

# On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems—Part II: Conditions Involving Circles in the Frequency Plane and Sector Nonlinearities

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**Abstract**—The object of this paper is to outline a stability theory based on functional methods. Part I of the paper was devoted to a general feedback configuration. Part II is devoted to a feedback system consisting of two elements, one of which is linear time-invariant, and the other nonlinear.

An attempt is made to unify several stability conditions, including Popov's condition, into a single principle. This principle is based on the concepts of conicity and positivity, and provides a link with the notions of gain and phase shift of the linear theory.

Part II draws on the (generalized) notion of a "sector nonlinearity." A nonlinearity  $N$  is said to be **INSIDE THE SECTOR**  $\{\alpha, \beta\}$  if it satisfies an inequality of the type  $\langle (Nx - \alpha x)_t, (Nx - \beta x)_t \rangle \leq 0$ . If  $N$  is memoryless and is characterized by a graph in the plane, then this simply means that the graph lies inside a sector of the plane. However, the preceding definition extends the concept to include nonlinearities with memory.

There are two main results. The first result, the **CIRCLE THEOREM**, asserts in part that: *If the nonlinearity is inside a sector  $\{\alpha, \beta\}$ , and if the frequency response of the linear element avoids a "critical region" in the complex plane, then the closed loop is bounded; if  $\alpha > 0$  then the critical region is a disk whose center is halfway between the points  $-1/\alpha$  and  $-1/\beta$ , and whose diameter is greater than the distance between these points.*

The second result is a method for taking into account the detailed properties of the nonlinearity to get improved stability conditions. This method involves the removal of a "multiplier" from the linear element. The frequency response of the linear element is modified by the removal, and, in effect, the size of the critical region is reduced. Several conditions, including Popov's condition, are derived by this method, under various restrictions on the nonlinearity  $N$ ; the following cases are treated:

- (i)  $N$  is instantaneously inside a sector  $\{\alpha, \beta\}$ .
- (ii)  $N$  satisfies (i) and is memoryless and time-invariant.
- (iii)  $N$  satisfies (ii) and has a restricted slope.

## 1. INTRODUCTION

THE feedback system of Fig. 1 consists<sup>1</sup> of a linear time-invariant element  $H$  and a (not necessarily linear or time-invariant) element  $N$ . It will be

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<sup>1</sup> A single input  $x$ , multiplied by real constants  $a_1$  and  $a_2$ , is added in at two points. By setting  $a_1$  or  $a_2$  to zero, it is possible to obtain a single-input system, in which the element closest to the input is either the linear element or the nonlinearity. The terms  $w_1$  and  $w_2$  are fixed bias functions, which will be used to account for the effects of initial conditions. The variables  $e_1$  and  $e_2$  are outputs.

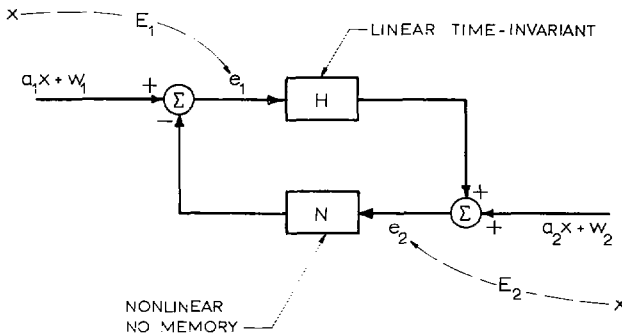


Fig. 1. A feedback system.

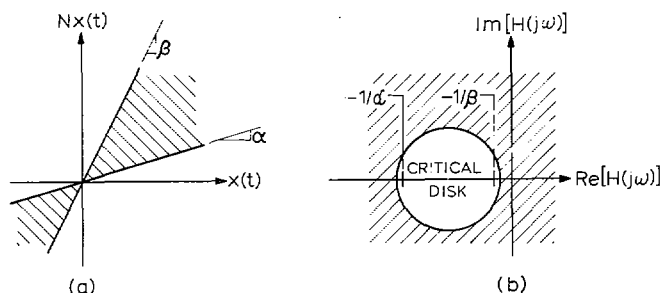


Fig. 2. If  $Nx(t)$  vs.  $x(t)$  and  $H(j\omega)$  lie in the shaded regions, and if the Nyquist diagram of  $H(j\omega)$  does not encircle the critical disk, then the closed loop is bounded.

supposed, for the moment, that  $N$  has no memory. These assumptions are among the simplest which ensure that the system is both

- (i) general enough to have many applications
- (ii) complicated enough to exhibit such characteristic nonlinear phenomena as jump resonances, subharmonics, etc.

The object here is to find stability conditions for the closed-loop system. For practical reasons, it is desirable to express these conditions in terms of quantities that can be measured experimentally, such as frequency responses, transfer characteristics, etc. In particular, the following question is of interest: Imagine that the graph of  $N$  lies inside a sector of the plane, as shown in Fig. 2(a), and that the frequency response of  $H$  is plotted in the complex plane; can the complex plane be divided

into regions that are "safe" or "unsafe" as far as stability is concerned?

It will be shown that, with certain qualifications, such a division is possible. In fact it has already been shown in Part I that such regions, called "conic sectors," exist in a quite general sense. Here these general results will be applied to some concrete situations, involving frequency responses, etc. (Fig. 2, which illustrates the simplest of the results to be obtained here, gives some idea of what is being sought.)

## 2. STATEMENT OF THE PROBLEM

The purpose of this section is to define  $H$  and  $N$ , and to write feedback equations.  $H$  and  $N$  will be represented by input-output relations or by operators, in keeping with the theory outlined in Part I.

**DEFINITION:**  $R[0, \infty)$  is the space of real-valued functions on the interval  $[0, \infty)$ .

$L_p$ , where  $p=1, 2, \dots$ , is the space consisting of those  $x$  in  $R[0, \infty)$  for which the integral  $\int_0^\infty |x(t)|^p dt$  is finite. In addition, for the case  $p=2$ , it is assumed that  $L_2$  is an inner-product space, with inner-product

$$\langle x, y \rangle = \int_0^\infty x(t)y(t)dt$$

and norm  $\|x\|_2$ . The symbol  $\|x\|$ , without subscript, will often be used instead of  $\|x\|_2$ .

$L_\infty$  is the space consisting of those functions  $x$  in  $R[0, \infty)$  that are measurable and essentially bounded.  $L_\infty$  is assumed to be a normed linear space, with norm

$$\|x\|_\infty = \text{ess sup}_{t \geq 0} |x(t)|.$$

No distinction will be made between functions differing over sets of zero measure.

Those definitions which were introduced in Part I will only be summarized here. Following the convention of Part I, the subscripted symbol  $x_t$  denotes a function in  $R[0, \infty)$  truncated after  $[0, t)$ . The space  $L_{pe}$ , where  $p=1, 2, \dots, \infty$ , is the extension of  $L_p$ , i.e.,

$$L_{pe} = \{x \mid x \in R[0, \infty) \text{ and } x_t \in L_p \text{ for all } t \geq 0\}.$$

An extended norm  $\|x\|_{pe}$  is defined on  $L_{pe}$ , where  $\|x_t\|_{pe} = \|x_t\|_p$  if  $x \in L_p$  and  $\|x\|_{pe} = \infty$  if  $x \notin L_p$ . The symbol  $\|x\|_{2e}$  will usually be abbreviated to  $\|x\|_e$ .

The concept of a relation  $H$  on  $L_{pe}$ , with domain  $Do(H)$  and range  $Ra(H)$  was introduced in Part I. A relation  $H$  on  $L_{pe}$  is  $L_p$ -bounded if  $H$  maps bounded subsets of  $L_{pe}$  into bounded subsets of  $L_{pe}$ .  $H$  is  $L_p$ -continuous if, given any  $x$  in  $Do(H)$  and any  $\Delta > 0$ , there is a  $\delta > 0$  such that, for any  $y$  in  $Do(H)$ , if  $\|x - y\|_{pe} < \delta$  then  $\|Hx - Hy\|_{pe} < \Delta$ .

Part II will be devoted entirely to finding  $L_2$  conditions (for boundedness and continuity), since these are easier to derive than the other  $L_p$  conditions. However, most of the results of this paper have been extended to the  $L_\infty$  norm, in [1b]. It has been found that, in most

cases, the  $L_2$  conditions imply  $L_\infty$ -boundedness or continuity. For physical applications the most appropriate definitions of boundedness and continuity are, of course, obtained in the  $L_x$  norm.

**DEFINITION:** Let  $\mathcal{R}_0$  be the class of relations on  $L_{2e}$  having the property that the zero element, denoted  $o$ , is in  $Do(H)$ , and  $Ho = o$ . An operator  $H$  on  $L_{2e}$  is any function of the type  $H: L_{2e} \rightarrow L_{2e}$ .

**DEFINITION:** An operator  $H$  on  $L_{2e}$  is TIME-INVARIANT if it commutes with all delays. That is, for  $t \geq 0$  let  $T_\tau$  be the operator on  $L_{2e}$  given by:  $T_\tau x(t) = x(t - \tau)$  for  $t \geq \tau$ , and  $T_\tau x(t) = 0$  for  $t < \tau$ . Then  $HT_\tau = T_\tau H$  for all  $\tau \geq 0$ .

$H$  is MEMORYLESS if  $Hx(t)$  is a function of  $x(t)$  (i.e., only of  $x(t)$ ) for all  $x$  in  $L_{2e}$  and for all  $t \geq 0$ .

