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Research Article

Additional Recursion Relations, Factoriz Diophantine Properties Associated with Polynomials of the Askey Scheme

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Abstract

In this paper, we apply to (almost) all the “named” polynomials three-term recursion relations, the machinery developed in previou at least one additional recursion relation involving a shift in some these polynomials characterized by special values of their paramet of their zeros—generally given by simple expressions in terms of findings generally are applicable for values of the Askey polynomia orthogonality relations hold. Most of these results are not (yet) rep

1. Introduction

Recently *Diophantine* findings and conjectures concerning the correspondingly the zeros of the polynomials associated with their the behavior of certain *isochronous* many-body problems of Toda (for a review of these and other analogous results, see [3, Appe

theoretical framework was then developed [4 - 6], involving poly
hence being, at least for appropriate ranges of the parameters
referred to as "Favard theorem," on the basis of [7]; however
"spectral theorem for orthogonal polynomials" [8]). Specific
coefficients, featuring a parameter ν, of these recursion relation
polynomials also satisfy a *second* three-term recursion relation
second recursion relation, *Diophantine* results of the kind indicate
make this paper essentially self-contained, these developments a
with the corresponding proofs relegated to an appendix to avoid
apply, in Section 3, this theoretical machinery to the "named" p
the basic three-term recursion relation they satisfy: this entails th
be done in more than one way, especially for the named polyno
identification of additional recursion relations satisfied by (most
(especially *after* they have been discovered) could also be obtain
relations of these polynomials with hypergeometric functions: we
cases) in the standard compilations [9 - 13], where they in our op
our machinery yields factorizations of certain of these polynomial
zeros, as well as factorizations relating some of these polynomial
most of these results seem *new* and deserving to be eventually re
generally require that the parameters of the named polynomials
the orthogonality property. To clarify this restriction let us remark
which might be considered the *prototype* of formulas reported
scheme—reads as follows:

$$L_n^{(-n)}(x) = \frac{(-x)^n}{n!}, \quad n = ($$

where $L_n^{(\alpha)}(x)$ is the standard (generalized) Laguerre polynomial of

$$\int_0^\infty dx\, x^\alpha \exp(-x) L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) =$$

it is, however, generally required that $\mathrm{Re}\,\alpha > -1$. This formula, (1.1
of the standard compilations reporting results for classical orthogo
book by Magnus and Oberhettinger [14] or [11, Equation 8.973.4
neat generalization of this formula, reading

$$L_n^{(-m)}(x) = (-1)^m \frac{(n-m)!}{n!} x^m L_{n-m}^{(m)}(x), \quad n$$

which qualifies as well as the *prototype* of formulas reported below
(Note, incidentally, that this formula can be inserted without d
generalized Laguerre polynomials, (1.1b), reproducing the stand
gets indeed neatly compensated by the term x^m appearing in
property—and the analogous version for Jacobi polynomials—
polynomials; e.g., a referee of this paper wrote "Although I hav
written it down nor saw it stated explicitly. It is clear from readi
(1.1c) and the more general case of Jacobi polynomials.") Most c
the named polynomials of the Askey scheme that are reported bel
they do not appear in the standard compilations where we sugg
their neatness and their *Diophantine* character. They could of cou
we followed to identify and prove them (it is indeed generally the
they have been discovered, are easily proven via several differen
the results reported below have been obtained by a rather s
polynomials of the Askey scheme, we do not claim that the results
these polynomials. And let us also note that, as it is generally
polynomials [9 - 13], we have treated *separately* each of the differ

though “in principle” it would be sufficient to only treat the mo: encompasses all the other classes via appropriate assignments features. Section 4 mentions tersely possible future developments.

2. Preliminaries and Notation

In this section we report tersely the key points of our approach, m indicated above—and also to establish its notation: previously k proofs, except for an extension of these findings whose proof is rel

Hereafter we consider classes of monic polynomials $p_n^{(\nu)}(x)$, of c parameter ν, defined by the three-term recursion relation:

$$p_{n+1}^{(\nu)}(x) = (x + a_n^{(\nu)})p_n^{(\nu)}(x) -$$

with the “initial” assignments

$$p_{-1}^{(\nu)}(x) = 0, \qquad p_0^{(\nu)}($$

clearly entailing

$$p_1^{(\nu)}(x) = x + a_0^{(\nu)}, \qquad p_2^{(\nu)}(x) = (x +$$

and so on. (In some cases the left-hand side of the first (2.1b) m account of possible indeterminacies of $b_0^{(\nu)}$.)

