceeds, the matrix equations will have constant coefficients with respect to the operators that have been considered.

Categories and Subject Descriptors

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General Terms

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1. INTRODUCTION

A Laurent-Ore algebra $\mathcal{L}$ over a field is a mathematical abstraction of common properties of linear partial differential and difference operators. Finite-dimensional $\mathcal{L}$-modules interpret finite-dimensional systems of linear partial differential and difference equations concisely and precisely. For example, a factor of a finite-dimensional system corresponds to a submodule of its module of formal solutions [7, 18]. A method for factoring finite-dimensional systems of linear PDE's is given in [14], and, recently, a method for factoring finite-dimensional $\mathcal{L}$-modules is presented in [18]. Both are generalizations of the associated equations method dated back to Beke [5]. A basic step in these methods is to compute one-dimensional submodules of some exterior powers of the given module. One approach for computing one-dimensional submodules is to identify all possible partial "logarithmic derivatives" of hyperexponential solutions with respect to each differential or difference operator, and then, glue the partial results together by common associates, as described in [11, 13, 14]. In this approach, one would have to compute hyperexponential solutions of several ordinary differential and difference equations over the ground field F, and in addition, it is only applicable when F is a field of rational functions and each operator acts on only one variable non-trivially.

In this paper, we describe a method that is recursive on the set of differential and difference operators acting on F. It computes hyperexponential solutions of an (ordinary) matrix equation and then proceeds by back-substitution. In doing so, one avoids computing all possible partial "logarithmic derivatives" of hyperexponential solutions, which may be costly. Each time an operator is carried on, we can reduce our problem to solving a first-order matrix equation whose coefficients are constants in F with respect to the operator. So the systems to be solved become simpler as the computation goes on. In particular, we are able to remove the restrictions imposed in [11, 13, 14, 18] on operators and now require only that they commute.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and define the notion of hyperexponential vectors. In Section 3, we describe a correspondence between the one-dimensional submodules of an $\mathcal{L}$-module and the hyperexponential solutions of associated systems. In Section 4, we demonstrate how to identify unspecified constants appearing in hyperexponential vectors to make these vectors extensible. In Section 5 we describe an

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algorithm for determining the one-dimensional submodules of an $\mathcal{L}$-module and give some examples.

2. PRELIMINARIES

Throughout the paper, F is a commutative field of characteristic zero. Let R be an F-algebra, not necessarily commutative. We present some basic facts about finite-dimensional R-modules, and the notion of hyperexponential vectors.

2.1 One-dimensional submodules

Let M be a (left) R-module that is a finite-dimensional vector space over F. A submodule of M is said to be *one-dimensional* if it is also a vector space of dimension one over F. Let N be a one-dimensional submodule of M and $\mathbf{v}$ a non-zero element of N. Then N is generated by $\mathbf{v}$ as a vector space over F. So we may write $N = F\mathbf{v}$. Moreover, for all $r \in R$, there exists $f \in F$ such that $r\mathbf{v} = f\mathbf{v}$.

We review some results concerning one-dimensional submodules, which will help us describe one-dimensional submodules of a finite-dimensional module over a Laurent-Ore algebra by a finite amount of information, as sketched in [16, page 111] and [8] for differential modules.

Lemma 1. *Let $N_1, \ldots, N_s$ be one-dimensional submodules of an R-module such that the sum $\sum_{i=1}^s N_i$ is direct. If N is a nontrivial submodule contained in $\sum_{i=1}^s N_i$, then there exists a one-dimensional submodule $N' \subset N$. Moreover, N' is isomorphic to some N_i.*

Proof. Every element of $\sum_{i=1}^s N_i$ can be (uniquely) expressed as a sum of elements in $N_1, \ldots, N_s$. Among all nonzero elements of N, choose a $\mathbf{v} \in N$ such that its additive expression is shortest. Without loss of generality, suppose that the additive expression of $\mathbf{v}$ is $\sum_{i=1}^t \mathbf{v}_i$ where $\mathbf{v}_i \in N_i$ is nonzero and $1 \leq t \leq s$. For any $r \in R$, $r\mathbf{v}_i \in N_i$, and, hence, $r\mathbf{v}_i = a_i\mathbf{v}_i$ for some $a_i \in F$, because N_i is one-dimensional. It follows that $r\mathbf{v} = \sum_{i=1}^t a_i\mathbf{v}_i$. By the selection of $\mathbf{v}$, $r\mathbf{v}-a_1\mathbf{v} = 0$, and, hence, $F\mathbf{v}$ is a one-dimensional submodule in N. Let π_1 be the projection from $\bigoplus_{i=1}^s N_i$ to N_1. Note that $\pi_1(\mathbf{v})=\mathbf{v}_1\neq 0$. So the restriction of π_1 on $F\mathbf{v}$ is bijective since $F\mathbf{v}$ and N_1 both have dimension one. $\square$

As a consequence, one can prove by induction on s that

Corollary 2. *If $N_1, \ldots, N_s$ are pairwise nonisomorphic one-dimensional submodules of an R-module, then $\sum_{i=1}^s N_i$ is direct.*

Let $\mathcal{M}_1$ be the set of all one-dimensional submodules of a finite-dimensional R-module M, and $\overline{\mathcal{M}}_1$ the set of equivalence classes of $\mathcal{M}_1$ modulo isomorphism. The cardinality of $\overline{\mathcal{M}}_1$ is finite by Corollary 2. For an equivalence class I in $\overline{\mathcal{M}}_1$, there exist a finite number of submodules $N_1 = F\mathbf{v}_1$, ..., $N_s = F\mathbf{v}_s$ in I such that $\mathbf{v}_1, \ldots \mathbf{v}_s$ are linearly independent over F, and moreover, for every $F\mathbf{v} \in I$, $\mathbf{v}$ is linearly dependent on $\mathbf{v}_1, \ldots \mathbf{v}_s$ over F. Then $\sum_{N\in I} N = \bigoplus_{i=1}^s N_i$. Setting the latter (direct) sum to be S_I, one can prove, using Lemma 1 and induction, the following

Proposition 3. *With the notation just introduced, we have $\sum_{N\in\mathcal{M}_1} N = \bigoplus_{I\in\overline{\mathcal{M}}_1} S_I$.*

2.2 Hyperexponential vectors

Let R be a ring and Δ be a finite set of commuting maps from R to itself. A map in Δ is assumed to be either a derivation on R or an automorphism of R. Recall that a derivation δ is an additive map satisfying the multiplicative rule $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$. The pair (R, Δ) is called a Δ-ring.

For a derivation $\delta \in \Delta$, an element c of R is called a *constant* with respect to δ if $\delta(c) = 0$. For an automorphism $\sigma \in \Delta$, c is called a *constant* with respect to σ if $\sigma(c) = c$. An element c of R is called a *constant* if it is a constant with respect to all maps in Δ. The set of constants of R, denoted by C_R, is a subring. The ring C_R is a subfield if R is a field.

Let (F, Δ) be a Δ-field and R a commutative ring containing F. If all the maps in Δ can be extended to R in such a way that all derivations (resp. automorphisms) of F become derivations (resp. automorphisms) of R and the extended maps commute pairwise, then (R, Δ), or simply R, is called a Δ-extension of F.

In a Δ-extension R of F, a non-zero element h is said to be *hyperexponential with respect to a map ϕ in* Δ if $\phi(h)=rh$ for some $r \in F$. The element r is denoted $\ell\phi(h)$. The element h is said to be *hyperexponential* over F if it is hyperexponential with respect to all the maps in Δ. A non-zero vector $V \in R^n$ is said to be *hyperexponential (with respect to a map ϕ)* if there exist $h \in R$, hyperexponential (with respect to ϕ), and $W \in F^n$ such that $V = hW$ (see [18, Chapter 4]). A straightforward calculation shows that

Lemma 4. *Let h_1, h_2 be two hyperexponential elements of a Δ-extension E of F. If $\ell\phi(h_1) = \ell\phi(h_2)$ for all $\phi \in \Delta$ and h_2 is invertible, then h_1/h_2 is a constant.*

Let Δ' be a nonempty subset of Δ, and let E and E' be Δ and Δ'-extensions of F, respectively. The F-algebra $E\otimes_F E'$ is a Δ'-extension, where $\delta(r \otimes r') = \delta(r) \otimes r' + r \otimes \delta(r')$ and $\sigma(r \otimes r') = \sigma(r) \otimes \sigma(r')$ for all derivation operators δ and automorphisms σ in Δ'. The canonical maps $E \longrightarrow E\otimes_F E'$ and $E' \longrightarrow E \otimes_F E'$ are injective since E and E' are F-algebras. Thus $E \otimes_F E'$ can be regarded as a Δ'-extension that contains both E and E'.

3. MODULES OVER LAURENT-ORE ALGEBRAS

In the sequel, we set $\Delta = \{\delta_1, \ldots, \delta_\ell, \sigma_{\ell+1}, \ldots, \sigma_m\}$ where $\delta_1, \ldots, \delta_\ell$ are derivation operators on F and $\sigma_{\ell+1}, \ldots, \sigma_m$ are automorphisms of F.

The *Laurent-Ore algebra* over F is a noncommutative ring $\mathcal{L} = F[\partial_1, \ldots, \partial_m, \partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}]$ whose multiplication rules are $\partial_s\partial_t = \partial_t\partial_s$, $\partial_j\partial_j^{-1} = 1$, $\partial_i a = a\partial_i+\delta_i(a)$, $\partial_j a = \sigma_j(a)\partial_j$, and $\partial_j^{-1} a = \sigma_j^{-1}(a)\partial_j^{-1}$, where $1 \leq s < t \leq m$, $1 \leq i\leq\ell$, $\ell + 1 \leq j \leq m$, and $a \in F$. The algebra $\mathcal{L}$ can be constructed from an Ore algebra over F (see [7]). For any finite-dimensional $\mathcal{L}$-module, its F-bases may be computed via the Gröbner basis techniques in [18, Chapter 3].

Let Δ' be a nonempty subset of Δ. Then Δ' corresponds to a Laurent-Ore algebra $\mathcal{L}'$. An $\mathcal{L}$-module M is also an $\mathcal{L}'$-module. To distinguish the different module structures, we write (M, Δ) and (M, Δ') to mean that M is an $\mathcal{L}$-module and an $\mathcal{L}'$-module, respectively.

Let M be an $\mathcal{L}$-module with a finite basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ over F. The module structure of M is determined by m matrices $A_1, \ldots, A_m$ in $F^{n\times n}$ such that

$$\partial_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^T = A_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)^T,\ i=1, \ldots, m. \quad (1)$$

Note that $A_{\ell+1}, \ldots, A_m$ are invertible because $\mathcal{L}$ contains $\partial_{\ell+1}^{-1}, \ldots, \partial_m^{-1}$. We call $A_1, \ldots, A_m$ *the structure matrices* with respect to $\mathbf{b}_1, \ldots, \mathbf{b}_n$. For a vector $Z = (z_1, \ldots, z_n)^T$ of unknowns,

$$\delta_i(Z) = -A_i^T Z,\ i \le \ell, \quad \sigma_j(Z) = \left(A_j^{-1}\right)^T Z,\ j > \ell, \qquad (2)$$

is called the system associated to M and the basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Systems associated to different bases are equivalent in the sense that the solutions of one system can be transformed to those of another by a matrix in $F^{n\times n}$. The commutativity of the maps in Δ implies that (2) is fully integrable [7, Definition 2]. A detailed verification of this assertion is presented in [18, Lemma 4.1.1]. On the other hand, every fully integrable system is associated to its module of formal solutions [7, Example 4], which is an $\mathcal{L}$-module of finite dimension.

A solution V of (2) is called a hyperexponential solution if V is a hyperexponential vector. It is called a rational solution if the entries of V are in F.

The next proposition connects one-dimensional submodules with hyperexponential vectors.

Proposition 5. *Let an $\mathcal{L}$-module M have a finite F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ with structure matrices given in (1) and the associated system in (2). Let $\mathbf{u} = \sum_{i=1}^n u_i \mathbf{b}_i$ with $u_i \in F$ not all zero.*

(i) If there exists a hyperexponential element h in some Δ-extension such that $h(u_1, \ldots, u_n)^T$ is a solution of (2), then $F\mathbf{u}$ is a submodule of M with

$$\partial_i(\mathbf{u}) = -\ell\delta_i(h)\mathbf{u},\ i \le \ell \text{ and } \partial_j(\mathbf{u}) = \ell\sigma_j(h)^{-1}\mathbf{u},\ j > \ell. \quad (3)$$

(ii) If $F\mathbf{u}$ is a submodule of M then there exists an invertible hyperexponential element h in some Δ-extension such that $h(u_1, \ldots, u_n)^T$ is a solution of (2).

Proof. Let $U = (u_1, \ldots, u_n)^T$ and $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)^T$. If hU is a solution of (2), then $\delta_i(U) = -A_i^T U - \ell\delta_i(h)U$ for $i \le \ell$. Therefore $\partial_i(\mathbf{u}) = \delta_i\left(U^T\right)\mathbf{b} + U^T A_i \mathbf{b} = -\ell\delta_i(h)\mathbf{u}$ for $i \le \ell$. Similarly, $\partial_j(\mathbf{u}) = \ell\sigma_j(h)^{-1}\mathbf{u}$ for $j > \ell$. So $F\mathbf{u}$ is a submodule and (3) holds.

Now let $F\mathbf{u}$ be a submodule. Then $\partial_i \mathbf{u} = f_i \mathbf{u}$ where $f_i \in F$ for $1 \le i \le m$, and $f_j \ne 0$ for $j > \ell$. The system associated to $F\mathbf{u}$ is $\{\,\delta_i(z) = -f_i z, i \le \ell, \sigma_j(z) = f_j^{-1} z, j > \ell\}$. By Theorem 1 in [7] it has an invertible solution h in certain Δ-extension. Thus h is hyperexponential over F. From $\partial_i(\mathbf{u}) = f_i \mathbf{u}$ it follows that $\delta_i(U) = f_i U - A_i^T U$, which together with $\ell\delta_i(h) = -f_i$ implies $\delta_i(hU) = -A_i^T hU$ for $i \le \ell$. Similarly, we get $\sigma_j(hU) = (A_j^{-1})^T hU$ for $j > \ell$. $\square$

Let h_1 and h_2 be two hyperexponential elements of a Δ-extension of F such that $h_1(u_1, \ldots, u_n)^T$ and $h_2(v_1, \ldots, v_n)^T$ are solutions of (2). From Proposition 5 (i), $F\mathbf{u}$ and $F\mathbf{v}$ with $\mathbf{u} = \sum_{i=1}^n u_i \mathbf{b}_i$ and $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{b}_i$ are one-dimensional submodules of M. Suppose $F\mathbf{u} = F\mathbf{v}$. Then $\mathbf{u} = r\mathbf{v}$ for some $r \in F$, which, together with (3), implies that $\ell\phi(rh_1) = \ell\phi(h_2)$ for all $\phi \in \Delta$. By Lemma 4, $rh_1 = c\,h_2$ with c a constant if we assume that h_2 is invertible. Consequently, $h_1(u_1, \ldots, u_n)^T = c\,h_2(v_1, \ldots, v_n)^T$. In the situation described in Proposition 5, we say that the hyperexponential vector $h(u_1, \ldots, u_n)^T$ corresponds to the submodule $F\mathbf{u}$ and understand that in any Δ-extension, this correspondence is unique up to constant multiples.

The next lemma tells us how to decide whether two one-dimensional submodules are isomorphic.

Lemma 6. *Let M be an $\mathcal{L}$-module with a finite F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Let $\mathbf{u} = \sum_{i=1}^n u_i \mathbf{b}_i$ and $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{b}_i$ where u_i, $v_j \in F$. Suppose that $F\mathbf{u}$ and $F\mathbf{v}$ are two one-dimensional submodules of M and that $\partial_i \mathbf{u} = f_i \mathbf{u}$ and $\partial_i \mathbf{v} = g_i \mathbf{v}$, where $f_i, g_i \in F$ and $i = 1, \ldots, m$. Then we have the following statements:*

(i) The map $\mathbf{u} \mapsto r\mathbf{v}$ from $F\mathbf{u}$ to $F\mathbf{v}$ is a module isomorphism if and only if r is a nonzero solution of the system

$$\delta_i(z) = (f_i - g_i)z,\ i \le \ell,\ \ \sigma_j(z) = f_j g_j^{-1} z,\ j > \ell. \ (4)$$

(ii) Suppose that $h(u_1, \ldots, u_n)^T$ is a solution of the system associated to M, where h is hyperexponential in some Δ-extension of F. Then $F\mathbf{u}$ and $F\mathbf{v}$ are isomorphic if and only if there exists a non-zero $r \in F$ such that $rh(v_1, \ldots, v_n)^T$ is a solution of the associated system.

Proof. Let $\psi : F\mathbf{u} \to F\mathbf{v}$ be a module isomorphism with $\psi(\mathbf{u}) = r\mathbf{v}$ for some non-zero $r \in F$. It follows that

$$\psi(\partial_i \mathbf{u}) = f_i r\mathbf{v} = \partial_i(r\mathbf{v}) = \begin{cases} (\delta_i(r) + g_i r)\mathbf{v}, & i \le \ell, \\ \sigma_i(r) g_i \mathbf{v}, & i > \ell. \end{cases}$$

Thus r is a non-zero solution of (4). Conversely, if r is a non-zero solution of (4), then $\mathbf{u} \mapsto r\mathbf{v}$ gives rise to a module isomorphism from $F\mathbf{u}$ to $F\mathbf{v}$ by a similar calculation.

To prove (ii), we assume that the module structure of M is given by (1) and the associated system is given by (2). Thus $f_i = -\ell\delta_i(h)$ and $f_j = \ell\sigma_j(h)^{-1}$ by Proposition 5 (i).

If $F\mathbf{u} \to F\mathbf{v}$ is an isomorphism given by $\mathbf{u} \mapsto r\mathbf{v}$ with $r \in F$, then r satisfies (4) by (i), hence

$$g_i = -\ell\delta_i(rh),\ i \le \ell, \ \text{ and } \ g_i = \ell\sigma_i(rh)^{-1},\ i > \ell. \qquad (5)$$

Set $V = (v_1, \ldots, v_n)^T$. From $\partial_i(\mathbf{v}) = g_i \mathbf{v}$, we get $\delta_i(V) = g_i V - A_i^T V$ for $i \le \ell$, which together with (5) implies

$$\delta_i(r\,hV) = \delta_i(rh)V + r\,h\,\delta_i(V) = -A_i^T r\,hV.$$

A similar calculation yields $\sigma_j(rhV) = (A_j^{-1})^T rhV$ for $j > \ell$. So rhV is a solution of (2).

Conversely, let $rh(v_1, \ldots, v_n)^T$ with $r \in F$ be a solution of (2). From Proposition 5 (i), both $F\mathbf{u}$ and $Fr\mathbf{v}$ ($= F\mathbf{v}$) are two submodules, and in addition, $\partial_i(\mathbf{u}) = -\ell\delta_i(h)\mathbf{u}$ and $\partial_i(r\mathbf{v}) = -\ell\delta_i(h)r\mathbf{v}$ for $i \le \ell$, and $\partial_j(\mathbf{u}) = \ell\sigma_j(h)^{-1}\mathbf{u}$ and $\partial_j(r\mathbf{v}) = \ell\sigma_j(h)^{-1} r\mathbf{v}$ for $j > \ell$. One can then verify easily that $\mathbf{u} \mapsto r\mathbf{v}$ is an isomorphism. $\square$

We now construct a Δ-extension E of F such that every one-dimensional submodule of M corresponds to a hyperexponential vector hV, where h is an invertible element of E and V is a column vector in F^n. Denote by $\mathcal{M}_1$ the set of one-dimensional submodules of M and by $\overline{\mathcal{M}_1} = \{I_1, \ldots, I_s\}$ the set $\mathcal{M}_1$ modulo isomorphism. For each k in $\{1, \ldots, s\}$, we select a one-dimensional submodule N_k in I_k. Assume that N_k corresponds to a hyperexponential vector $h_k V_k$, where h_k is in some Δ-extension of F and V_k is a vector with entries in F. We can verify directly that the system

$$\begin{cases} \delta_i(Z) = \operatorname{diag}(\,\ell\delta_i(h_1), \ldots, \ell\delta_i(h_s)\,)Z, & 1 \le i \le \ell, \\ \sigma_j(Z) = \operatorname{diag}(\ell\sigma_j(h_1), \ldots, \ell\sigma_j(h_s))Z, & \ell < j \le m \end{cases}$$

where $Z = (z_1, \ldots, z_s)^T$, is fully integrable. By Theorem 1 in [7], there exists a Δ-extension E containing a fundamental matrix $\operatorname{diag}(h_1', \ldots, h_s')$ and the inverse of its determinant.

That is, for every k with $1 \le k \le s$, $\ell\phi(h_k) = \ell\phi(h'_k)$ for all $\phi \in \Delta$. Consequently, $h'_k V_k$ also corresponds to N_k. By Lemma 6 (ii), we need only to search for hyperexponential solutions of the system associated to M in E^n to determine $\mathcal{M}_1$. Observe that the construction of E is independent of the choices of F-bases, since all associated systems are equivalent. The ring E is therefore called a *hyperexponential extension* relative to (M, Δ).

Next, we represent all one-dimensional submodules of M by a finite amount of information. Suppose that M has a basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ and the associated system (2), with $\mathcal{M}_1$ and $\overline{\mathcal{M}}_1 = \{I_1, \ldots, I_s\}$ denoted as above. Let E be a hyperexponential extension of M. By Lemma 6 (ii), for each k with $1 \le k \le s$ there exists an invertible hyperexponential element h_k of E such that every $N \in I_k$ corresponds to a solution $h_k V_{k,N}$ of (2), where $V_{k,N}$ is a column vector in F^n. Let V_k be a matrix whose column vectors form a maximal set of F-linearly independent vectors among all $V_{k,N}$ for $N \in I_k$. We call $\{(h_1, V_1), \ldots, (h_s, V_s)\}$ a *representation of hyperexponential solutions* of (2) and $\{V_1, \ldots, V_s\}$ a *representation of* $\mathcal{M}_1$ relative to the given basis.

PROPOSITION 7. *With the notation just introduced, let $\bar{E}$ be a Δ-extension containing E, and H the set of hyperexponential solutions of (2) in $\bar{E}$. If $\{(h_1, V_1), \ldots, (h_s, V_s)\}$ is a representation of hyperexponential solutions of (2), then*

(i) N is a one-dimensional submodule of M if and only if N is generated by $U^T(\mathbf{b}_1, \ldots, \mathbf{b}_n)^T$ over F, where U is a C_F-linear combination of the column vectors in some V_k;

(ii) H is the disjoint union $\cup_{k=1}^s H_k$, where $H_k = \{c\, h_k V_k C\}$ with c an arbitrary non-zero constant in $\bar{E}$ and C an arbitrary non-zero column vector over C_F.

PROOF. Observe that for every $k \in \{1, \ldots, s\}$ and $N \in I_k$, the vector $V_{k,N}$ defined above is a rational solution of the system obtained by substituting $h_k(z_1, \ldots, z_n)^T$ into (2). Hence by Lemma 1.7 in [16] and its difference analogue, the column vectors in V_k also form a maximal set of C_F-linearly independent vectors among all $V_{k,N}$ for $N \in I_k$. Consequently, $V_{k,N}$ is a C_F-linear combination of the column vectors in V_k, which proves the first assertion.