### 2.1. The Operator Classes $\mathfrak{N}$ and $\mathfrak{L}$

**DEFINITION:**  $\mathfrak{N}$  is the class of operators on  $L_{2e}$  having the following property: If  $N$  is in  $\mathfrak{N}$  then there is a function,  $N: \text{Reals} \rightarrow \text{Reals}$ , satisfying<sup>2</sup>

$$Nx(t) = N(x(t)) \quad (x \in L_{2e}; t \geq 0) \quad (1)$$

and having the following properties: (i)  $N(0) = 0$ , (ii)  $|N(x)| \leq \text{const.} |x|$ , and (iii) for any real  $x$ ,  $\int_0^\infty N(x') dx'$  is finite.

An operator in  $\mathfrak{N}$  is memoryless, time-invariant, not necessarily linear, and can be characterized by a graph in the plane. The letter  $N$  will indicate the graph of  $N$ .

**DEFINITION:**  $\mathfrak{L}$  is the class of those operators  $H$  on  $L_{2e}$  satisfying an equation of the type<sup>2</sup>

$$Hx(t) = h_\infty x(t) + \int_0^t h(t - \tau)x(\tau)d\tau \quad (x \in L_{2e}; t \geq 0) \quad (2)$$

in which  $h_\infty$  is a real constant, and the impulse response  $h$  is a function in  $L_1$  with the property that, for some  $\sigma_0 < 0$ ,  $h(t) \exp(-\sigma_0 t)$  is also in  $L_1$ .

Operators in  $\mathfrak{L}$  are linear and time-invariant.

### 2.2. Feedback Equations

Consider the feedback system of Fig. 1, but with two modifications: (i)  $N$  is not necessarily memoryless; (ii)  $a_1$  and  $a_2$  are operators on  $L_{2e}$ , multiplying  $x$ . (This amount of generality will be needed for some of the intermediate results; ultimately, the interesting case is that in which  $N$  has no memory, and  $a_1$  and  $a_2$  are real constants.) The equations of this system are

$$e_1 = a_1 x + w_1 - Ne_2 \quad (3a)$$

$$e_2 = a_2 x + w_2 + He_1 \quad (3b)$$

in which it is assumed that:

$H$  is an operator in  $\mathfrak{L}$

$N$  is a relation in  $\mathcal{R}_0$

<sup>2</sup> It can be verified that every mapping of the type  $N: L_{2e} \rightarrow R[0, \infty)$  satisfying (1) is in fact an operator on  $L_{2e}$ . Similarly, every mapping  $H: L_{2e} \rightarrow R[0, \infty)$  satisfying (2) is an operator on  $L_{2e}$  [see (B1) of Appendix B].

$x$  in  $L_{2e}$  is an input  
 $e_1$  and  $e_2$  in  $L_{2e}$  are outputs  
 $w_1$  and  $w_2$  in  $L_2$  are fixed biases  
 either  $a_1$  and  $a_2$  are real constants, or, more generally,<sup>3</sup>  
 $a_1$  is a relation on  $L_{2e}$  having the property that  
 $\|a_1x\|_e \leq \text{const.} \|x\|_e$ , and similarly for  $a_2$ .

REMARK: If, to begin with, the linear element satisfies a state equation, then  $H_{e_1}$  is set equal to the "zero-initial-condition response" of the state equation, and  $w_2$  is set equal to the "zero-input response."

The closed-loop relations which map  $x$  into  $e_1$  and  $e_2$  will be denoted  $E_1$  and  $E_2$ . The objective here is: Find conditions on  $N$  and  $H$  which ensure that  $E_1$  and  $E_2$  are  $L_2$ -bounded and  $L_2$ -continuous.

3. CONDITIONS FOR CONICITY AND POSITIVITY

This section has some preliminary results, which will be needed later in the analysis of stability. The following questions are fundamental to this analysis. Under what conditions is an operator conic or positive? Under what conditions is the composition product of two operators conic or positive?

The definitions of conicity and positivity were introduced in Part I. They are repeated here, for the special case of relations on  $L_{2e}$ .

DEFINITION: Let  $H$  be a relation in  $\mathcal{R}_0$ .  $H$  is INTERIOR CONIC if  $c$  and  $r \geq 0$  are real constants and the inequality

$$\|(Hx)_t - cx_t\| \leq r\|x_t\| \quad (x \in \text{Do}(H); t \geq 0) \quad (4)$$

holds.  $H$  is exterior conic if (4) holds with the inequality sign reversed.  $c$  will be called the center parameter of  $H$ , and  $r$  will be called the radius parameter of  $H$ .

$H$  is INSIDE (OUTSIDE) THE SECTOR  $\{a, \beta\}$  if  $a \leq \beta$  and if the inequality

$$\langle (Hx - \alpha x)_t, (Hx - \beta x)_t \rangle \leq 0 \quad (x \in \text{Do}(H); t \geq 0) \quad (5)$$

holds (with the inequality sign reversed).

$H$  is POSITIVE if it satisfies the inequality  $\langle x_t, (Hx)_t \rangle \geq 0$  for all  $x$  in  $\text{Do}(H)$  and all  $t \geq 0$ .

In Part I, the concepts of conicity and positivity were subdivided into categories such as "instantaneous" conicity, "incremental" conicity, etc. The definitions of these terms are listed in Appendix A.

REMARK: The following conditions are equivalent: (i)  $H$  is interior conic with parameters  $c$  and  $r$ . (ii)  $H$  is inside  $\{c-r, c+r\}$ .

3.1. Memoryless, Time-Invariant Nonlinearities

Consider the operator class  $\mathcal{N}$ ; the conditions for  $N$  in  $\mathcal{N}$  to be conic, positive, etc., are simply the "instantaneous" conditions of Appendix A. Some of these conditions are illustrated in Fig. 3. In particular,  $N$  is inside the sector  $\{\alpha, \beta\}$  if its graph lies in a sector of the plane bounded by lines of slopes  $\alpha$  and  $\beta$ ;  $N$  is incrementally

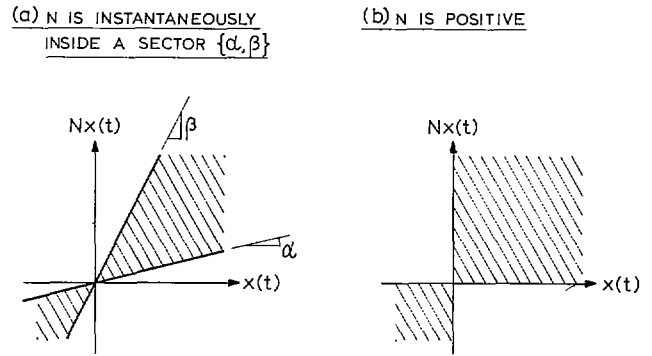


Fig. 3. Permissible regions (shaded) for instantaneously confined nonlinearities.

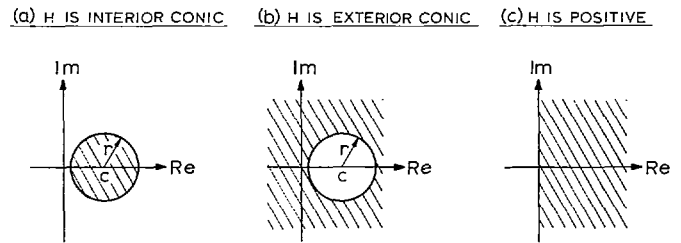


Fig. 4. Permissible regions (shaded) for the frequency response  $H(j\omega)$ .

inside  $\{\alpha, \beta\}$  if, in addition,  $N$  satisfies the slope restrictions  $\alpha \leq [N(x) - N(y)] / (x - y) \leq \beta$ .  $N$  is positive if its graph lies in the first and third quadrants;  $N$  is incrementally positive if, in addition,  $N$  is nondecreasing.

3.2. Linear Time-Invariant Operators

Consider the operator class  $\mathcal{L}$ ; it will be shown, roughly speaking, that a conic sector has a counterpart in the frequency plane, in the form of a circular disk (see Fig. 4). This disk degenerates into a half-plane in the case of a positive operator.

DEFINITION: Let  $s = \sigma + j\omega$  denote a point in the complex plane. The LAPLACE TRANSFORM  $H(s)$  of  $H$  in  $\mathcal{L}$  is

$$H(s) = h_\infty + \int_0^\infty h(t) \exp(-st) dt \quad (\sigma \geq 0) \quad (6)$$

(The integral on the right-hand side of (6) exists and is analytic for  $\sigma \geq 0$  [See (B1) of Appendix B].)

DEFINITION: The NYQUIST DIAGRAM of  $H(s)$  is a curve in the complex plane consisting of: (i) the image of the  $j\omega$ -axis under the mapping  $H(s)$ , and (ii) the point  $h_\infty$ .

LEMMA 1. Let  $H$  be an operator in  $\mathcal{L}$ , and let  $c$  and  $r \geq 0$  be real constants.

(a) If  $H(s)$  satisfies the inequality

$$|H(j\omega) - c| \leq r \quad (\omega \in (-\infty, \infty)) \quad (7)$$

then  $H$  is incrementally interior conic with center parameter  $c$  and radius parameter  $r$ .

<sup>3</sup> The more general assumption will be needed in Section 5 only.

(b) If  $H(s)$  satisfies the inequality

$$|H(j\omega) - c| \geq r \quad (\omega \in (-\infty, \infty)) \quad (8)$$

and if the Nyquist diagram of  $H(s)$  does not encircle the point  $(c, 0)$ , then  $H$  is incrementally exterior conic with center parameter  $c$  and radius parameter  $r$ .

(c) If  $\text{Re}\{H(j\omega)\} \geq 0$  for  $\omega \in (-\infty, \infty)$  then  $H$  is incrementally positive.

The Proof of Lemma 1 is in Appendix B.

REMARK. The gains  $g(H)$  and  $\bar{g}(H)$  were defined in Part I. It follows from Lemma 1(a) that if  $|H(j\omega)| \leq c$ , then  $\bar{g}(H) = g(H) \leq c$ .

### 3.3. Composition Products and Sector Products

The composition product of two positive operators need not be positive. Those special cases in which the product is positive are of interest because they give a tighter bound on the composite behavior than would be obtained in general. (They form the basis of the factorization method of Section 5.2.)

Similarly, those special cases in which the product of two sector operators lies inside the "product sector" are of interest.

DEFINITION: The PRODUCT SECTOR  $\{\alpha_1, \beta_1\} \times \{\alpha_2, \beta_2\}$  is the sector  $\{\alpha, \beta\}$ , where  $[\alpha, \beta]$  is the interval of the reals defined by  $[\alpha, \beta] = \{xy | x \in [\alpha_1, \beta_1] \text{ and } y \in [\alpha_2, \beta_2]\}$ .

In other words, product sectors behave like point-wise products of the corresponding real intervals (see Fig. 5).

It is easy to show that if both operators are memoryless, say if both operators are in  $\mathfrak{X}$ , then their product has the above mentioned properties. [This can be shown by expressing the ratio  $N_1(N_2(x))/x$  as the product  $(N_1(y)/y) \times (N_2(x)/x)$ , where  $y \triangleq N_2(x)$ .] More difficult are cases in which one operator is in  $\mathfrak{X}$  and the other is in  $\mathfrak{L}$ , as in Lemmas 2 and 3.

### 3.4. A Memoryless Nonlinearity and a First-Order Multiplier

The following lemma is the basis for Popov's condition (Section 5.1).

LEMMA 2. Let  $N$  be an operator in  $\mathfrak{X}$ ,  $K$  be an operator in  $\mathfrak{L}$ , and let the Laplace transform of  $K$  be  $K(s) = k\lambda/s + \lambda$  where  $k > 0$  and  $\lambda > 0$ .

- (a) If  $N$  is positive<sup>4</sup> then  $NK$  is positive.
- (b) If  $N$  is inside<sup>5</sup> a sector  $\{\alpha, \beta\}$  then  $NK$  is inside the product sector  $\{\alpha, \beta\} \times \{0, k\}$ .

The proof of Lemma 2 is in Appendix C. (Note that  $K$  itself is positive and inside  $\{0, k\}$ , since  $K(j\omega)$  lies entirely in the right-half plane and since  $|K(j\omega) - \frac{1}{2}k| = \frac{1}{2}k$ .)

<sup>4</sup> i.e.,  $xN(x) \geq 0$ .  
<sup>5</sup> i.e.,  $\alpha \leq N(x)/x \leq \beta$ .

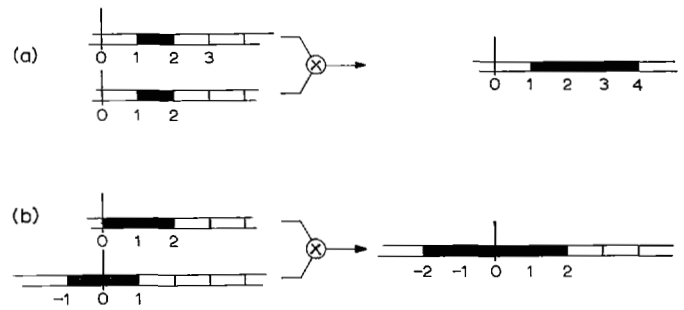


Fig. 5. Products of intervals.

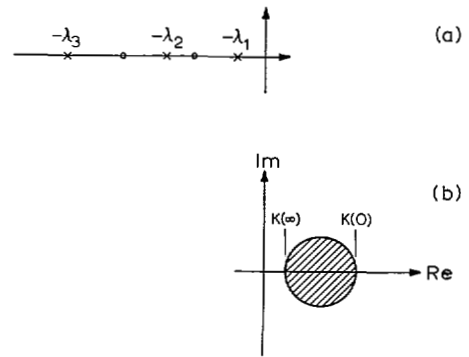


Fig. 6.  $\mathfrak{RC}$  operators. (a) Typical pole-zero pattern. (b) Circle of confinement of  $K(j\omega)$ .

### 3.5. A Memoryless Nonlinearity and an $\mathfrak{RC}$ Multiplier

A situation resembling Lemma 2, but with  $N$  more restricted and  $K$  more general, is considered next.  $K$  is taken to be a sum of first-order terms, of a type that can be realized as the driving-point impedance of an RC network (see Guillemin [2], ch. 4).

DEFINITION: Let  $\mathfrak{RC}$  be the class of those operators  $K$  in  $\mathfrak{L}$  having Laplace transforms of the form

$$K(s) = \sum_{i=1}^n \frac{k_i \lambda_i}{s + \lambda_i} + K(\infty) \quad (9)$$

where  $k_i \geq 0$ ,  $\lambda_i > 0$ , and  $K(\infty) \geq 0$  are real constants.

An operator  $K$  in  $\mathfrak{RC}$  has poles and zeros alternating on the negative-real axis, with a pole nearest but not at the origin. The frequency response of  $K$  lies inside a circle in the right-half plane, located as shown in Fig. 6(b); it follows that  $K$  is positive and inside the sector  $\{K(\infty), K(0)\}$ . (Observe that

$$K(0) = K(\infty) + \sum_{i=1}^n k_i \geq K(\infty).)$$

LEMMA 3. Let  $N$  be an operator in  $\mathfrak{X}$ , and  $K$  be an operator in  $\mathfrak{RC}$ .

- (a) If  $N$  is incrementally positive then  $NK$  is positive.
- (b) If  $N$  is incrementally inside the sector  $\{\alpha, \beta\}$  then  $NK$  is inside the product sector  $\{\alpha, \beta\} \times \{K(\infty), K(0)\}$ .

In other words, multiplication by  $K$  affects the composite sector as if  $K$  had no memory. The proof of Lemma 3 is in Appendix D.

## 4. CIRCLE CONDITIONS FOR STABILITY

Consider now the main problem of this paper, namely, the problem of stability for the loop of Fig. 1. Suppose that  $N$  is a relation (which may or may not be memoryless) inside a sector  $\{\alpha, \beta\}$ . What conditions on the frequency response  $H(j\omega)$  are sufficient to ensure boundedness of the closed loop? It will appear that the following "circle conditions" are sufficient:

**DEFINITION:**  $H(j\omega)$  will be said to SATISFY THE CIRCLE CONDITIONS FOR THE SECTOR  $\{\alpha, \beta\}$ , WITH OFFSET  $\delta$ , where  $\alpha \leq \beta$ ,  $\beta > 0$ , and  $\delta \geq 0$  are real constants, if the following conditions hold:

CASE 1A. If  $\alpha > 0$ , then

$$\left| H(j\omega) + \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \right| \geq \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) + \delta \quad (\omega \in (-\infty, \infty)) \quad (10)$$

and the Nyquist diagram of  $H(j\omega)$  does not encircle the point  $-\frac{1}{2}(1/\alpha + 1/\beta)$ .

CASE 1B. If  $\alpha < 0$ , then

$$\left| H(j\omega) + \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \right| \leq \frac{1}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) - \delta \quad (\omega \in (-\infty, \infty)). \quad (11)$$

CASE 2. If  $\alpha = 0$ , then  $\text{Re}\{H(j\omega)\} \geq -(1/\beta) + \delta$  for  $\omega \in (-\infty, \infty)$ .

In other words, the complex plane is divided into two regions, shaped either like a circular disk and its complement, or like two half-planes. (The case  $\alpha > 0$  is illustrated in Fig. 2.) One of the regions will be called "permissible" and the other will be called "critical." If  $H(j\omega)$  does not enter or encircle the critical region, then the closed loop is bounded. If, in addition,  $N$  is incrementally inside  $\{\alpha, \beta\}$ , then the closed loop is continuous. These results are formalized in the following theorem:

**A CIRCLE THEOREM.** Suppose that

- (I)  $N$  is a relation in  $\mathcal{R}_0$ , (incrementally) inside the sector  $\{\alpha + \Delta, \beta - \Delta\}$ , where  $\beta > 0$ .
- (II)  $H$  is an operator in  $\mathcal{L}$ , which satisfies the circle conditions for the sector  $\{\alpha, \beta\}$  with offset  $\delta$ .
- (III)  $\delta$  and  $\Delta$  are non-negative constants, at least one of which is greater than zero.

Then the closed-loop operators  $E_1$  and  $E_2$  are  $L_2$ -bounded ( $L_2$ -continuous).

The Circle Theorem is based on Theorem 2 of Part I. It was assumed in Theorem 2 that  $a_1$  and  $a_2$  were real constants. However, with only minor changes in the proof, it can be shown that Theorem 2 holds more generally if  $a_1$  and  $a_2$  are relations on  $L_{2e}$ , provided  $a_1$  and  $a_2$  satisfy inequalities of the type  $\|a_1 x\|_e \leq \text{const.} \|x\|_e$ . The Circle Theorem then follows immediately with the aid of Lemma 1 of Part II.

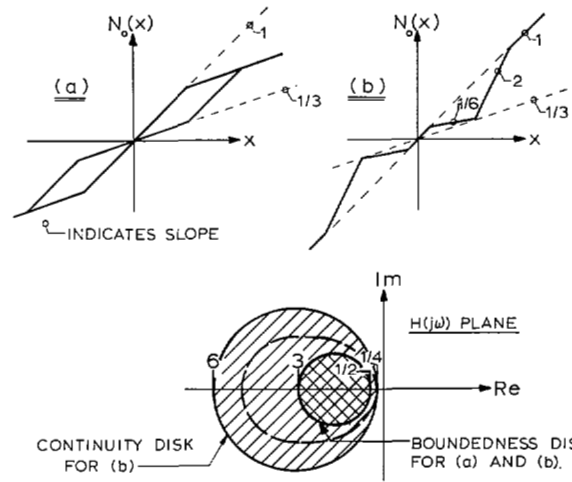


Fig. 7. Critical disks for Example 1. (Broken curve indicates edges of jump region in  $H(j\omega)$  plane.)

The Circle Theorem can be viewed as a generalization of Nyquist's criterion,<sup>6</sup> in which a critical region replaces the critical point. For a given  $N$  there are two critical regions, one for boundedness and one for continuity. It can be shown that the continuity region always contains the boundedness region (see Example 1 and Fig. 7).

The Circle Theorem will serve as the generating theorem for the rest of this paper; i.e., the remaining results will be obtained as corollaries to the Circle Theorem by variously constraining the form of  $N$ . In particular, the following corollary is obvious.

4.1. A Circle Condition for Instantaneous Nonlinearities<sup>7</sup>

**COROLLARY 1.** If (I)  $N$  in  $\mathcal{R}_0$  is instantaneously (incrementally) inside the sector  $\{\alpha + \Delta, \beta - \Delta\}$  where  $\beta > 0$ , and if conditions (II) and (III) of the Circle Theorem hold, then  $E_1$  and  $E_2$  are  $L_2$ -bounded ( $L_2$ -continuous).

**EXAMPLE 1.** (a) Let  $N_0$  be the relation shown in Fig. 7(a), and  $N$  be the relation in  $\mathcal{R}_0$  defined by the equation  $Nx(t) = [1 + \sin^2(t)] \cdot N_0(x(t))$ . Find the critical regions for boundedness and continuity. (b) Repeat for the function shown in Fig. 7(b).

(a) Observe that  $N_0$  is inside the sector  $\{1/3, 1\}$  and that the time-varying gain  $[1 + \sin^2(t)]$  is inside the sector  $\{1, 2\}$ . It follows that  $N$  is inside the product sector,  $\{1/3, 1\} \times \{1, 2\} = \{1/3, 2\}$ . Corollary 1 therefore implies that the critical region for boundedness is a disk, as shown in Fig. 7. However, since  $N$  is multi-valued, and therefore not incrementally in any sector, Corollary 1 provides no information about continuity.

(b) The same results as in (a) are obtained for boundedness. In addition,  $N$  is incrementally inside  $\{1/6, 4\}$ , and a continuity disk is obtained, as shown in Fig. 7.

<sup>6</sup> More accurately, of the sufficient part of Nyquist's criterion.

<sup>7</sup> Similar or closely related circle conditions were found independently by the author [1a], Sandberg [5], Narendra and Goldwyn [6], and Kudrewicz [7].

Observe that the nonlinearity  $N$  in Corollary 1 can be time-varying and can have memory. In fact, very little has been assumed about the detailed character of  $N$ . The price paid for this is that Corollary 1 is often conservative, i.e., the critical region is too large. This is especially true of the boundedness condition (see Example 2). The continuity condition probably gives a quite fair estimate of what to expect. In fact, an approximate analysis, based on the harmonic balance method (cf. Hatanaka [3]), suggests that continuity breaks down in the following way: There is a zone, inside the critical continuity disk, in which jump-resonance phenomena occur. The zone is not much smaller than the continuity disk. Furthermore, the magnitudes of jump resonances depend on the Nyquist diagram behavior inside the continuity disk.

### 5. CONDITIONS WITH TRANSFERRED MULTIPLIERS

The next two corollaries can be viewed as attempts to reduce the size of the critical region, at the cost of added restrictions on  $N$ . In certain cases, it will be possible to remove a "multiplier"  $K$  from the linear element, before applying the Circle Theorem. The removal of  $K$  will shift the frequency response of the remainder,  $H_1(j\omega)$ , away from the critical region. Thus the effective size of the critical region will be reduced.

#### 5.1. Popov's Condition

Consider the feedback system of Fig. 1, under the same conditions as in Corollary 1, but with the added constraint that  $N$  is a memoryless, time-invariant operator. The following condition for boundedness (not continuity) involves the removal of a first-order multiplier from  $H$ .

COROLLARY 2. If

- (I)  $N$  is an operator in  $\mathfrak{X}$ , inside the sector  $\{\alpha, \beta\}$  where  $\beta > 0$ .
- (II)  $H$  is an operator in  $\mathfrak{L}$  that can be factored into a product  $H = KH_1$ , where  $H_1$  and  $K$  are in  $\mathfrak{L}$ , and  $K(s) = \lambda/(s + \lambda)$  where  $\lambda > 0$ .
- (III)  $H_1$  satisfies the circle conditions for the product sector  $\{\alpha, \beta\} \times \{0, 1\}$  with offset  $\delta$ , where  $\delta > 0$ .
- (IV)  $a_2 = 0$  and  $\dot{w}_2$  is in  $L_2$ , where  $\dot{w}_2$  denotes the derivative on  $[0, \infty)$ .

Then  $E_1$  and  $E_2$  are  $L_2$ -bounded.

REMARKS: (i) For  $\alpha > 0$ , Condition III simply means that  $\text{Re}\{(j\omega + \lambda)H(j\omega)\} \geq -\lambda/\beta + \delta$ . (ii) Condition IV limits the result to that configuration in which the directions of flow is from the input to  $H$  to  $N$ .

PROOF OF COROLLARY 2. The feedback equations will be transformed, as illustrated in Fig. 8; i.e.,  $H$  will be split into a product,  $H = KH_1$ , and the multiplier  $K$  will be transferred into a composition with  $N$ . It will then be shown, in Step 1, that the transformed equations are bounded, and, in Step 2, that they are equivalent to the original equations as far as stability is concerned.

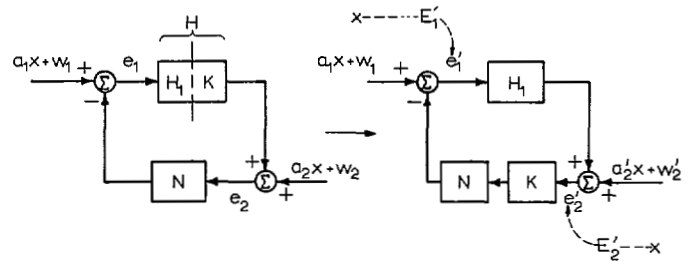


Fig. 8. A transformation.

Letting  $w_2' = w_2 + \lambda^{-1}\dot{w}_2$ , and recalling that  $a_2 = 0$ , consider the equations of the transformed system of Fig. 8,

$$e_1' = w_1 + a_1x - NK e_2' \quad (12a)$$

$$e_2' = w_2' + H_1 e_1' \quad (12b)$$

Let  $E_1'$  and  $E_2'$  be the closed-loop relations for (12a)–(12b).

STEP 1.  $E_1'$  and  $E_2'$  are  $L_2$ -bounded. This follows from the Circle Theorem whose hypotheses are satisfied because:  $w_2'$  is in  $L_2$  by assumption IV;  $NK$  is in the product sector  $\{\alpha, \beta\} \times \{0, 1\}$  by Lemma 2; and  $NK$  satisfies the appropriate circle conditions.

STEP 2. It will be shown (below) that

$$E_1 = E_1' \quad (13a)$$

$$E_2 = KE_2' \quad (13b)$$

Since  $E_1'$  and  $E_2'$  have been proved  $L_2$ -bounded, and since  $K$  is certainly  $L_2$ -bounded, it follows that  $E_1$  and  $E_2$  are  $L_2$ -bounded.

To prove (13b), recall that  $E_2$  and  $E_2'$  are subsets (of a product space), so that it is enough to establish that each contains the other. Suppose that  $(x, e_2)$  is an element of  $E_2$ ; by definition of  $E_2$ , there is an  $e_1$  in  $L_2$  satisfying (3a)–(3b); let  $e_1' = e_1$  and  $e_2' = w_2' + H_1 e_1$ . Direct substitution shows that  $(x, e_1', e_2')$  satisfies (12a)–(12b), so that  $(x, e_2')$  belongs to  $E_2'$ . Substitution also shows that  $e_2 = KE_2'$ , so that  $(x, e_2)$  is in  $KE_2'$ . Since  $(x, e_2)$  is an arbitrary element of  $E_2$ , it follows that  $KE_2'$  contains  $E_2$ . It can similarly be shown that  $E_2$  contains  $KE_2'$ , so that (13b) holds. The proof of (13a) is similar. Q.E.D.

EXAMPLE 2. Let  $N$  be the operator in  $\mathfrak{X}$  whose graph is shown in Fig. 9(a), and let  $H(s) = k/(s + \lambda)(s + \mu)$ . For what values of  $k$  is the closed-loop  $L_2$ -bounded? Compare Corollaries 1 and 2.

Here  $N$  is inside  $\{0, 1\}$ , so the critical region is a half-plane,  $\text{Re}\{\cdot\} \leq -1 + \delta$ , in both corollaries. In Corollary 2 let  $K(s) = \lambda/(s + \lambda)$  and  $H_1(s) = k/(\lambda(s + \mu))$ ; the following estimates are obtained:

$$\text{Corollary 1: } -\lambda\mu < k < (\lambda + \mu)(\lambda + \mu + 2\sqrt{\lambda\mu})$$

$$\text{Corollary 2: } -\lambda\mu < k$$

Corollary 2 is less conservative than Corollary 1, as it shows the closed loop to be bounded for all positive  $k$ .

Figure 10 also illustrates the following point: For  $a \leq 0$ , both corollaries predict the same critical region; however, in many cases of interest,  $H_1(j\omega)$  is further from the critical region than  $H(j\omega)$ .

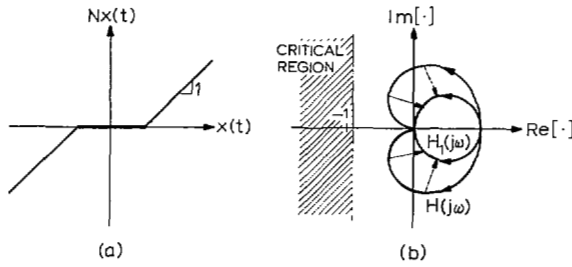


Fig. 9. An example of Popov's Condition. Arrows indicate shift away from critical region.

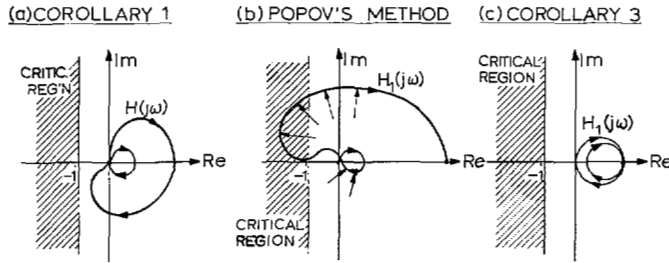


Fig. 10. Nyquist diagrams for Corollaries 1-3.

5.2. The Factorization Method

The proof of Corollary 2 suggests a method for generating a class of Popov-like conditions. The method consists of a factorization of  $H$  into  $H = KH_1$ , followed by the transformation of (3a)-(3b) into (12a)-(12b) followed by an application of the Circle Theorem. Various stability conditions are produced by variously choosing the multiplier  $K$ .

The method has two preconditions:

- (Ia) Either  $K^{-1}$  exists or  $a_2 = 0$ .
- (Ib) There is a  $w_2'$  in  $L_2$  such that  $K(w_2') = w_2$ .

These preconditions ensure that (3) are transformable into (12). Note that if  $K^{-1}$  exists, then  $a_2$  need not be zero; however, in that case, (12b) must be modified by the addition of a term  $a_2'x$  where  $a_2'x = K^{-1}a_2x$  (that is,  $a_2'$  is a relation on  $L_2$ ).

The method is worthwhile only if it gives a smaller effective critical region than Corollary 1. This happens if:

- (IIa)  $NK$  lies in a sector not greater than the product of the sectors of  $N$  and  $K$ .
- (IIb)  $KH_1$  lies in a sector greater than the product of the sectors of  $K$  and  $H_1$ .

If Requirements (IIa)-(IIb) are satisfied, then it is advantageous to transfer  $K$  from a composition with  $H_1$  into a composition with  $N$ . Requirement (IIa) usually means that the multiplier  $K$  has a very special form, and the difficulty in finding suitable multipliers is the main problem in applying this method. Once  $K$  is fixed, Requirement (IIb) defines a (limited) class of operators  $H$  for which this method is useful.

As an illustration of this method, a condition resembling Popov's is derived next.

5.3. A Slope-Restricted Nonlinearity and an RC Multiplier

Consider the feedback system of Fig. 1, under the conditions of Corollary 2, but with an added slope restriction on  $N$ , and a more general type of multiplier  $K$ .

COROLLARY 3.<sup>8</sup> If:

- (I)  $N$  is an operator in  $\mathfrak{N}$ , incrementally inside<sup>9</sup>  $\{\alpha, \beta\}$  where  $\beta > 0$ .
- (II)  $H$  is an operator in  $\mathfrak{L}$ , which can be factored into a product  $H = KH_1$ , where  $K$  is in  $\mathfrak{RC}$  and  $H_1$  is in  $\mathfrak{L}$ .
- (III)  $H_1$  satisfies the circle conditions for the product sector  $\{\alpha, \beta\} \times \{K(\infty), K(0)\}$  with offset  $\delta$ , where  $\delta > 0$ .
- (IV) There is<sup>10</sup> a  $w_2'$  such that  $K(w_2') = w_2$ .
- (V) Either  $K(\infty) > 0$  or  $a_2 = 0$ .

Then  $E_1$  and  $E_2$  are  $L_2$ -bounded.

Corollary 3 is obtained immediately by the factorization method, with the help of Lemma 3.

REMARK: For suitably restricted  $N$ , Corollary 3 has several advantages over Popov's method:

- (i) The shift away from the critical region, which depends on  $K(j\omega)$ , can be controlled more flexibly as a function of  $\omega$ . This is likely to be useful where a negative magnitude slope  $d/d\omega |H(j\omega)|$  is followed by a positive slope at a larger  $\omega$ .
- (ii)  $a_2$  need not be zero if  $K(\infty) > 0$ .
- (iii) If  $\alpha > 0$ , the critical region predicted by Corollary 3 (a disk) is sometimes smaller than by Popov's method (which always gives a half-plane for  $\alpha > 0$ ).

EXAMPLE 3. Let  $N$  be the operator in  $\mathfrak{N}$  whose graph is shown in Fig. 9(a) (the same as in Example 2), and let  $H$  have the Laplace transform

$$H(s) = kr^{-1} \left\{ \frac{s+1}{s+r^{-1}} \right\} \cdot \left\{ \frac{r^{-1}s}{(s+r^{-2})(s+r^{-1})} + \frac{r^2s}{(s+r)(s+r^2)} \right\}$$

where  $r \gg 1$ . For what values of  $k$  is the closed loop bounded? Compare Corollaries 1, 2, and 3.

Figure 10(a) illustrates the significant features of the Nyquist diagram of  $H(j\omega)$  (not drawn to scale). Observe that  $H(j\omega)$  has two "pass bands," one for  $r^{-2} < \omega < r^{-1}$  and the other for  $r < \omega < r^2$ ; these "pass bands" produce the two loops in Fig. 10(a). Note that the critical region is the same half-plane in all corollaries, namely,  $\text{Re}\{\cdot\} \geq -1 + \delta$ .

<sup>8</sup> Corollary 3 and the factorization method, in a functional setting, were introduced by the author in Reference [1a]. A related method in a Liapunov setting has been exploited by Brockett, Willems, and Forsys [4a]-[4b].

<sup>9</sup> i.e.,  $\alpha \leq \frac{N(x) - N(y)}{x - y} \leq \beta$ .

<sup>10</sup> This condition is satisfied automatically if  $K(\infty) > 0$ . If  $K(\infty) = 0$ , then it is satisfied if  $w_2$  is in  $L_2$ .



Corollary 1 predicts boundedness for  $-1 < k < 8$  approximately.

Popov's method is useless here. A comparison of Figs. 10(a) and 10(b) shows the effect of removing the multiplier  $\lambda/s + \lambda$ :  $H_1(j\omega)$  is moved away from the critical region in the lower left half-plane (in the decaying edge of the lower pass band,  $r^{-1} < \omega < 1$ ); however, this improvement is more than offset by the bulge introduced in the upper left half-plane (in the rising edge of the upper pass band,  $1 < \omega < r$ ).

What is obviously needed here is a multiplier that acts like Popov's in the lower pass band, but has no effect in the upper pass band. The RC multiplier  $K(s) = r^{-1}(s+1)(s+r^{-1})$  accomplishes just this. Its removal shifts  $H_1(j\omega)$  entirely into the right half-plane [Fig. 10(c)]. Corollary 3 therefore implies that the closed loop is bounded for all positive  $k$ , in fact, for  $k > -1$ .

## 6. COMMENTS AND CONCLUSIONS

### 6.1. Circle Conditions

The main result here is the Circle Theorem. The Circle Theorem is a sufficient condition for closed-loop stability, which requires the nonlinearity  $N$  to lie inside a sector, but which leaves  $N$  free otherwise. The other conditions are all corollaries of the Circle Theorem.

Corollary 1 is probably the most useful result, since it roughs out the region of stability, with a minimum of restrictions on  $N$ . However, it is often conservative.

Corollaries 2 and 3 provide a tradeoff between limitations on  $N$  and limitations on  $H(j\omega)$ . Probably more significant than the actual conditions is the fact that there is a method of generating them, namely, the factorization method.

The results derived in Part II hold for nonzero initial conditions in the linear element, provided the "zero-input response"  $w_2$  satisfies the indicated restrictions.

### 6.2. Extensions of the Theory

The theory has been extended in several directions (see [1b]), notably,

- 1) to  $L_\infty$ ,
- 2) to systems with a limited rate of time variation.

The extension to  $L_\infty$  involves the use of exponential weighting factors, which transform  $L_\infty$  functions into  $L_2$  functions. The extension to time-varying systems involves the use of a shifted Nyquist diagram,  $H(\sigma + j\omega)$ , in which  $\sigma$  depends on the rate of time variation.

### 6.3. Gain and Phase Shift in Relation to Nonlinear Time-Varying Systems

The stability of a linear time-invariant feedback system depends on the amounts of gain and phase shift introduced by the open loop. Are similar considerations involved in nonlinear, time-varying problems? Of course the classical definitions of gain and phase shift, in terms

of frequency response, have no strict meaning in nonlinear or time-varying systems. However, stability does seem to depend on certain measures of signal amplification and signal shift. Thus the norm ratio  $\|Hx\|/\|x\|$  plays a role similar to the role of gain. Furthermore, the inner product  $\langle x, Hx \rangle$ , a measure of input-output cross-correlation, is closely related to the notion of phase shift. For example, for linear time-invariant operators in  $\mathcal{L}$  the condition of positivity,  $\langle x, Hx \rangle \geq 0$ , is equivalent (by Lemma 1) to the phase condition,

$$|\text{Arg} \{H(j\omega)\}| \leq 90^\circ.$$

It may be worthwhile to see what the theorems of Part I mean in terms of gain and phase shift. This can be done with the help of Lemma 1. Theorem 1 of Part I can be viewed as a generalization to nonlinear time-varying systems of the rule that, "if the open-loop gain is less than one, then the closed loop is stable." Theorem 3 can be viewed as the generalization of, "if the open-loop absolute phase shift is less than  $180^\circ$  then<sup>11</sup> the closed loop is stable." Theorem 2 places gain and phase shift in competition, permitting large gains at small phase shifts, etc.

### 6.4. Conclusions

Some of the salient features of the functional theory are:

(i) It provides an alternative to the method of Liapunov, an alternative resembling the classical Nyquist-Bode theory.

(ii) It is well suited to input-output problems.

(iii) It is free of state-space restrictions, and is therefore useful for distributed systems, hysteretic systems, etc. It also lends itself well to multivariable systems.

(iv) It unifies several results in stability theory. In particular, it is noteworthy that Popov's condition, the slope-restricted- $N$  result, etc., can all be derived from the Circle Theorem.

(v) It has led to some new results, notably Corollary 3 and [1b].

The theory outlined here is probably still far from its definitive form. Nevertheless, it provides enough insight to make possible a reasonably systematic design of stabilizers.

## APPENDIX A

### DEFINITIONS OF CONICITY AND POSITIVITY

It will be assumed that  $H$  is a relation in  $\mathcal{R}_0$  and that  $c, r \geq 0$ , and  $\alpha \leq \beta$  are real constants.

#### GROUP I. "Incremental" Conditions

$H$  is incrementally interior conic if

$$\| (Hx - Hy)_t - c(x - y)_t \| \leq r \| (x - y)_t \|;$$

<sup>11</sup> There are two positive elements in the open loop; each contributes an absolute phase shift of less than  $90^\circ$ ; the open-loop absolute phase shift is therefore less than  $180^\circ$ .



$H$  is incrementally inside the sector  $\{\alpha, \beta\}$  if  $\langle (Hx - Hy)_t, -\alpha(x - y)_t, (Hx - Hy)_t - \beta(x - y)_t \rangle \leq 0$ ;

$H$  is incrementally positive if

$$\langle (x - y)_t, (Hx - Hy)_t \rangle \geq 0$$

(where the inequalities of Group I hold for all  $x$  and  $y$  in  $Do(H)$  and all  $t \geq 0$ ).

The definitions of an operator that is "exterior" conic or "outside" a sector, are identical to the preceding ones except for a reversal of the inequality sign, and will therefore be omitted.

REMARK: If  $H$  is incrementally inside  $\{\alpha, \beta\}$  then  $H$  is inside  $\{\alpha, \beta\}$ . Similarly, each inequality in Group I implies a corresponding inequality in Section 3.

GROUP II. "Instantaneous" Conditions

$H$  is instantaneously inside the sector  $\{\alpha, \beta\}$  if

$$\alpha \leq Hx(t)/x(t) \leq \beta \quad (x \in L_{2e}; t \geq 0; x(t) \neq 0)$$

$H$  is instantaneously positive if

$$x(t) \cdot Hx(t) \geq 0 \quad (x \in L_{2e}; t \geq 0)$$

GROUP III. "Instantaneous Incremental" Conditions

$H$  is instantaneously incrementally inside the sector  $\{\alpha, \beta\}$  if

$$\alpha \leq \frac{Hx(t) - Hy(t)}{x(t) - y(t)} \leq \beta \quad (x \in L_{2e}; t \geq 0; x(t) - y(t) \neq 0)$$

$H$  is instantaneously incrementally positive if

$$[Hx(t) - Hy(t)] \cdot [x(t) - y(t)] \geq 0 \quad (x \in L_{2e}; t \geq 0)$$

REMARK: If  $H$  is instantaneously inside  $\{\alpha, \beta\}$ , then  $H$  is inside  $\{\alpha, \beta\}$ . Similarly, each inequality in Group II implies a corresponding inequality in Section 3. Also each inequality in Group III implies a corresponding inequality in Group II.

APPENDIX B

LEMMA 1

The proof of Lemma 1(b) will be based on the Principle of the Argument, a theorem of Paley and Wiener, and Parseval's theorem. The proofs of Lemmas 1(a) and 1(c), being straightforward applications of Parseval's theorem, will be omitted.

Some preliminary lemmas and properties will now be introduced.

DEFINITION: If  $x$  is a function in  $L_2$  then its L.I.M. TRANSFORM is

$$X(s) = \text{l.i.m.}_{T \rightarrow \infty} \int_0^T x(t) \exp(-st) dt. \quad (\sigma \geq 0) \quad (B1)$$

The l.i.m. transform of  $X(s)$  is

$$x_1(\tau) = \frac{1}{2\pi} \text{l.i.m.}_{W \rightarrow \infty} \int_{-W}^W X(j\omega) \exp(j\omega\tau) d\omega. \quad (\tau \text{ real}) \quad (B2)$$

The limits in the mean in (B1)–(B2) exist, and  $x_1(\tau) = x(\tau)$ .<sup>12</sup>

B.1. Properties of Transforms

In the following properties,  $H$  is an operator in  $\mathfrak{L}$  having a Laplace transform  $H(s)$ .

(A) The integral defining  $H(s)$  [see (6)] converges and is bounded for  $\sigma > \sigma_0$ , since

$$\int_0^\infty h(t) \exp(-st) dt \leq \int_0^\infty |h(t)| \exp(-\sigma_0 t) dt = \text{const.}$$

(B)  $H(s)$  is analytic for  $\sigma > \sigma_0$ .<sup>13</sup>

(C) For  $\sigma \geq 0$ ,  $\lim_{|s| \rightarrow \infty} H(s) = h_\infty$ .<sup>14</sup>

(D) If  $x$  is in  $L_2$  then  $Hx$  is in  $L_2$ .<sup>15</sup>

(E) If the l.i.m. transforms of  $x \in L_2$  and  $Hx \in L_2$  are  $X(s)$  and  $Y(s)$  respectively, then<sup>15</sup>

$$Y(s) = H(s)X(s) \quad (B3)$$

One consequence of Property D is that every mapping of the type defined by the right-hand side of (2) is an operator<sup>16</sup> on  $L_{2e}$ , and belongs to  $\mathfrak{R}_0$ .

B.2. Some Consequences of Nonencirclement

A contour in the complex plane will be said to have Property N if it does not pass through or encircle the origin.

LEMMA 4. If the Nyquist diagram of  $H(s)$  has Property N then: (a)  $1/H(s)$  is analytic for  $\sigma \geq 0$ . (b) If the inequality  $1/|H(\sigma + j\omega)| \leq M$  holds for  $\sigma = 0$  then it holds for all  $\sigma \geq 0$ .

PROOF: (a) Since  $H(s)$  is analytic for  $\sigma \geq 0$ , it is enough to show that  $H(s) \neq 0$  for  $\sigma \geq 0$  to prove (a). For this purpose several contours are defined: Let  $\Gamma_\omega$  denote the  $j\omega$ -axis (as shown in Fig. 11); for  $R \geq 0$  a constant, let  $\Gamma_R$  denote the clockwise contour bounding the semi-circular region  $|\sigma + j\omega| \leq R$  where  $\sigma \geq 0$ ; let  $\Gamma_\omega - \Gamma_R$  denote the difference contour; and let  $H(\Gamma_\omega)$  and  $H(\Gamma_\omega - \Gamma_R)$  denote images of the respective contours, each augmented with the point  $h_\infty$ .

It will be shown that  $H(\Gamma_R)$  has Property N for  $R \geq R_0$ ; since by hypothesis  $H(s)$  is analytic for  $\sigma \geq 0$ , and has no zeros on the  $j\omega$ -axis, Lemma 4(a) follows by the Principle of the Argument.

<sup>12</sup> See Widder [8], ch. II, Theorem 10.

<sup>13</sup> *Ibid.*, Theorem 5a.

<sup>14</sup> The special case  $s = j\omega$  is implied by the Riemann-Lebesgue theorem (Titchmarsh [9a], Theorem 1). The general case follows from the special case and from Properties A and B by a theorem of Phragmén-Lindelöf (Titchmarsh [9b], sec. 5.64).

<sup>15</sup> This follows from Theorem 65 of Titchmarsh [9a], which implies that the convolution of an  $L_2$  function with an  $L_1$  function is in  $L_2$ , and has a transform of the type (B3).

<sup>16</sup> Suppose that  $x$  is in  $L_2$ ;  $x_t$  is certainly in  $L_2$ , and  $H(x_t)$  is in  $L_2$  by Property D; since  $[H(x_t)]_t = [Hx]_t$ , it follows that  $[Hx]_t$  is in  $L_2$ ; i.e.,  $Hx$  is in  $L_{2e}$ . Thus  $H$  maps  $L_{2e}$  into  $L_{2e}$ ; since  $H$  also maps  $o$  into  $o$ , it follows that  $H$  is an operator on  $L_{2e}$  and in  $\mathfrak{R}_0$ .