Notation. Here and hereafter the index n is a *nonnegative integ* make little sense for $n = 0$, requiring a—generally quite obvious—sp this index n and of the parameter ν. They might—indeed they oft (see below); but this parameter ν plays a crucial role, indee identification of special values of it (generally simply related to the

Let us recall that the theorem which guarantees that these polyr relation (2.1), are orthogonal (with a positive definite, albeit a p coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ be *real* and that the latter be *negative*, b_n

2.1. Additional Recursion Relation

Proposition 2.1. *If the quantities $A_n^{(\nu)}$ and $\omega^{(\nu)}$ satisfy the nonlinea*

$$[A_{n-1}^{(\nu)} - A_{n-1}^{(\nu-1)}][A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \omega^{(\nu)}] = [A_{n-1}^{(\nu-1)} -$$

with the boundary condition

$$A_0^{(\nu)} = 0$$

(where, without significant loss of generality, this constant is set value A : see [5, Equation (4a)]; and we also replaced, for notatic [5] with $\omega^{(\nu)}$), and if the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are defined in t

$$a_n^{(\nu)} = A_{n+1}^{(\nu)} - A_n^{(\nu}$$

$$b_n^{(\nu)} = [A_n^{(\nu)} - A_n^{(\nu-1)}][A_n^{(\nu)} - A$$

then the polynomials $p_n^{(\nu)}(x)$ identified by the recursion relation (2.
(involving a shift both in the order n of the polynomials and in the

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)}$$

with

$$g_n^{(\nu)} = A_n^{(\nu)} - A_n^{(\nu-}$$

This proposition corresponds to [5, Proposition 2.3]. (As suggested a parameter—albeit of a very special type and different from that paper by Dickinson et al. [17].)

Alternative conditions sufficient for the validity of Proposition 2.1 a and $g_n^{(\nu)}$ read as follows (see [5, Appendix B]):

$$a_n^{(\nu)} - a_n^{(\nu-1)} = g_{n+1}^{(\nu)}$$

$$b_{n-1}^{(\nu-1)} g_n^{(\nu)} - b_n^{(\nu)} g_{n-}^{(\nu}$$

with

$$g_n^{(\nu)} = -\frac{b_n^{(\nu)} - b_n^{(\nu}}{a_n^{(\nu)} - a_{n-}^{(\nu}}$$

and the “initial” condition

$$g_1^{(\nu)} = a_0^{(\nu)} - a_0^{(\nu-}$$

entailing via (2.5c) (with $n = 1$)

$$b_1^{(\nu)} - b_1^{(\nu-1)} + (a_0^{(\nu)} - a_0^{(\nu-1)})($$

and via (2.5a) (with $n = 0$)

$$g_0^{(\nu)} = 0.$$

Proposition 2.2. *Assume that the class of (monic, orthogonal) p satisfies Proposition 2.1, hence that they also obey the (“second the relations:*

$$p_n^{(\nu)}(x) = [x - x_n^{(1,\nu)}] p_{n-1}^{(\nu-1)}(x) +$$

$$x_n^{(1,\nu)} = -[a_{n-1}^{(\nu-1)} + g$$

in addition to

$$p_n^{(\nu)}(x) = [x - x_n^{(2,\nu)}] p_{n-1}^{(\nu-2)}(x)$$

$$x_n^{(2,\nu)} = -[a_{n-1}^{(\nu-2)} + g_n^{(\nu)} +$$

$$c_n^{(\nu)} = b_{n-1}^{(\nu-2)} + g_n^{(\nu)} g_{...}$$

as well as

$$p_n^{(\nu)}(x) = [x - x_n^{(3,\nu)}] p_{n-1}^{(\nu-3)}(x) + d_n^{(\nu)} p_{...}$$

$$x_n^{(3,\nu)} = -[a_{n-1}^{(\nu-3)} + g_n^{(\nu)} + g_n^{(\nu ...}$$

$$d_n^{(\nu)} = b_{n-1}^{(\nu-3)} + g_n^{(\nu)} g_{n-1}^{(\nu-2)} + g_n^{(\nu-1)} g ...$$

$$e_n^{(\nu)} = g_n^{(\nu)} g_{n-1}^{(\nu-1)} g_{...}$$

These findings correspond to [6, