Inherently Improper Surface Parametric Supports

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Abstract

We identify a class of monomial supports that are inherently improper because any surface rational parametrization defined on them is improper. A surface support is inherently improper if and only if the gcd of the normalized areas of the triangular sub-supports is non-unity. The constructive proof of this claim can be used to detect all and correct almost all improper surface parametrizations due to improper supports.

Key words: Inherently improper supports, improper parametrizations, support transformation, reparametrization

1 Introduction

Rational parametrizations are fundamental curve and surface representations in computer shape modeling and processing. Perhaps the most basic property of a rational parametrization is whether it is proper (one-to-one) or improper (many-to-one). Improper parametrizations are undesirable because the parametric degree could be unnecessarily high. A high degree parametrization costs more to analyze and process. What is worst, operations based on a proper parametrization simply fail. For example, closed form inversion formulas are

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impossible for an improper parametrization. Therefore it is important to be able to detect improper parametrizations efficiently, and when one is detected, to attenuate or remove the improperness.

For curves, Lüroth's theorem settles the improper parametrization problem thoroughly. The theorem says that given a curve rational parametrization $(X(t), Y(t))$ with coefficient field k, there is a rational function $s(t) \in k(t)$ such that $k(X(t), Y(t)) = k(s(t))$. If $s(t)$ is linear in t, the given parametrization is already proper. Regardless, there are rational functions ϕ, ψ such that $X(t) = \phi(s(t))$, $Y(t) = \psi(s(t))$. Consequently, the reparametrization $(\phi(s), \psi(s))$ is always proper and generates the same curve. Furthermore, practical algorithms to find ϕ, ψ, and $s(t)$ using polynomial gcd, involution, and elimination are described respectively in (Sommerville, 1959; Sederberg, 1986; Gao and Chou, 1992).

For surfaces the situation is not so satisfactory. When k is the field of complex numbers a theorem analogous to Lüroth's was proved by Castelnuovo but the proof was non-constructive. When confined to rational or real coefficients, it is disappointing that a reparametrization similar to that of curves to avoid improperness may not be possible. These facts are surveyed in (Schinzel, 2000). Though correcting an improper surface rational parametrization is difficult or impossible, there are practical algorithms to detect one (Chionh and Goldman, 1992; Pérez Díaz, Schicho, and Sendra, 2002).

However, the situation greatly improves, if instead of dealing with the general problem of improper surface parametrizations, we study a class of surface rational parametrizations arising from some special monomial supports (for definition, please refer Section 2.1). These monomial supports are special because any surface rational parametrization constructed from them is always improper whatever the coefficients are. To emphasize this characteristic, it seems appropriate to refer to such monomial supports as inherently improper surface parametric supports, or simply improper supports. The main results of the paper are thus two-fold:

- highlighting the existence of inherently improper surface parametric supports, and
- proving a theorem constructively for the detection and correction, wholly or partially, of improper surface parametrizations due to improper supports.

As surfaces in the 3-dimensional space are the main objects of interest, the paper focuses on rational parametrizations from the 2-dimensional space to the 3-dimensional space. An advantage of doing so is the simplification of notation and derivation without losing generality. We expect that the discussions can easily be rephrased for rational parametrizations from the n-dimensional space to the $(n+1)$-dimensional space, $n > 2$. This is because the algebraic tools of implicitization, the BKK degree bound, Jacobians, and n-volume determinantal expressions are all valid in the general n-dimensional setting.

The paper develops as follows. Section 2 defines two key concepts: the improper index and support transformations. Section 3 proves that the improper index of a support is the gcd of the normalized areas of all its triangular sub-supports. Section 4 explains why any surface parametrization defined on an improper support must be improper. Section 5 studies the likelihood of encountering an improper surface parametric support. Finally, Section 6 concludes the paper with a summary and suggests possible future research.

2 Preliminaries

In this section we introduce all terminology and notation needed in the entire paper. The most important concepts defined are the improper index and support transformations. Support transformations are important because they reparametrize and improper supports can be shrunk by support transformations.