Clearly, $\cup_{k=1}^s H_k \subset H$ and $H_i \cap H_j = \emptyset$ for all $i \ne j$. Assume $hW \in H$ with $h \in \bar{E}$ and W a column vector in F^n. By Proposition 5 (i), hW corresponds to a one-dimensional submodule N, which by (i) also corresponds to a hyperexponential solution $h_k V_{k,N} \in H_k$ for some k with $1 \le k \le s$. Thus the two solutions differ from a constant multiple according to the discussion following the proof of Proposition 5. □

4. PARAMETRIC HYPEREXPONENTIAL VECTORS

As before, let $\Delta = \{\delta_1, \ldots, \delta_\ell, \sigma_{\ell+1}, \ldots, \sigma_m\}$ where the δ_i and σ_j are derivation operators and automorphisms of F, respectively, and $\mathcal{L}$ be the Laurent-Ore algebra over F. Let M be an n-dimensional $\mathcal{L}$-module with an associated system given in (2). For the purpose of this article, it suffices to find the hyperexponential solutions of (2) in a hyperexponential extension relative to M. We plan to proceed as follows:

First, compute hyperexponential solutions of a matrix equation in (2), say, $\delta_1(Z) = -A_1^T Z$. The set of solutions is partitioned into finitely many groups by Proposition 7. Each group is given as $c\,h\,V\,C$, where c is a constant with respect to δ_1, h is hyperexponential with respect to δ_1, V is a matrix over F, and C is a column vector whose entries are arbitrary constants with respect to δ_1.

Second, substitute $Z = c\,hV\,C$ into another matrix equation, say $\delta_2(Z) = -A_2^T Z$ to find c and C so that Z is a hyperexponential solution of both $\delta_1(Z) = -A_1^T Z$ and $\delta_2(Z) = -A_2^T Z$. There arise several questions in this process:

(a) In what extensions do we compute hyperexponential solutions of $\delta_1(Z) = -\,A_1^T Z$?

(b) Does the substitution introduce coefficients outside F ? Note that h is not necessarily hyperexponential with respect to δ_2.

(c) How do we determine c and C so that Z is hyperexponential with respect to both δ_1 and δ_2?

These questions will be answered in Proposition 11 at the end of this section.

Let Θ be the commutative monoid generated by the δ_i and σ_j. The multiplication in Θ is the composition of maps. For $\theta = \delta_1^{k_1} \ldots \delta_\ell^{k_\ell} \sigma_{\ell+1}^{k_{\ell+1}} \ldots \sigma_m^{k_m} \in \Theta$, the sum $\sum_{i=1}^m k_i$ is called the *order* of θ. The set of elements of Θ of order less than or equal to s is denoted Θ_s.

LEMMA 8. *Let F be a Δ-field, $f_1, \ldots, f_s$ in F and E a Δ-ring extension of F. Then $f_1, \ldots, f_s$ are linearly dependent over C_E if and only if the matrix $W(f_1, \ldots, f_s) = (\theta f_i)_{\theta \in \Theta_{s-1}, 1 \le i \le s}$ has rank less than s. In particular, if $f_1, \ldots, f_s \in F$ are linearly dependent over C_E, then they are linearly dependent over C_F.*

PROOF. If there exist $d_1, \ldots, d_s \in C_E$, not all zero, such that $d_1 f_1 + \ldots + d_s f_s = 0$, then $d_1\theta f_1 + \ldots + d_s \theta f_s = 0$ for all $\theta \in \Theta_{s-1}$. The matrix $W(f_1, \ldots, f_s)$ has rank less than s by Corollary 4.17 in [12, Chapter XIII §4].

Assume that $W(f_1, \ldots, f_s)$ has rank less than s. The proof follows the similar arguments concerning Wronskians and Casoratians, and proceeds by induction on s. The statement holds when $s = 1$. Assume that $s > 1$ and that the statement holds for lower values of s. We can find in F a nontrivial solution $c_1, \ldots, c_s$ to the equations $\sum_{k=1}^s c_k \theta(f_k) = 0$ for all $\theta \in \Theta_{s-1}$. Since F is a field, we can assume $c_1 = 1$. Applying δ_i (resp. σ_j) to each equation indexed by $\theta \in \Theta_{s-2}$ and then subtracting from the equation indexed by $\delta_i\theta$ (resp. $\sigma_j\theta$, and noting that σ_j is an automorphism), we have $\sum_{k=2}^s \delta_i(c_k)\theta(f_k) = 0$ and $\sum_{k=2}^s (c_k - \sigma_j^{-1}(c_k))\theta(f_k) = 0$ for all $\theta \in \Theta_{s-2}$. Either the c_k are constants or some $\delta_i(c_k) \ne 0$ or some $\sigma_j(c_k) - c_k \ne 0$. In the former case, we have the conclusion. In the latter two cases, the matrix $W(f_2, \ldots, f_s)$ has rank less than $s-1$. The induction hypothesis then implies that $f_2, \ldots, f_s$ are already linearly dependent over C_E. The conclusion of the lemma is again satisfied. □

LEMMA 9. *Let K be a field and R a commutative K-algebra. Let*

$$\begin{cases} \sum_{j=1}^n a_{ij} X_j = 0, & 1 \le i \le p \\ \sum_{j=1}^n b_{kj} X_j \ne 0, & 1 \le k \le q \end{cases}$$

be a system of equations with coefficients in K. This system has a non-zero solution in K if and only if it has a non-zero solution in R.

PROOF. Let $\{\alpha_j\}$ be a K-basis of R and let $c_i = \sum_j d_{ij}\alpha_j$ with $d_{ij} \in K$ be a solution of the above system in R. Substituting in the system and equating the coefficients of the α_i, we find a solution in K. $\square$

Notation: In the rest of this article Δ' is a nonempty subset of Δ. For a Δ-ring R, the ring of constants with respect to the maps in Δ' is denoted $C'(R)$.

LEMMA 10. *Let F be a Δ-field, E a Δ-extension of F, and E' a Δ'-extension of E. Let $V_1, \ldots, V_s, W$ be non-zero column vectors in F^n, $c_1, \ldots, c_s \in C'(E')$, $g \in E$, $h \in E'$ with g, h invertible, and $gW = h\sum_{i=1}^s c_iV_i$. If h is hyperexponential over F with respect to Δ', and g is hyperexponential over F with respect to Δ, then there exist $d_1, \ldots, d_s \in C'(F)$, $\bar{h} \in E$ with $\bar{h}$ invertible such that the following statements hold:*

(i) $\bar{h}$ is hyperexponential over F with respect to Δ.

(ii) $\ell\phi(\bar{h}) = \ell\phi(h)$ for all $\phi \in \Delta'$.

(iii) $gW = \bar{h}\sum_{i=1}^s d_iV_i$.

PROOF. Let $W = (w_1, \ldots, w_n)^T$ and $V_i = (v_{1i}, \ldots, v_{ni})^T$. Assume that $w_1, \ldots, w_t$ are non-zero while $w_{t+1}, \ldots, w_n$ are all zero. The equation $gW = h\sum_i c_iV_i$ translates to

$$gh^{-1} = \sum_i c_i \frac{v_{ji}}{w_j}, \quad \text{for } j = 1, \ldots, t, \tag{6}$$

and $0 = \sum_i c_iv_{ki}$ for $k = t+1, \ldots, n$. Note that the equations (6) imply that $\sum_i c_i\frac{v_{ji}}{w_j} = \sum_i c_i\frac{v_{li}}{w_l}$ for $1 \le j, l \le t$. Furthermore, we have that for any $\phi \in \Delta'$ there is a $u_\phi \in F$ such that $\ell\phi(\sum_i c_i\frac{v_{ji}}{w_j}) = \ell\phi(gh^{-1}) = u_\phi$ for $1 \le j \le t$. Consider the equations

$$\begin{aligned}
\sum_i c_i\frac{v_{ji}}{w_j} &= \sum_i c_i\frac{v_{li}}{w_l} \neq 0, \quad 1 \le j,\, l \le t && (7)\\
0 &= \sum_i c_iv_{ki}, \quad k = t+1, \ldots, n && (8)\\
\sum_i c_i\phi\left(\frac{v_{ji}}{w_j}\right) &= \sum_i c_i\frac{v_{ji}}{w_j}u_\phi, \quad 1 \le j \le t, \phi \in \Delta'. && (9)
\end{aligned}$$