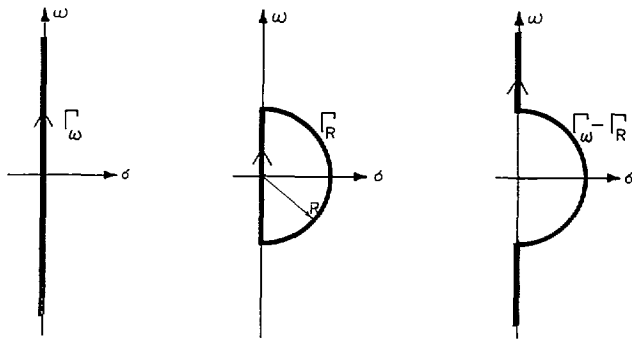


Fig. 11. Contours.

To prove that  $H(\Gamma_R)$  has Property N observe that  $H(\Gamma_R) = H(\Gamma_\omega) + H(\Gamma_R - \Gamma_\omega)$ ; since  $H(\Gamma_\omega)$  has Property N by hypothesis, it is enough now to show that  $H(\Gamma_R - \Gamma_\omega)$  has Property N. This can be accomplished by showing that  $H(\Gamma_R - \Gamma_\omega)$  lies in a circle centered at  $h_\infty$  and not including the origin. The last assertion is a consequence of two facts.

- (i) There is an  $R_0 > 0$  for which, for  $R \geq R_0$  and  $s$  in  $(\Gamma_R - \Gamma_\omega)$ ,  $|H(s) - h_\infty| \leq \frac{1}{2}|h_\infty|$ .
- (ii)  $h_\infty \neq 0$ .

(i) is obtained from Property C of Appendix B.1 for  $|s| \geq R_0$ , and therefore certainly holds for  $s$  in  $(\Gamma_R - \Gamma_\omega)$ . (ii) holds since

$$h_\infty = \lim_{\omega \rightarrow \infty} H(j\omega),$$

and since  $H(j\omega) \neq 0$  by Property N.

(b) This is a special case of the Maximum Modulus Theorem of Phragmén-Lindelöf.<sup>17</sup> The theorem implies that a function analytic in a half-plane, and bounded on the boundary, is bounded throughout the half-plane.

### B.3. A Paley-Wiener Lemma

A complex-valued function  $W(s)$  will be said to satisfy the Paley-Wiener conditions if

$$(PW1) \quad W(s) \text{ is analytic for } \sigma > 0, \text{ and}$$

$$(PW2) \quad \int_{-\infty}^{\infty} |W(\sigma + j\omega)|^2 d\omega \leq \text{const.} \quad (\sigma > 0).$$

The following lemma is a modification of Theorem 5 of Paley-Wiener [10], and is stated without proof.

LEMMA 5. (a) If  $w$  is in  $L_2$ ,  $w_t = 0$ , and  $W(s)$  is the l.i.m. transform of  $w(\tau)$ , then  $W(s) \exp(st)$  satisfies the Paley-Wiener conditions. (b) Conversely, if  $t \geq 0$  and  $W(s) \exp(st)$  satisfies the Paley-Wiener conditions, then there is a function  $w$  in  $L_2$  having the properties that  $w_t = 0$  and that  $W(s)$  is the l.i.m. transform of  $w$ .

### B.4. Proof of Lemma 1(b)

Let  $x$  in  $L_{2e}$  and  $t \geq 0$  be given. Since  $H$  is linear, it is enough to show that  $H$  is conic with parameters  $c$  and  $r$ ;

<sup>17</sup> Titchmarsh [9b], sec. 5.61.

that is,

$$\| (Hx - cx)_t \| \geq r \| x_t \|. \quad (B4)$$

For this purpose, let  $y \triangleq (Hx - cx)$  and  $\delta \triangleq H(x_t) - [Hx]_t$ . Hence

$$y_t = (Hx - cx)_t = H(x_t) - cx_t - \delta. \quad (B5)$$

Now  $x_t$  is in  $L_2$ . Hence  $H(x_t)$  is in  $L_2$  by Property D of Appendix B.1. Since  $[H(x_t)]_t = (Hx)_t$ , it follows that  $(Hx)_t$  is in  $L_2$ . Thus  $\delta$  is in  $L_2$ , all terms in (B5) are in  $L_2$ , and by Property E,

$$Y_t(s) = H(s)X_t(s) - cX_t(s) - \Delta(s) \quad (B6)$$

where  $Y_t(s)$ ,  $X_t(s)$ , and  $\Delta(s)$  are the l.i.m. transforms of  $y_t$ ,  $x_t$ , and  $\delta$ . Hence

$$\{X_t(s)\} - \left\{ \frac{\Delta(s)}{H(s) - c} \right\} = \left\{ \frac{Y_t(s)}{H(s) - c} \right\}. \quad (B7)$$

Now the braced terms in (B7) are l.i.m. transforms of functions in  $L_2$ ; for  $X_t(s)$  this is true by definition; for the remaining terms, this can be proved by the reasoning given below in Assertion 1. Suppose that  $\Delta(s)/H(s) - c$  is the l.i.m. transform of a function  $q(\tau)$ ; it follows that

$$\begin{aligned} \|x_t - q\|^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| x_t(j\omega) - \frac{\Delta(j\omega)}{H(j\omega) - c} \right|^2 d\omega \\ &\quad \text{(Parseval's Theorem)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{Y_t(j\omega)}{H(j\omega) - c} \right|^2 d\omega \quad [\text{by (B7)}] \\ &\leq \frac{1}{2\pi r^2} \int_{-\infty}^{\infty} |Y_t(j\omega)|^2 d\omega \\ &\quad \text{[by inequality (8) of Section 3.2]} \\ &= \frac{1}{r} \|y_t\|^2 \quad \text{(Parseval's Theorem)} \quad (B8) \end{aligned}$$

It will be shown, in Assertion 1, that  $q(\tau) = 0$  for almost all  $\tau < t$ . Therefore

$$\|x_t - q\|^2 = \|x_t\|^2 + \|q\|^2 \geq \|x_t\|^2. \quad (B9)$$

(B8) and (B9) imply (B4).

ASSERTION 1. The expression  $\Delta(s)/[H(s) - c]$  is the l.i.m. transform of a function  $q$  in  $L_2$ ; furthermore  $q(\tau) = 0$  for almost all  $\tau < t$ .

PROOF OF ASSERTION 1. By Lemma 5b, it is enough to show that  $[\Delta(s) \exp(st)]/[H(s) - c]$  satisfies the Paley-Wiener conditions.

To prove (PW1), observe that the following three terms are analytic for  $\sigma > 0$ :  $\Delta(s)$ , because it is the l.i.m. transform of an  $L_2$  function;  $\exp(st)$ , because it is analytic throughout the plane;  $[H(s) - c]^{-1}$ , by Lemma 4 and the hypothesis on the Nyquist diagram. The product of these terms must therefore also be analytic for  $\sigma > 0$ .

To prove (PW2), observe that

$$\int_{-\infty}^{\infty} \left| \frac{\Delta(s) \exp(st)}{H(s) - c} \right|^2 d\omega \leq \frac{1}{r^2} \int_{-\infty}^{\infty} |\Delta(s) \exp(st)| d\omega$$

$$\leq \text{const. } (\sigma > 0) \text{ [by inequality (8)]}$$

(Since  $\delta \in L_2$  and  $\delta_i = 0$  by construction, the last inequality is implied by Lemma 5(a).) Q.E.D.

APPENDIX C

LEMMA 2

A preliminary assertion will be proved first.

ASSERTION 2. If  $N$  in  $\mathfrak{X}$  is positive,  $K$  is in  $\mathfrak{L}$ , and  $y = Kx$ , then

$$\int_0^t \frac{dy}{d\tau} \cdot N(y(\tau)) d\tau \geq 0. \tag{C1}$$

PROOF OF ASSERTION 2.

$$\int_0^t \frac{dy}{d\tau} \cdot N(y(\tau)) d\tau = \int_{y(0)}^{y(t)} N(y') dy'. \tag{C2}$$

Since  $K$  is in  $\mathfrak{L}$ ,  $y(t)$  is given by a convolution integral, whose kernel is fixed for a fixed  $x$ , and whose limits of integration are 0 and  $t$ ; therefore  $y(0) = 0$ . Furthermore, since  $N$  is positive, its graph lies in the first and third quadrants. It follows that the right-hand side of (C2) is non-negative. Q.E.D.

C.1. Proof of Lemma 2

Part a) It is required to show that, for any given  $x$  in  $L_{2e}$  and any given  $t \geq 0$ , the inequality

$$\int_0^t x(\tau) \cdot [NKx(\tau)] d\tau \geq 0 \tag{C3}$$

holds. For this purpose, make the following substitutions: Write  $y \triangleq Kx$ , and observe that, since  $K(s) = k\lambda/(s + \lambda)$ ,  $y$  is differentiable and

$$x(\tau) = \frac{1}{k\lambda} \frac{dy}{d\tau} + \frac{1}{k} y(\tau). \tag{C4}$$

(C3) is therefore equivalent to

$$\int_0^t \left\{ \frac{1}{k\lambda} \frac{dy}{d\tau} + \frac{1}{k} y(\tau) \right\} \cdot N(y(\tau)) d\tau \geq 0$$

or

$$\frac{1}{k\lambda} \int_0^t \frac{dy}{d\tau} \cdot N(y(\tau)) d\tau + \frac{1}{k} \int_0^t y(\tau) \cdot N(y(\tau)) d\tau \geq 0. \tag{C5}$$

Now  $k$  and  $\lambda$  are positive by hypothesis; the first integral in (C5) is non-negative by Assertion 2; the second integral is non-negative, since  $N$  is a positive operator; therefore (C5) is true. Q.E.D.

Part b) It will be assumed, for simplicity, that  $\beta > 0$ .

CASE A. Suppose  $\alpha \geq 0$ . It must be shown that  $NK$  is inside  $\{\alpha, \beta\} \times \{0, k\}$ . This is equivalent to saying that  $NK$  is inside  $\{0, k\beta\}$ , or that

$$\langle (NKx)_t, (NKx - k\beta x)_t \rangle \leq 0. \tag{C6}$$

(C6) is equivalent to the inequality

$$k\beta \langle x_t, (Ny)_t \rangle \geq \|(Ny)_t\|^2 \tag{C7}$$

where  $y = Kx$ . Now recalling that  $\langle x_t, (Ny)_t \rangle$  equals the left-hand side of (C5), we get

$$k\beta \langle x_t, (Ny)_t \rangle \geq \beta \int_0^t y(\tau) \cdot N(y(\tau)) d\tau.$$

Observing that, for  $\alpha \geq 0$ ,  $N$  satisfies the inequality  $\beta y(\tau) \cdot N(y(\tau)) \geq [N(y(\tau))]^2$ , we get

$$k\beta \langle x_t, (Ny)_t \rangle \geq \int_0^t [N(y(\tau))]^2 d\tau,$$

which implies (C7). Q.E.D.

CASE B. Suppose  $\alpha < 0$ . Decompose  $N(x)$  into two parts,  $N(x) = N_+(x) + N_-(x)$ ; let  $N_+(x) = N(x)$  for  $N(x) \geq 0$  and  $N_+(x) = 0$  elsewhere, and let  $N_-(x)$  be similarly defined.

Since  $N_+$  is clearly inside  $\{0, \beta\}$ , Case A implies that  $N_+K$  is inside  $\{0, k\beta\}$ . Similarly  $N_-K$  is inside  $\{k\alpha, 0\}$ . On summing the sectors of  $N_+K$  and  $N_-K$  (by the Sum Rule of Part I) it is found that  $NK$  is inside  $\{k\alpha, k\beta\}$ ; that is, inside  $\{\alpha, \beta\} \times \{0, k\}$ . Q.E.D.

APPENDIX D

LEMMA 3

Before proving Lemma 3, a few related assertions will be introduced.

ASSERTION 3. Let  $K$  be an operator in  $\mathfrak{RC}$ ,  $x$  a fixed element of  $L_{2e}$ , and  $y \triangleq Kx$ . Then  $x$  has a "Foster expansion" in  $y$ ;<sup>18</sup> that is,  $x$  can be expressed as a finite sum,

$$x = \sum_{i=0}^m F_i y,$$

in which  $F_i$  are operators mapping the image under  $K$  of  $L_{2e}$  into  $R[0, \infty)$ , as follows:

CASE 1.  $F_0 y = K^{-1}(0) \cdot y$ .

CASE 2. If  $i = 1, 2, \dots, (m-1)$ , then  $F_i$  is in  $\mathfrak{L}$  and has a Laplace Transform,

$$F_i(s) = h_i s / (s + \theta_i), \quad (h_i > 0, \theta_i > 0).$$

CASE 3.  $F_m y = h_m \dot{y}$  if  $K(\infty) = 0$  and  $F_m y = 0$  otherwise, where  $h_m \geq 0$ .

ASSERTION 4. If  $N$  is incrementally positive, and  $\langle x_t, [Ny]_t \rangle \geq 0$  then  $\langle x_t, [N(x+y)]_t \rangle \geq 0$ .

PROOF OF ASSERTION 4. It is enough to show that

$$\langle x_t, [N(x+y)]_t \rangle - \langle x_t, [Ny]_t \rangle \geq 0 \tag{D1}$$

But the left-hand side of (D1) can be expressed as

$$\langle (x+y)_t - y_t, [N(x+y)]_t - (Ny)_t \rangle$$

which has the form  $\langle x_{1t} - x_{2t}, Nx_{1t} - Nx_{2t} \rangle$ , and is non-negative, since  $N$  is an incrementally positive operator. Therefore (D1) holds. Q.E.D.

<sup>18</sup> See Guillemin [2], p. 115.

ASSERTION 5. If  $N$  is an operator in  $\mathfrak{A}$ , incrementally inside a sector  $\{\alpha, \beta\}$  where  $\alpha < 0$  and  $\beta > 0$ , then  $N$  can be decomposed into  $N = N_+ + N_-$ , where  $N_-$  is inside  $\{\alpha, 0\}$ , and  $N_+$  is inside  $\{0, \beta\}$ .

PROOF OF ASSERTION 5. Since  $N$  is incrementally inside a sector, its graph  $N$  is continuous and has bounded variation on every finite interval. Consequently  $N$  can be expressed as an integral,  $N(x) = \int_0^x n(x') dx'$ . Let  $n_+(x) = n(x)$  if  $n(x) \geq 0$ , and  $n_+(x) = 0$  if  $n(x) < 0$ ; let  $N_+(x) = \int_0^x n_+(x') dx'$ . Clearly  $N_+$  has the desired property.  $N_-$  is constructed similarly. Q.E.D.

#### D.1. Proof of Lemma 3

Part a) Let  $x$  be any given element of  $L_2$ , and  $t$  any given point in  $[0, \infty)$ ; it is required to show that

$$\langle x_t, (NKx)_t \rangle \geq 0. \quad (D2)$$

Letting  $y \triangleq Kx$ , and recalling that  $x$  can be expressed by the Foster expansion

$$x = \sum_{i=0}^m F_i y$$

(see Assertion 3), (D2) is equivalent to

$$\sum_{i=0}^m \langle (F_i y)_t, (Ny)_t \rangle \geq 0. \quad (D3)$$

It will be shown that each component on the left-hand side of (D3) is non-negative.

CASE 1. Here  $F_0 y = K^{-1}(0) \cdot y$ . Hence  $\langle (F_0 y)_t, (Ny)_t \rangle = [K^{-1}(0)] \cdot \langle y_t, (Ny)_t \rangle$ ; this is non-negative since  $N$  is a positive operator, and since  $K(0)$  is necessarily positive.

CASE 2. Here  $F_i(s) = h_i s / (s + \theta_i)$ . Let

$$z(t) = h_i^{-1} \int_0^t F_i y(\tau) d\tau.$$

It follows that  $y = \dot{z} + \theta_i z$  almost everywhere, and that  $F_i y = h_i \dot{z}$  almost everywhere. Hence

$$\langle (F_i y)_t, (Ny)_t \rangle = h_i \langle \dot{z}_t, [N(\dot{z} + \theta_i z)]_t \rangle. \quad (D4)$$

Now, observing that  $\theta_i > 0$ , Assertion 2 implies that  $\langle \dot{z}, [N(\theta_i z)]_t \rangle$  is non-negative. Observing that  $h_i$  is positive, the right-hand side of (D4) is non-negative by Assertion 4. Thus Case 2 is proved.

CASE 3. Here  $F_m y = h_m \dot{y}$  if  $K(\infty) = 0$ . Hence  $\langle (F_m y)_t, (Ny)_t \rangle = h_m \langle \dot{y}_t, (Ny)_t \rangle$ . Case 3 follows by Assertion 2.

Since the inner product is non-negative in all three cases, (D3) holds. Q.E.D.

Part b) Assume, for simplicity, that  $\beta > 0$ .

CASE A. If  $\alpha = 0$  then, by reasoning similar to that used in Lemma 2(b), (C5)–(C7), the following inequality is obtained:

$$\langle x_t, (NKx)_t \rangle \geq [\beta K(0)]^{-1} \| (Ny)_t \|^2.$$

Hence  $NK$  is inside  $\{0, \beta K(0)\}$ , which equals  $\{0, \beta\} \times \{0, K(0)\}$ .

CASE B. If  $\alpha > 0$  then  $NK$  is decomposed into three parts

$$NK = \{[N - \alpha I]K\} + \{\alpha[K - K(\infty) \cdot I]\} + \{\alpha K(\infty) \cdot I\}. \quad (D5)$$

Now the three parts lie in the sectors  $\{0, (\beta - \alpha)K(0)\}$ ,  $\{0, \alpha[K(0) - K(\infty)]\}$ , and  $\{\alpha K(\infty), \alpha K(\infty)\}$ , respectively. (The first two of these sectors are determined by the rule formed in Case A, after observing that  $[N - \alpha I]$  is inside  $\{0, \beta - \alpha\}$ , and that  $[K - K(\infty) \cdot I]$  is inside  $\{0, [K(0) - K(\infty)]\}$ ; the third sector is simply the sector of a constant times the identity.) On summing the three sectors (by the Sum Rule of Part I), it is found that  $NK$  is inside  $\{\alpha K(\infty), \beta K(0)\}$ ; that is, inside  $\{\alpha, \beta\} \times \{K(\infty), K(0)\}$ .

CASE C. If  $\alpha < 0$ ,  $N$  is decomposed into  $N_+$  and  $N_-$ , as in Assertion 5. This case then follows by the reasoning used in Case B of Lemma 2. Q.E.D.

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