2.1 Surface Parametric Supports, Surface Parametrizations

Let $\mathbf{Z}$ be the set of integers and $\mathbf{R}$ be the set of reals. The set $\mathbf{Z}^2$ is the set of lattice points and the set $\mathbf{R}^2$ is the Euclidean plane. For any set $S \subseteq \mathbf{Z}^2$, the Newton polygon $NP(S)$ is the convex hull of S in the Euclidean plane. For any polygon $P \subseteq \mathbf{R}^2$, the normalized area $NA(P)$ is twice the usual Euclidean area of P.

A finite set $S \subseteq \mathbf{Z}^2$ is called a *surface parametric support*, or simply *surface support*, if $NA(NP(S)) > 0$. For example, total degree n surface supports are

$$T_n = \{(i,j) : 0 \leq i, 0 \leq j, i+j \leq n\}, \; n \geq 1; \tag{1}$$

and bidegree $m \times n$ surface supports are

$$B_{m,n} = \{(i,j) : 0 \leq i \leq m, 0 \leq j \leq n\}, \; m \geq 1, n \geq 1. \tag{2}$$

Note that

$$NA(NP(T_n)) = n^2, \quad NA(NP(B_{m,n})) = 2mn. \tag{3}$$

Let S be a surface support in the following discussions.

Any set $S' \subseteq S$ is a *surface parametric sub-support*, or simply *surface sub-support*, of S if S' is also a surface parametric support. In particular, a sub-support S' is triangular if $|S'| = 3$. Triangular sub-supports turn out to be significant in the study of improper supports.

Let $\mathbf{C}$ be the field of complex numbers. A *rational parametrization on* S is defined to be a map from $\mathbf{C}^2$ to $\mathbf{C}^3$:

$$(X(s,t), Y(s,t), Z(s,t)) = \left(\frac{x(s,t)}{w(s,t)}, \frac{y(s,t)}{w(s,t)}, \frac{z(s,t)}{w(s,t)}\right) \tag{4}$$

$$= \frac{\sum_{(i,j)\in S}(x_{i,j}, y_{i,j}, z_{i,j})s^i t^j}{\sum_{(i,j)\in S} w_{i,j} s^i t^j} \tag{5}$$

where $(w_{i,j}, x_{i,j}, y_{i,j}, z_{i,j}) \neq (0,0,0,0)$ are coefficients from some field $k \subseteq \mathbf{C}$.

It is easily checked that a rational parametrization on S is invariant under integer translations of S. In practice, for any surface support S, we may want to translate S such that the coordinates of points of S are non-negative and S intersects both the X and Y axes.

Note that a rational parametrization on S need not define a surface. It defines a point if there are constants w', x', y', z' such that

$$\frac{w_{i,j}}{w'} = \frac{x_{i,j}}{x'} = \frac{y_{i,j}}{y'} = \frac{z_{i,j}}{z'}, \ (i,j) \in S; \tag{6}$$

or it may define a curve if $X(s,t)$, $Y(s,t)$, $Z(s,t)$ satisfy more than one implicit equation $f(X,Y,Z) = 0$. One way to detect if a rational parametrization on a surface support degenerates to a curve is to use u-resultants (Chionh and Goldman, 1991). Fortunately, the theory of BKK degree bounds ensures that a general rational parametrization on a surface support indeed defines a surface.

A *surface parametrization* on a surface support S is a rational parametrization on S that does not degenerate to a point or a curve. For the rest of the paper, all rational parametrizations on S are assumed to be surface parametrizations; that is, they are non-degenerate. The discussion in the preceding paragraph ensures there is no loss of generality in making this assumption.

2.2 Shrinking Support Transformations and Shrinkable Supports

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be an invertible affine transformation with

$$T(x,y) = (ax + by + c, dx + ey + f). \tag{7}$$

The transformation T is a *support transformation with respect to a surface parametric support* S if $T(S)$ is also a surface parametric support; that is, $T(S) \subseteq \mathbf{Z}^2$. The absolute value of the determinant of the Jacobian matrix of T is written

$$J(T) = abs \begin{vmatrix} a & d \\ b & e \end{vmatrix}. \tag{8}$$

Given a support transformation T for a surface support S, it is a well-known fact in calculus that we have

$$J(T) = \frac{NA(NP(T(S)))}{NA(NP(S))}. \tag{9}$$

For obvious reasons, a support transformation T with $J(T) < 1$ is called a *shrinking support transformation*, and a support that admits a shrinking transformation is called a *shrinkable support*.