Letting $\{\alpha_s\}$ be a $C'(F)$-basis of F, there exist a_s^{ji}, b_s^{ki}, $c_s^{ji\phi}$, $d_s^{ji\phi}$ in $C'(F)$ such that $\frac{v_{ji}}{w_j} = \sum_s a_s^{ji}\alpha_s$, $v_{ki} = \sum_s b_s^{ki}\alpha_s$, $\phi\left(\frac{v_{ji}}{w_j}\right) = \sum_s c_s^{ji\phi}\alpha_s$ and $\frac{v_{ji}}{w_j}u_\phi = \sum_s d_s^{ji\phi}\alpha_s$. Substitute these into equations (7), (8) and (9). Using Lemma 8 and equating coefficients of the α_s, we see that $X_i = c_i$ satisfy the following system of equations for all s:

$$\begin{aligned}
\sum_i X_ia_s^{ji} &= \sum_i X_ia_s^{li}, \quad 1 \le j,\, l \le t\\
0 &= \sum_i X_ib_s^{ki}, \quad k = t+1, \ldots, n\\
\sum_i X_ic_s^{ji\phi} &= \sum_i X_id_s^{ji\phi}, \quad 1 \le j \le t,\, \phi \in \Delta',
\end{aligned}$$

and that for $1 \le j \le t$ there is an s such that $\sum_i X_ia_s^{ji} \neq 0$. Lemma 9 implies that this system will have a solution $X_i = d_i$ in $C'(F)$. Let $S = \sum_i d_i\frac{v_{1i}}{w_1} = \ldots = \sum_i d_i\frac{v_{ti}}{w_t} \neq 0$. Note that $\phi(S) = u_\phi S$ for all $\phi \in \Delta'$. Therefore $\ell\phi\left(\frac{g}{hS}\right) = 0$ for all derivations $\phi \in \Delta'$ and $\ell\phi\left(\frac{g}{hS}\right) = 1$ for all automorphisms $\phi \in \Delta'$ and so $g = Shd$ for some $d \in C'(E')$, that is, $w_jg = h(\sum_i d_iv_{ji})d$ for all $1 \le j \le t$. Letting $\overline{h} = hd = w_jg/(\sum_i d_iv_{ji}) \in E$, we have that $\ell\phi(\overline{h}) = \ell\phi(h)$ for $\phi \in \Delta'$ and $\ell\phi(\overline{h}) \in F$ for all $\phi \in \Delta$. $\square$

We now consider how to have some information about $\bar{h}$ given in the conclusion of Lemma 10 without knowing gW. Let $\Delta' = \{\delta_1, \ldots, \delta_p, \sigma_{\ell+1}, \ldots, \sigma_q\}$, $r_i = \ell\delta_i(h)$ and $r_j = \ell\sigma_j(h)$ where $1 \le i \le p$ and $\ell+1 \le j \le q$. Note that $r_i = \ell\delta_i(\bar{h})$ and $r_j = \ell\sigma_j(\bar{h})$ by Lemma 10. Assume that $\phi \in \Delta \setminus \Delta'$. We want to compute an element r of F such that $r = \ell\phi(\bar{h})$.

Case 1. ϕ is a derivation operator. On one hand, we have $\phi\circ\delta_i(\bar{h}) = (\phi(r_i)+r_ir)\bar{h}$ and $\phi\circ\sigma_j(\bar{h}) = (\phi(r_j)+r_jr)\bar{h}$. On the other hand, we have $\delta_i \circ \phi(\bar{h}) = (\delta_i(r) + r_ir)\bar{h}$ and $\sigma_j \circ \phi(\bar{h}) = \sigma_j(r)r_j\bar{h}$. By the commutativity of the maps in Δ, r is a solution of the system

$$\begin{cases} \delta_i(z) = \phi(r_i), & 1 \le i \le p \\ \sigma_j(z) - z = \ell\phi(r_j), & \ell+1 \le j \le q. \end{cases} \tag{10}$$

Consequently, if there exists h such that $gW = h\sum_i c_iV_i$, (10) has a solution r in F and $\ell\phi(\bar{h}) = r+c$ for some $c \in C'(F)$.

Case 2. ϕ is an automorphism. A similar calculation shows that r is a non-zero solution of the system

$$\begin{cases} \delta_i(z) = (\phi(r_i)-r_i)z, & 1 \le i \le p \\ \sigma_j(z) = \ell\phi(r_j)z, & \ell+1 \le j \le q. \end{cases} \tag{11}$$

Consequently, if there is h such that $gW = h\sum_i c_iV_i$, (11) has a solution r in F and $\ell\phi(\bar{h}) = cr$ for some $c \in C'(F)$.

Let h be a hyperexponential element with respect to Δ'. We say that h is *extensible* for a map $\phi \in \Delta \setminus \Delta'$ if there exists $\bar{h}$, hyperexponential with respect to both Δ' and ϕ, such that $\ell\psi(h) = \ell\psi(\bar{h})$ for all $\psi \in \Delta'$.

By the above discussion, h is extensible for a derivation (resp. an automorphism) if and only if (10) (resp. (11)) has a rational solution.

PROPOSITION 11. *Let $\Delta\setminus\Delta'$ have one element and M be an $\mathcal{L}$-module of finite dimension. Let E and E' be hyperexponential extensions relative to (M, Δ) and (M, Δ'), respectively. Let $\mathcal{A}$ and $\mathcal{A}'$ be the systems associated to (M, Δ) and (M, Δ'), respectively. Let $\{(h_1', V_1'), \ldots, (h_t', V_t')\}$ be a representation of hyperexponential solutions of $\mathcal{A}'$ in E' with respect to Δ'. Then there exist a Δ'-extension R of F containing both E and E', and invertible hyperexponential elements $h_1, \ldots, h_s$ in R, with $s \le t$, such that, for every hyperexponential solution gW of $\mathcal{A}$ with coordinates in R, $gW = h_kV_k'D$, where k is unique and D is a hyperexponential vector over $C'(R)$.*

PROOF. Let $\Delta\setminus\Delta' = \{\phi\}$. Assume that $h_1', \ldots, h_s'$ are extensible to $h_1, \ldots, h_s$ for ϕ, respectively, while $h_{s+1}', \ldots, h_t'$ are not extensible. We can regard $h_1, \ldots, h_s$ as invertible elements in a Δ-extension E'', as we did in the construction of hyperexponential extensions. Let $R = E \otimes_F E' \otimes_F E''$. Since gW is a hyperexponential solution of $\mathcal{A}$, it is a hyperexponential solution of $\mathcal{A}'$. By Proposition 7, there exist k with $1 \le k \le t$ and a column vector C with entries in $C'(R)$ such that $gW = h_k'V_k'C$. By Lemma 10 we have $gW = \bar{h}V_k'D'$ where $\bar{h} \in R$ is hyperexponential such that $\ell\psi(h_k') = \ell\psi(\bar{h})$ for all $\psi \in \Delta'$, and D' is a column vector with entries in $C'(F)$. Hence, h_k' is extensible and the ratio $d = \bar{h}/h_k$ is in $C'(R)$ by Lemma 4. Setting $D = dD'$ yields the proposition. $\square$

5. ALGORITHM DESCRIPTION

Let M be an $\mathcal{L}$-module with an F-basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ and let $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)^T$. We will compute one-dimensional submodules of M recursively. The key step for recursion proceeds as follows.

Assume that we have obtained all one-dimensional submodules of (M, Δ'), where $|\Delta \setminus \Delta'| = 1$. Let $\mathcal{A}$ and $\mathcal{A}'$ be the systems associated to (M, Δ) and (M, Δ'), respectively. Let U be an $n \times s$ matrix over F such that the set

$$S = \{\, F\mathbf{u} \mid \mathbf{u} = (U\,C)^T\mathbf{b} \text{ and } C \text{ is a nonzero column vector over } C'(F)\}$$

is an equivalence class of one-dimensional submodules of (M, Δ') with respect to isomorphism. If $W^T\mathbf{b}$ with $W{\in}F^n$ generates a one-dimensional submodule of (M, Δ) that is in S, then there exists an element g in a hyperexponential extension relative to (M, Δ) such that gW is a solution of $\mathcal{A}$. By Proposition 11 there exists a hyperexponential element h in some Δ'-extension R such that $gW = hUD$ for some hyperexponential vector D with entries in $C'(R)$. Moreover, h can be found by computing rational solutions of equations (10) or (11). Substituting hUD into the matrix equation corresponding to the map in $\Delta \setminus \Delta'$, we get an ordinary differential or difference matrix equation in D over F. This system translates to a system over $C'(F)$ by the technique used in the proof of Lemma 10, since we only look for hyperexponential solutions in $C'(R)$. In this way we obtain all one-dimensional submodules of (M, Δ) that are in S.

To make this idea effective, we will need several assumptions. Define $\Delta_0 = \emptyset$, $\Delta_i = \{\phi_1, \ldots, \phi_i\}$ and C_i to be the set of all elements of F that are constants with respect to Δ_i. Note that $C_0 = F$, $C_m = C_F$ and that each C_i is a $(\Delta\backslash\Delta_i)$-field. The above algorithm can be formalized if we assume that, for each i,

1. One is able to identify the field C_i and effectively carry out computations in C_i as a $(\Delta\backslash\Delta_i)$-field. Furthermore, we assume that we can find a C_i-basis of F and express any element in F in this basis.

2. Assuming that ϕ_{i+1} is a derivation, we can decide if systems of the form $\{L_j(z) = a_j \mid a_j \in F\}_{j=1}^{i}$ have solutions in F where $L_j(z) = \phi_j(z)$ if ϕ_j is a derivation and $L_j(z) = \phi_j(z) - z$ if ϕ_j is an automorphism, and, if so, find one.

3. Assuming that ϕ_{i+1} is an automorphism, we can decide if systems of the form $\{\phi_j(z) = a_j z \mid a_j \in F\}_{j=1}^{i}$ have solutions in F, and, if so, find one.

4. Given an equation $\phi_{i+1}(Z) = AZ$ with $A \in C_i^{n\times n}$, we can find all hyperexponential solutions over C_i.

By conditions 2 and 3, we can find rational solutions of (10) and (11). In condition 4, if ϕ_{i+1} is a differential operator, methods for solving such an equation or reducing the system to a scalar equation and solving the scalar equation for certain fields are discussed in [3, 4, 6, 9, 17]. Methods to find hypergeometric solutions for scalar difference equations are discussed in [2, 10, 15]. We will discuss below a method to reduce systems to scalar equations in the difference case.

5.1 Ordinary case

Let ϕ be a difference operator. Consider a system

$$\phi(Z) = AZ \quad \text{with } A \in F^{n\times n} \text{ and } Z = (z_1, \ldots, z_n)^T. \quad (12)$$

From (12), we construct by linear algebra a linear difference equation with minimal order, say,

$$L(z_1) = \phi^k(z_1) + a_{k-1}\phi^{k-1}(z_1) + \cdots + a_0 z_1 = 0$$

where $a_i \in C'(F)$. If $k = n$, then each of the z_i is a linear combination of $z_1, \phi(z_1), \ldots, \phi^{k-1}(z_1)$ over F. So we need only to compute hyperexponential solutions of $L(z_1) = 0$. If $k < n$, then we compute hyperexponential solutions of (12), in which $z_1 \neq 0$ and $z_1 = 0$, separately. In the former case, let h be a hyperexponential solution of $L(z_1) = 0$, then, all hyperexponential solutions of (12) of the form $h(v_1, \ldots, v_n)^T$ can be found by substituting hZ into (12) and computing the rational solutions of the resulting equation. There are methods for computing rational solutions of linear functional matrix equations in [1, 3]. In the latter case $z_1 = 0$, we compute P, Q and a partition of $(z_2, \ldots, z_n)^T$ into two sub-vectors Y_1 and Y_2 such that $\phi(Y_1) = PY_1$ and $Y_2 = QY_1$, by an ordinary version of the algorithm LinearReduction described in [18, Section 2.5.3]. Then we apply the same method to $\phi(Y_1) = PY_1$, recursively.

In Section 5.2, one will encounter a matrix equation of form $V\phi(Y){=}UY$ where Y is a vector of unknowns, U and V are matrices over F, and V has full column rank. A similar reduction transforms the equation into $\{\,\phi(Y_1) = U'Y_1,\ Y_2 = V'Y_1\}$, where Y_1 and Y_2 form a partition of Y into two sub-vectors of unknowns, and U' and V' are some matrices over F. So we can find hyperexponential solutions of $V\phi(Y) = UY$.

Example 1. *Let $F = \mathbb{C}(x, m, n)$ and σ_n be the shift operator with respect to n. We now compute hyperexponential solutions of the matrix difference equation $\mathcal{A} : \sigma_n(Z){=}AZ$ where $Z = (z_1, z_2, z_3)^T$ and*

$$A=\begin{pmatrix} \frac{n(2nx+x-2x^2-1)}{2(nx-1)} & \frac{x(-n-3+2x+2nx)}{2(nx-1)} & 0 \\ \frac{n(n-1-x+nx)}{2(nx-1)} & \frac{-2n-2+x+2nx+n^2x}{2(nx-1))} & 0 \\ \frac{n^2x+3nx+2nm^2-n^2-n+2m^2}{2(nx-1)} & \frac{x+2m^2-n^2x+2xm^2+2x^2n}{2(1-nx)} & x \end{pmatrix}.$$

By linear algebra, we find a linear difference equation: $L(z_1) = \sigma_n^2(z_1) + \frac{4x^2n+11-18nx-6n^2x-12x+14n+3n^2+4x^2}{2(-n-3+2x+2nx)}\sigma_n(z_1) - \frac{n(8x^2+4-12x-8nx+4x^2n-2n^2x+5n+n^2)}{2(-n-3+2x+2nx)}z_1 = 0$. All hyperexponential solutions of L are of the form $c\Gamma(n)$ for $c \in \mathbb{C}(x, m)$. Substituting $Z = hY$ with $h = \Gamma(n)$ and $Y = (y_1, y_2, y_3)^T$ into $\mathcal{A}$, we get a $\mathbb{C}(x, m)$-basis $V = \left(\frac{n+1}{x}, \frac{(1+x)n}{x^2}, \frac{nx+m^2}{x^2}\right)^T$ of rational solutions of the resulting system. So $\{(\Gamma(n), V)\}$ is among the representation of hyperexponential solutions of $\mathcal{A}$. In addition, z_1 is not a cyclic vector as the order of L is less than the size of $\mathcal{A}$. By substituting $z_1 = 0$ into $\mathcal{A}$ we get $z_1 = 0$, $z_2 = 0$ and $\sigma_n(z_3) = xz_3$. Thus $z_3 = c\,x^n$ for any $c \in \mathbb{C}(x, m)$, hence $c\,x^n(0, 0, 1)^T$ is a hyperexponential solution of $\mathcal{A}$. So a representation of hyperexponential solutions of $\mathcal{A}$ is $\left\{\, (\Gamma(n),\, V)\,,\ (x^n,\, (0, 0, 1)^T) \,\right\}$.