2.3 Improper Indices and Improper Supports

Consider a general surface (5) with generic coefficients (such as indeterminate coefficients) $w_{i,j}$, $x_{i,j}$, $y_{i,j}$, $z_{i,j}$, $(i,j) \in S$. The algebraic degree of the general surface is denoted $AD(S)$. The *improper index* of S is defined to be

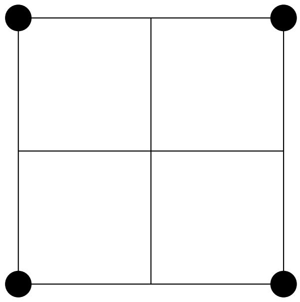
$$IX(S) = \frac{NA(NP(S))}{AD(S)}. \tag{10}$$

It is known that a general surface point corresponds to μ parametric points (Zariski, 1971). Since $NA(NP(S))$ is the expected surface degree, or the BKK degree (Cox, Little, and O'Shea, 1998), for a general surface parametrization on S, the improper index $IX(S)$ gives μ, the number of parameter points (s,t) corresponding to a general surface point $(X(s,t), Y(s,t), Z(s,t))$. We state this fundamental property as a proposition.

Proposition 1 *Let S be a surface parametric support with improper index $IX(S)$. For a general surface parametrization defined on S, there are $IX(S)$ parametric points corresponding to a general surface point.*

Consequently, we call a surface support S proper if $IX(S) = 1$ and improper if $IX(S) > 1$.

2.3.1 Example: $IX(\{(0,0),(2,0),(0,2),(2,2)\})$

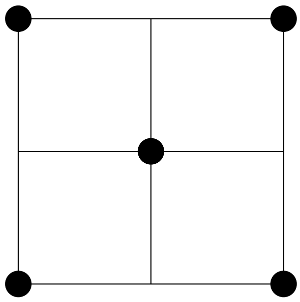


Let $S = \{(0,0),(2,0),(0,2),(2,2)\}$. By the technique of random coefficients or otherwise, we find $AD(S) = 2$. Thus

$$IX(\{(0,0),(2,0),(0,2),(2,2)\}) = \frac{2 \times 2 \times 2}{2} = 4. \tag{11}$$

Indeed, four parameter points $(\pm s, \pm t)$ correspond to a general surface point (X, Y, Z).

2.3.2 Example: $IX(\{(0,0),(2,0),(1,1),(0,2),(2,2)\})$



Let $S = \{(0,0),(2,0),(1,1),(0,2),(2,2)\})$. Again by the technique of random coefficients or otherwise, we find $AD(S) = 4$. Thus

$$IX(\{(0,0),(2,0),(1,1),(0,2),(2,2)\}) = \frac{2 \times 2 \times 2}{4} = 2. \tag{12}$$

Indeed, two parameter points (s,t), $(-s,-t)$ correspond to a general surface point (X, Y, Z).

2.3.3 Example: Triangular supports $|S| = 3$

Let $S = \{(i,j),(k,l),(p,q)\}$ be a surface support consisting of three lattice points. It is easily verified that a general surface rational parametrization on

S gives a plane, thus $AD(S) = 1$ and

$$IX(\{(i,j),(k,l),(p,q)\}) = NA(NP(S)) = abs \begin{vmatrix} i & j & 1 \\ k & l & 1 \\ p & q & 1 \end{vmatrix}. \tag{13}$$

This means the improper index of a triangular support is simply its normalized area. Consequently, a triangular support is improper if and only if its normalized area is non-unity.

2.4 Support Transformations Reparametrize

Proposition 2 *A support transformation induces a reparametrization.*

Proof

Let $T(i,j) = (ai+bj+c, di+ej+f) = (i',j')$ be a support transformation with respect to the surface support S. To obtain the reparametrization induced by T, let

$$s = u^a v^d, \quad t = u^b v^e. \tag{14}$$

We obtain a new rational parametrization:

$$\frac{\sum_{(i,j)\in S}(x_{i,j}, y_{i,j}, z_{i,j})s^i t^j}{\sum_{(i,j)\in S} w_{i,j} s^i t^j} = \frac{\sum_{(i,j)\in S}(x_{i,j}, y_{i,j}, z_{i,j})u^{ai+bj}v^{di+ej}}{\sum_{(i,j)\in S} w_{i,j} u^{ai+bj}v^{di+ej}} \tag{15}$$

$$= \frac{\sum_{(i,j)\in S}(x_{i,j}, y_{i,j}, z_{i,j})u^{ai+bj+c}v^{di+ej+f}}{\sum_{(i,j)\in S} w_{i,j} u^{ai+bj+c}v^{di+ej+f}} \tag{16}$$

$$= \frac{\sum_{(i',j')\in T(S)}(x'_{i',j'}, y'_{i',j'}, z'_{i',j'})u^{i'}v^{j'}}{\sum_{(i',j')\in T(S)} w'_{i',j'} u^{i'}v^{j'}} \tag{17}$$

where $w'_{i',j'} = w_{i,j}$, $x'_{i',j'} = x_{i,j}$, $y'_{i',j'} = y_{i,j}$, $z'_{i',j'} = z_{i,j}$. $\square$

2.5 A Shrinkable Support is Improper

Proposition 3 *If there is a shrinking support transformation T for a surface parametric support S; that is, $J(T) < 1$, then S is an improper surface* *parametric support and*

$$\frac{IX(S)}{IX(T(S))} = \frac{1}{J(T)}. \tag{18}$$

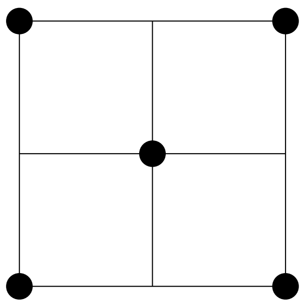
Proof

By Proposition 2, the induced parametrization on the support $T(S)$ is a reparametrization of the parametrization on the support S. Thus $AD(T(S)) = AD(S)$ and

$$\frac{IX(S)}{IX(T(S))} = \frac{NA(NP(S))AD(T(S))}{NA(NP(T(S)))AD(S)} = \frac{NA(NP(S))}{NA(NP(T(S)))} = \frac{1}{J(T)}. \tag{19}$$

Since $IX(T(S)) \geq 1$, we have $IX(S) > 1$ and S is improper. $\square$

2.5.1 Example: $S = \{(0,0),(2,0),(1,1),(0,2),(2,2)\}$



Instead of computing the non-trivial $AD(S)$ to find $IX(S)$ (see Example 2.3.2), we can check that the support is improper because there is a shrinking support transformation

$$T(x,y) = \left(\frac{x+y-2}{2}, \frac{y-x}{2}\right), \quad J(T) = 1/2. \tag{20}$$

2.5.2 *Example: Triangular supports* $|S| = 3$

In addition to the method of Example 2.3.3, we can check if a triangular support $S = \{(i, j), (k, l), (p, q)\}$ is improper by considering the transformation

$$T(x, y) = \left(\frac{\begin{vmatrix} i & j & 1 \\ x & y & 1 \\ p & q & 1 \end{vmatrix}}{\begin{vmatrix} i & j & 1 \\ k & l & 1 \\ p & q & 1 \end{vmatrix}}, \frac{\begin{vmatrix} i & j & 1 \\ k & l & 1 \\ x & y & 1 \end{vmatrix}}{\begin{vmatrix} i & j & 1 \\ k & l & 1 \\ p & q & 1 \end{vmatrix}} \right) \tag{21}$$

Since $T(S) = \{(0, 0), (1, 0), (0, 1)\}$, T is a support transformation for S with

$$J(T) = abs \begin{vmatrix} i & j & 1 \\ k & l & 1 \\ p & q & 1 \end{vmatrix}^{-1} = NA(NP(S))^{-1} = IX(S)^{-1}. \tag{22}$$

The equality

$$IX(S)J(T) = 1 \tag{23}$$

leads to a statement stronger than Proposition 3: a triangular support is improper if and only if it is shrinkable.

3 The GCD of Normalized Areas of Triangular Sub-Supports

The main result of this section is

$$IX(S) = \gcd\{NA(NP(S')) : S' \subseteq S, |S'| = 3\}. \tag{24}$$

The result is significant because it finds the improper index of a support with elementary means without having to compute the non-trivial general surface degree.

First we show that if $S' \subseteq S$ is a sub-support then $IX(S) \mid IX(S')$.

Proposition 4 *Let S be a surface parametric support. If $S' \subseteq S$ is a sub-support then $IX(S)|IX(S')$.*

Proof

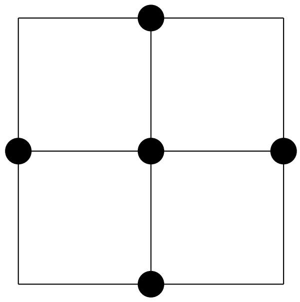
The proof is based on the Factors of Zero Theorem (Chionh and Goldman, 1991). The normalized implicit equation, one that takes into account the BKK degree, of the general surface with support S has the form $f^n = 0$, where $n = IX(S)$ and f is the actual irreducible implicit polynomial in X, Y, Z and all the indeterminate coefficients $w_{i,j}$, $x_{i,j}$, $y_{i,j}$, $z_{i,j}$, $(i,j) \in S$. To obtain S' from S, we set $w_{i,j}$, $x_{i,j}$, $y_{i,j}$, $z_{i,j}$ to zero successively for each $(i,j) \in S \setminus S'$. To illustrate the process, assume w_{i_0,j_0} is the very first indeterminate coefficient that is set to zero. We can write $f = w_{i_0,j_0} g_1 + g_2$ where the indeterminate w_{i_0,j_0} does not divide the polynomial g_2. After setting w_{i_0,j_0} to zero, the Factors of Zero Theorem assures that g_2 can be factored as $g_3 f_1^{m_1}$, $m_1 \geq 1$, such that g_3 is an extraneous factor (not satisfied by the modified parametrization) but the irreducible f_1 is satisfied by the modified parametrization. Thus the normalized implicit polynomial of the modified parametrization is $f_1^{nm_1}$. It can be seen that if this is done for the rest of indeterminate coefficients $w_{i,j}$, $x_{i,j}$, $y_{i,j}$, $z_{i,j}$, one after another for each $(i,j) \in S \setminus S'$, we obtain the normalized implicit polynomial for S' which has the form $f_p^{nm_p}$ for some integer $m_p \geq 1$. This means $IX(S') = nm_p$ and thus $IX(S)|IX(S')$. □

The following corollaries are direct consequences of the above theorem.

Corollary 1 *Every sub-support of an improper surface support is improper. Equivalently, if a sub-support of a surface support is proper, then the support is also proper.*

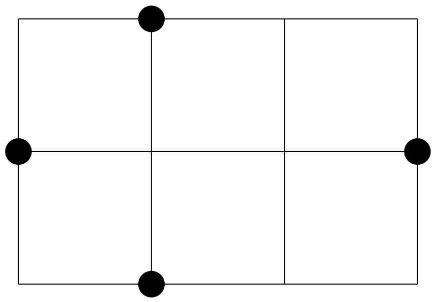
Corollary 2 *Let S_1, ..., S_n be sub-supports of a surface support S. If* gcd $(IX(S_1), \ldots, IX(S_n)) = 1$, *then S is a proper surface support. Equivalently, If S is improper then* $\gcd(IX(S_1), \ldots, IX(S_n)) > 1$.

3.1.1 Example: Support $\{(1,0),(1,1),(0,1),(2,1),(1,2)\}$ *is proper*



The support is proper because any three non-linear adjacent points form a proper surface support.

3.1.2 Example: Support $\{(1,0),(2,1),(1,2),(0,1),\}$ *is proper*



The improper indices of the bottom, right, top, left triangular sub-supports are 3, 4, 3, 2 respectively. Since $\gcd(3,4,3,2)=1$, the given support is proper.

3.2 The Main Result

Now we are ready to prove the main result.

Theorem 1 *Let S be a surface parametric support. We have*

$$IX(S) = \gcd\{NA(NP(S')) : S' \subseteq S, |S'| = 3\}. \tag{25}$$

Proof

Without loss of generality we may assume $(0,0) \in S$. For any $(x,y),(x_i,y_i) \in S$, we have

$$NA(NP(\{(0,0),(x,y),(x_i,y_i)\})) = abs\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x_i & y_i & 1 \end{vmatrix} = abs(y_i x - x_i y). \quad (26)$$

Let $\gcd\{NA(NP(S')) : S' \subseteq S, |S'| = 3\} = g$. For any $(x,y),(x_i,y_i) \in S$, there is an integer f_i such that

$$y_i x - x_i y = g f_i. \quad (27)$$

Let

$$\gcd\{y_i : (x_i,y_i) \in S\} = g_y. \quad (28)$$

There exist integers b_i such that $\sum_i b_i y_i = g_y$. Thus

$$\frac{g_y x - (\sum_i b_i x_i) y}{g} = \sum b_i f_i \quad (29)$$

is an integer and the transformation

$$T(x,y) = \left(\frac{g_y x - (\sum_i b_i x_i) y}{g}, \frac{y}{g_y}\right) \quad (30)$$

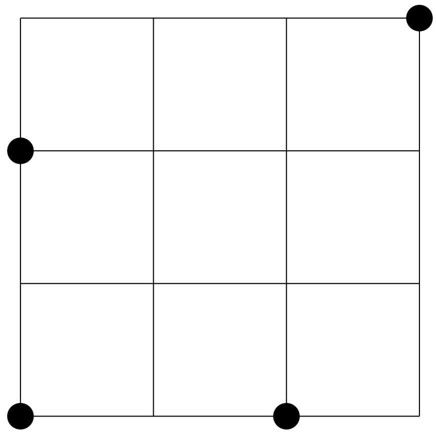
is a support transformation with Jacobian $J(T) = 1/g \leq 1$.

Since T is invertible, we have

$$\begin{aligned}
&\gcd\{NA(NP((S'')) : S'' \subseteq T(S), |S''| = 3\} \\
&= \gcd\{NA(NP((T(S')) : S' \subseteq S, |S'| = 3\} \\
&= J(T)\gcd\{NA(NP((S')) : S' \subseteq S, |S'| = 3\} \\
&= \frac{\gcd\{NA(NP(S')) : S' \subseteq S, |S'| = 3\}}{g} \\
&= 1.
\end{aligned}$$

But $IX(S'') = NA(NP(S''))$ when $S'' \subseteq T(S)$ is a triangular sub-support. By Proposition 4 we conclude $IX(T(S)) = 1$. But by Proposition 3, $IX(T(S)) = IX(S)J(T)$. Thus we have $IX(S) = g$. $\square$

3.2.1 Example: $\{(0,0),(2,0),(3,3),(0,2)\}$ *is improper*



The improper indices of the four triangular sub-supports are 4, 6, 6, 8. Their gcd is 2 and thus the given support is improper.

The theorem may also be phrased in the following form:

Corollary 3 *For a surface parametric support S, there exists a support transformation T such that $IX(S)J(T) = 1$ and $T(S)$ is proper.*

For any four points O, A, B, C, the normalized area $NA(ABC)$ is a sum or difference among the values $NA(OAB)$, $NA(OBC)$, $NA(OAC)$, thus instead of computing the gcd of all triangular sub-supports we need only compute the gcd of triangular sub-supports anchored at some chosen point O. This observation leads to the following result.

Corollary 4 *Let S be a surface parametric support and $(a,b) \in S$. We have*

$$IX(S) = \gcd\{NA(NP(S')) : S' \subseteq S, |S'| = 3, (a,b) \in S'\}. \tag{31}$$

Note that Corollary 4 provides an algorithm to compute $IX(S)$ with $O(|S|^2)$ integer gcd computations.

4 Arbitrary Surface Parametrizations on Improper Supports

All the preceding results hold for a surface defined by a rational parametrization with generic coefficients on a surface support. We now investigate the situation when the coefficients are specialized to some values in the coefficient field. It turns out that any surface parametrization defined on an improper support is indeed improper.

For a set of coefficients C of (5) taken from the coefficient field k, we use

$IX(S,C)$ to denote the *improper index* of the rational parametrization when it defines a surface.

Theorem 2 *Let $IX(S,C)$ be the improper index of a surface parametrization with coefficients C on a surface parametric support S, and T the transformation (30). Then $IX(S,C) = IX(T(S), T(C))\ IX(S)$, where $T(C)$ is the set of coefficients of the reparametrization induced by T. Consequently, if $IX(S) > 1$ then $IX(S,C) > 1$.*

Proof

The reparametrization introduced by T in (30) is

$$s = u^{g_y/g}, \quad t = u^{-(\sum_i b_i x_i)/g} v^{1/g_y}. \tag{32}$$

Since $g_y | g$, we can write $g = f_y g_y$ for some integer f_y. The reparametrization is then simplified to

$$u = s^{f_y}, \quad v = s^{(\sum_i b_i x_i)} t^{g_y}. \tag{33}$$

Recall that the improper index of a rational parametrization is the number of parameter values corresponding to a general point on the surface (Zariski, 1971). Since transformation T leads to a reparametrization of the same surface, the new rational parametrization RP of (5) after the transformation T has the same implicit surface with (5). A general point on the implicit surface $f = 0$ of RP with coefficients $T(C)$ corresponds to $IX(T(S), T(C))$ parameter values (u, v). We may assume $uv \neq 0$, since these points correspond to at most some curves on the implicit surface. By (33), one point (u, v) with $uv \neq 0$ leads to $f_y g_y = g$ points (s, t). Thus a general point on the implicit surface $f = 0$ corresponds to $IX(T(S), T(C))g$ parameter values (s, t). This proves the theorem since $g = IX(S)$. $\square$

The significance of Theorem 2 is that any surface parametrization on an improper support is improper. But, unlike the case of a general surface parametrization, transformation T given in Theorem 1 though reduces but may not completely remove the improperness of a specialized surface parametrization. However, the following theorem shows that the transformation T in Theorem 1 does give a proper reparametrization in almost all cases.

Theorem 3 *Let S be a proper surface parametric support; that is, $IX(S) = 1$. For coefficients C of (5) taken from a Zariski open set in the coefficients space $k^{4|S|}$, rational parametrization (5) is proper; that is, $IX(S,C) = 1$.*

Proof

When (5) has base points, the following homogenous equations in r, s, t have nonzero solutions

$$\begin{array}{ll} \sum_{(i,j)\in S} x_{i,j} r^{n-i-j} s^i t^j = 0, & \sum_{(i,j)\in S} y_{i,j} r^{n-i-j} s^i t^j = 0, \\ \sum_{(i,j)\in S} z_{i,j} r^{n-i-j} s^i t^j = 0, & \sum_{(i,j)\in S} w_{i,j} r^{n-i-j} s^i t^j = 0, \end{array} \tag{34}$$

where $n = \max\{i + j \mid (i, j) \in S\}$. Take the resultant R (Cox, Little, and O'Shea, 1998) of any three of the four equations. If $R(C) \neq 0$ for a set of numerical coefficients C, equations (34) with coefficients C have no non-zero solutions. In other words, if C is taken from the Zariski open set $k^{4|S|} \setminus Zero(R)$ then (5) has no base points. Thus there is no loss of generality by assuming the parametrization on S has no base points.

By $IX(S) = 1$ and Proposition 1, the rational parametrization (5) with indeterminate coefficients is proper. The implicit polynomial f must involve at least one of X, Y, or Z. Thus one of the discriminants D_X, D_Y, D_Z of f as a univariate polynomial in X, Y, Z respectively is not identically zero as f is irreducible. When the indeterminate coefficients are specialized to C such that $m = IX(S, C) > 1$, the implicit polynomial becomes $f = g^m$ where g is the implicit equation of (5) with coefficients C. As the specialized f is no longer square-free, the chosen non-identically zero discriminant, say D_Z, vanishes when it is also specialized to C. But D_Z is a non-zero polynomial in X, Y, and $x_{i,j}$, $y_{i,j}$, $z_{i,j}$, $w_{i,j}$, the coefficients of D_Z as polynomials in X, Y should be zero. Let T be such a coefficient. From the above argument, we see that for a set C of numerical values of the coefficients of (5), if $R(C)T(C) \neq 0$, (5) must be proper. The required Zariski open set can be taken as $k^{4|S|} \setminus Zero(RT)$. □

Since a Zariski open set is the whole coefficient space minus a set with lower dimensions, Theorem 3 means that for almost all numerical coefficients, transformation T in Theorem 1 gives a proper reparametrization. We state this result as a corollary.

Corollary 5 *Let S be an improper surface parametric support. For coefficients C of (5) taken from a Zariski open set in the coefficients space, the rational parametrization obtained with transformation (30) is proper.*

To find the exact conditions for the coefficients C such that the $IX(S, C) = 1$ is computationally complicated and is beyond the scope of this paper.

We clarify the situation with surface parametrizations on the support $S = \{(0, 0), (2, 0), (1, 1), (0, 2), (2, 2)\}$ of Example 2.3.2. A transformation T constructed with Theorem 1 is $T(i, j) = \left(\frac{i-j}{2}, j\right)$. Thus

$$T(S) \oplus (1, 0) = \{(1, 0), (1, 1), (2, 0), (0, 2), (1, 2)\} \tag{35}$$

where $\oplus$ denotes the Minkowski sum. We use the polynomial

$$p(s,t) = p_{0,0} + p_{2,0}s^2 + p_{1,1}st + p_{0,2}t^2 + p_{2,2}s^2t^2 \tag{36}$$

to generate the desired rational parametrization

$$(X(s,t), Y(s,t), Z(s,t)) = \left(\frac{x(s,t)}{w(s,t)}, \frac{y(s,t)}{w(s,t)}, \frac{z(s,t)}{w(s,t)}\right) \tag{37}$$

on the support S.

4.1 *Example:* $IX(S) = IX(S,C) = 2, IX(T(S),T(C)) = 1$

Random integer values are generated to construct four different $p(s,t)$ and arbitrarily take them to be $w(s,t)$, $x(s,t)$, $y(s,t)$, $z(s,t)$. By implicitization using the Dixon resultant before and after the transformation, we verify that $IX(S) = IX(S,C) = 2$ and obtain a proper reparametrization as expected; that is, $IX(T(S),T(C)) = 1$.

4.2 *Example:* $IX(S) = 2$*,* $IX(S,C) = 8, IX(T(S),T(C)) = 4$

Random integer values are generated to construct three different $p(s,t)$ and arbitrarily take them to be $w(s,t)$, $x(s,t)$, $z(s,t)$. We then set $y(s,t) = x(s,t)$. By implicitization using the Dixon resultant, we find both the before and after surface parametrizations to be improper, but the reparametrization has only half of the original improper index.

5 Likelihood of a Surface Parametric Support to be Improper

This section reports some counts to highlight that improper supports are quite pervasive for some popular total degree and bidegree configurations. But the counts seem to suggest that improper supports may become rare when the degree increases.

5.1 *Total Degree Supports*

A support $S \subseteq \mathbf{Z}^2$ is considered to be total degree n if

$$\min\{x_i : (x_i, y_i) \in S\} = \min\{y_i : (x_i, y_i) \in S\} = 0, \tag{38}$$

and

$$\max\{x_i + y_i : (x_i, y_i) \in S\} = n. \tag{39}$$

We have the following numbers

n	# of Supports	# of Improper Supports	%
2	36	10	28%
3	836	118	14%

(40)

5.2 *Bidegree Supports*

A support $S \subseteq \mathbf{Z}^2$ is considered to be bidegree degree $m \times n$ if

$$\min\{x_i : (x_i, y_i) \in S\} = \min\{y_i : (x_i, y_i) \in S\} = 0, \tag{41}$$

and

$$\max\{x_i : (x_i, y_i) \in S\} = m, \max\{y_i : (x_i, y_i) \in S\} = n. \tag{42}$$

We have the following numbers

$m \times n$	# of Supports	# of Improper Supports	%
2×2	318	65	20%
3×3	51464	501	1%

(43)

6 Conclusion

Instead of looking at surface rational parametrizations that are improper, we examine monomial supports from which any rational surface parametrization is improper. We call them inherently improper supports or simply improper supports. The main result of the paper is Equation (24) which we repeat below:

$$IX(S) = \gcd\{NA(NP(S')) : S' \subseteq S, |S'| = 3\}. \tag{44}$$

The significance of this formula is that it calculates the improper index of a support S with elementary means and avoids the difficulty of finding the degree of the generic surface defined on the support. The proof of this equality can be used to detect and reduce the improper index of an improper surface parametrization due to an improper support with $O(|S|^2)$ integer gcd computations.

We have assessed the likelihood of encountering an improper support of some popular degree configurations and found the presence of improper supports to be quite significant.

Just like monomial ideals are a special but significant special case of general ideals, we hope but do not know now if the results on the special case of improper supports will be useful in the investigation of the general problem of improper parametrizations.

Finally, we plan to generalize our results to rational parametrizations from the m dimensional space to the n dimensional space in a forthcoming paper.

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