5.2 Partial case

Let $|\Delta'|{=}m{-}1$ and $\Delta\backslash\Delta' = \{\phi\}$. Assume that $\{\, (h_1', V_1'), \ldots, (h_t', V_t') \,\}$ is a representation for hyperexponential solutions of the system associated to (M, Δ'). We decide whether $h'_1, \ldots, h'_t$ are extensible for ϕ. If none of them is extensible, then the system $\mathcal{A}$ associated to (M, Δ) has no hyperexponential solution by the proof of Proposition 11. Otherwise, we may further assume that $h'_1, \ldots, h'_s$ are extensible to $h_1, \ldots, h_s$, respectively, while $h'_{s+1}, \ldots, h'_t$ are not extensible. By Propositions 11, for every hyperexponential solution of $\mathcal{A}$, there exists a unique k in $\{1, \ldots, s\}$ such that the solution is of the form $h_k V'_k D_k$, where D_k is a hyperexponential vector with constant entries with respect to Δ'.

Let $\phi(Z) = BZ$ be the equation corresponding to ϕ in $\mathcal{A}$. For $1 \le k \le s$, substituting $h_k V'_k D_k$ into $\phi(Z) = BZ$ yields an equation $Q_k \phi(D_k) = B_k D_k$ for some matrices Q_k, B_k over F. In addition, Q_k has full column rank. As in the proof of Lemma 10, we choose a $C'(F)$-basis $\{\alpha_i\}$ of F, and write $Q_k = \sum_i Q_{ki}\alpha_i$ and $B_k = \sum_i B_{ki}\alpha_i$, where Q_{ki} and B_{ki} are matrices over $C'(F)$. Let U_k and W_k be matrices formed by the stacking of the non-zero matrices Q_{ki} and B_{ki}, respectively. By Lemma 8

$$U_k \phi(D_k) = W_k D_k, \tag{13}$$

where U_k has full column rank since Q_k has. We compute hyperexponential solutions of (13) over $C'(F)$.

Assume that, for $1 \le k \le l$, $\{(g_{k1}, G_{k1}), \ldots, (g_{ki_k}, G_{ki_k})\}$ is a representation of hyperexponential solutions of (13), while (13) has no hyperexponential solutions for any k with $l < k \le s$. Then a representation of hyperexponential solutions of $\mathcal{A}$ consists of (f_{kj}, V_{kj}), where $f_{kj} = g_{kj}h_k$, the set of the column vectors of V_{kj} is a maximal set of linearly independent column vectors of the matrix $V'_k G_{kj}$, $j = 1, \ldots, i_k$, and $k = 1, \ldots, l$. To prove this assertion, we need only to show that a hyperexponential solution of $\mathcal{A}$ cannot be represented by both (f_{kj}, V_{kj}) and $(f_{kj'}, V_{kj'})$ with $j \ne j'$. Suppose the contrary, then there exists $r \in F$ such that $r f_{kj} = f_{kj'}$. It follows that $r g_{kj} = g_{kj'}$, so (g_{kj}, G_{kj}) and $(g_{kj'}, G_{kj'})$ would also represent the same set of hyperexponential vectors, a contradiction.

We illustrate the algorithm by two examples. The first one cannot be handled directly by the method in [11].

EXAMPLE 2. *Consider the field* $F{=}C(x,y)$ *with* $\Delta{=}\{\delta,\sigma\}$ *where* $C = \mathbb{Q}(e)$, $\delta = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ *and* σ *is defined by* $\sigma(x) = x+1$ *and* $\sigma(y) = y$. *The constants of F are C, as the constants with respect to σ are $C(y)$ and the constants of $C(y)$ with respect to δ are C. Let us compute hyperexponential solutions of the system* $\{\sigma(Z) = A_s Z,\, \delta(Z) = A_d Z\}$ *where* $Z = (z_1, z_2, z_3, z_4)^T$,

$$A_s = \begin{pmatrix} 0 & \frac{1}{y} & -x\,e & e+1 \\ -y\,e & e+1 & 0 & y\,e \\ 0 & 0 & 0 & \frac{1}{x+1} \\ 0 & 0 & -x\,e & e+1 \end{pmatrix}$$

and

$$A_d = \begin{pmatrix} -\frac{1}{y} & -\frac{4}{2y-1} & \frac{x}{y} & \frac{2y-1+4y^2}{y(2y-1)} \\ 0 & 0 & 0 & 1 \\ -\frac{4y}{x(2y-1)} & 0 & \frac{4yx-2y+1}{x(2y-1)} & \frac{4y}{x(2y-1)} \\ 0 & -\frac{4}{2y-1} & 0 & \frac{4y}{2y-1} \end{pmatrix}.$$

We obtain a representation $\{(h_1 = 1, V_1),\, (h_2 = e^x, V_2)\}$ *of hyperexponential solutions of the first matrix equation* $\sigma(Z) = A_s Z$ *with*

$$V_1 = \begin{pmatrix} \frac{1}{y} & 1 \\ 1 & 0 \\ 0 & \frac{1}{x} \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \frac{1}{y\,e} & 1 \\ 1 & 0 \\ 0 & \frac{1}{x\,e} \\ 0 & 1 \end{pmatrix}.$$

Both h_1 and h_2 are extensible for δ. Suppose that $h_1 V_1 D$ is a solution of $\mathcal{A}$ for some hyperexponential vector D over $C(y)$. To decide D, substitute $h_1 V_1 D$ for Z into the second matrix equation $\delta(Z) = A_d Z$ to yield

$$\begin{pmatrix} \frac{1}{y} & 1 \\ 1 & 0 \\ 0 & \frac{1}{x} \\ 0 & 1 \end{pmatrix} \delta(D) = \begin{pmatrix} -\frac{4}{2y-1} & \frac{2y-1+4y^2}{y(2y-1)} \\ 0 & 1 \\ -\frac{4}{x(2y-1)} & \frac{4y}{x(2y-1)} \\ -\frac{4}{2y-1} & \frac{4y}{2y-1} \end{pmatrix} D,$$

which translates to a matrix equation of size two

$$\delta(D) = \begin{pmatrix} 0 & 1 \\ -\frac{4}{2y-1} & \frac{4y}{2y-1} \end{pmatrix} D.$$

A representation of hyperexponential solutions of the above system is $\{(1, U_1 := (y,1)^T),\, (e^{2y}, U_2 := (1,2)^T)\}$. *Hence the original system has hyperexponential solutions given by* $\{(1, V_1U_1),\, (e^{2y}, V_1U_2)\}$.

Similarly, substituting h_2V_2D for Z into $\delta(Z) = A_d Z$ finally yields a matrix equation of size two

$$\delta(D) = \begin{pmatrix} -1 & 1 \\ -\frac{4}{2y-1} & \frac{2y+1}{2y-1} \end{pmatrix} D.$$

A representation of hyperexponential solutions of the above system is $\{(e^y, W_1 := (e, 2e)^T),\, (e^{-y}, W_2 := (y\,e, e)^T)\}$. *So the original system has hyperexponential solutions given by* $\{(e^{x+y}, V_2W_1),\, (e^{x-y}, V_2W_2)\}$. *Accordingly,*

$$(1,\, (2, y, \frac{1}{x}, 1)^T), \quad (e^{2y}, (\tfrac{1}{y} + 2, 1, \tfrac{2}{x}, 2)^T),$$
$$(e^{x+y}, (\frac{1}{y} + 2e, e, \frac{2}{x}, 2e)^T), \quad (e^{x-y}, (1+e, ye, \tfrac{1}{x}, e)^T)$$

form a representation of hyperexponential solutions of the original system.

EXAMPLE 3. *Let* $F = \mathbb{C}(x,y,k)$, *and* δ_x, δ_y *and* σ_k *denote partial differentiations with respect to x, y and the shift operator with respect to k, respectively. Let* $\mathcal{L}{=}F[\partial_x, \partial_y, \partial_k, \partial_k^{-1}]$ *be the Laurent-Ore algebra over F and M be an $\mathcal{L}$-module with an F-basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *whose structure matrices are* $-A_x^T$, $(A_k^{-1})^T$ *and* $-A_y^T$ *where*

$$A_x {=} \begin{pmatrix} \frac{x+y}{xy} & -\frac{k(2x+k)}{x(x+k)} & 0 \\ 0 & \frac{-y+x+k}{y(x+k)} & 0 \\ \frac{3x+2y}{x+y} & -\frac{k(3x+2y)}{x+y} & \frac{x}{y(x+y)} \end{pmatrix},$$

$$A_k {=} \begin{pmatrix} \frac{k(y+k)}{y+k+1} & \frac{k(k^2+2xk+xy+x+k)}{(y+k+1)(x+k+1)} & 0 \\ 0 & \frac{k(x+k)}{x+k+1} & 0 \\ -\frac{x(2k+y+1)}{y+k+1} & \frac{xk(2k+y+1)}{y+k+1} & k+1 \end{pmatrix},$$

and

$$A_y = \begin{pmatrix} -\frac{y^2+xy+xk}{(y+k)y^2} & \frac{k(2y+k)}{y(y+k)} & 0 \\ 0 & -\frac{x-y}{y^2} & 0 \\ -\frac{x(2xy+y^2+xk)}{y(y+k)(x+y)} & \frac{xk(2xy+y^2+xk)}{y(y+k)(x+y)} & -\frac{x^2}{y^2(x+y)} \end{pmatrix}.$$

We compute all hyperexponential solutions of the associated system $\mathcal{A}: \{\delta_x(Z){=}A_x Z, \sigma_k(Z) = A_k Z, \delta_y(Z) = A_y Z\}$ *of* M *where* $Z = (z_1, z_2, z_3)^T$. *A representation of hyperexponential solutions of* $\delta_x(Z) = A_x Z$ *is* $\{(e^{\frac{x}{y}}, V)\}$ *where*

$$V = \begin{pmatrix} \frac{k}{x+k} & 0 & x \\ \frac{1}{x+k} & 0 & 0 \\ 0 & \frac{1}{x+y} & x^2 \end{pmatrix}.$$

Clearly, $h = e^{\frac{x}{y}}$ *is extensible for* σ_k. *Suppose that* hVD *is a solution of* $\{\,\delta_x(Z){=}A_x Z,\ \sigma_k(Z) = A_k Z\}$ *for some hyperexponential vector* D *over* $\mathbb{C}(k, y)$ *with respect to* $\{\delta_x, \sigma_k\}$. *To identify* D, *substitute* hVD *into the second matrix equation* $\sigma_k(Z) = A_k Z$ *to yield*

$$\begin{pmatrix} \frac{k+1}{x+k+1} & 0 & x \\ \frac{1}{x+k+1} & 0 & 0 \\ 0 & \frac{1}{x+y} & x^2 \end{pmatrix} \sigma_k(D) {=} \begin{pmatrix} \frac{k(k+1)}{x+k+1} & 0 & \frac{k(y+k)x}{y+k+1} \\ \frac{k}{x+k+1} & 0 & 0 \\ 0 & \frac{k+1}{x+y} & \frac{x^2 k(y+k)}{y+k+1} \end{pmatrix} D,$$

which translates to the system

$$\sigma_k(D) = \begin{pmatrix} k & 0 & 0 \\ 0 & k+1 & 0 \\ 0 & 0 & \frac{k(y+k)}{y+k+1} \end{pmatrix} D.$$

Its hyperexponential solutions are given by $\{(\Gamma(k), U)\}$ *where*

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & \frac{1}{y+k} \end{pmatrix}.$$

Hence hyperexponential solutions of the first two matrix equations have a representation $\{(e^{\frac{x}{y}}\Gamma(k), VU)\}$.

Carrying on the above process, we find a representation $\{(e^{\frac{x}{y}}\Gamma(k), W)\}$ *of hyperexponential solutions of the original system where*

$$W = \begin{pmatrix} \frac{ky}{x+k} & 0 & \frac{x}{y+k} \\ \frac{y}{x+k} & 0 & 0 \\ 0 & \frac{ky}{x+y} & \frac{x^2}{y+k} \end{pmatrix}.$$

So $\{W\}$ *is a representation of one-dimensional submodules of* M *relative to the given basis. For this example,* M *is a direct sum* $F\mathbf{w}_1 \oplus F\mathbf{w}_2 \oplus F\mathbf{w}_3$ *where* $\mathbf{w}_1 = \frac{ky}{x+k}\mathbf{e}_1 + \frac{y}{x+k}\mathbf{e}_2$, $\mathbf{w}_2 = \frac{ky}{x+y}\mathbf{e}_3$ *and* $\mathbf{w}_3 = \frac{x}{y+k}\mathbf{e}_1 + \frac{x^2}{y+k}\mathbf{e}_3$.

To a finite-dimensional linear functional system, one can associate a fully integrable system. Proposition 2 in [7] describes a one-to-one correspondence between the solutions of the given system and those of the associated one (see also Proposition 2.4.12 in [18]). Consequently, the algorithm in this section can be used for computing hyperexponential solutions of finite-dimensional linear functional systems.

6. REFERENCES

[1] S. Abramov and M. Bronstein. On solutions of linear functional systems. In B. Mourrain, ed. *Proc. ISSAC'2001*, pp. 1–6. ACM Press, 2001.

[2] S. Abramov, P. Paule, and M. Petkovšek. q-hypergeometric solutions of q-difference equations. *Discrete Math.*, 180:3–22, 1998.

[3] M. Barkatou. On rational solutions of systems of linear differential equations. *J. Symbolic Comput.*, 28(4/5):547–568, 1999.

[4] M. Barkatou and E. Pflügel. An algorithm computing the regular formal solutions of a system of linear differential equations. *J. Symbolic Comput.*, 28:569–587, 1999.

[5] E. Beke. Die irreducibilität der homogenen differentialgleichungen. *Math. Annal.*, 45:278–294, 1894.

[6] M. Bronstein. Linear ordinary differential equations: breaking through the order 2 barrier. In P. S. Wang, ed. *Proc. ISSAC'1992*, pp. 42–48. ACM Press, 1992.

[7] M. Bronstein, Z. Li, and M. Wu. Picard–Vessiot extensions for linear functional systems. In M. Kauers, ed. *Proc. ISSAC'2005*, pp. 68–75. ACM Press, 2005.

[8] D. Grigoriev and F. Schwarz. Generalized Loewy-decomposition of D-modules. In M. Kauers, ed. *Proc. ISSAC'2005*, pp. 163–170. ACM Press, 2005.

[9] M. van Hoeij. Factorization of differential operators with rational functions coefficients. *J. Symbolic Comput.*, 24(5):537–561, 1997.

[10] M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. *Journal of Pure and Applied Algebra*, (139):109–131, 1999.

[11] G. Labahn and Z. Li. Hyperexponential solutions of finite-rank ideals in orthogonal Ore algebras. In J. Gutierrez, ed. *Proc. ISSAC'2004*, pp. 213–220. ACM Press, 2004.

[12] S. Lang. *Algebra, Graduate Texts in Mathematics*, volume 211. Springer, 2002.

[13] Z. Li and F. Schwarz. Rational solutions of Riccati-like partial differential equations. *J. Symbolic Comput.*, 31:691–716, 2001.

[14] Z. Li, F. Schwarz, and S. Tsarev. Factoring systems of linear PDE's with finite-dimensional solution spaces. *J. Symbolic Comput.*, 36:443–471, 2003.

[15] M. Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symbolic Comput.*, **14**:243–264, 1992.

[16] M. van der Put and M.F. Singer. *Galois Theory of Linear Differential Equations, Grundlehren der Mathematischen Wissenschaften*, volume 328. Springer, Heidelberg, 2003.

[17] M.F. Singer. Liouvillian solutions of linear differential equations with liouvillian coefficients. *J. Symbolic Comput.*, **11**:251–273, 1991.

[18] M. Wu. *On Solutions of Linear Functional Systems and Factorization of Modules over Laurent-Ore Algebras*. PhD thesis, Academia Sinica and Université de Nice, `http://www.mmrc.iss.ac.cn/~mwu/Thesis/Wu-thesis.pdf`, 2005.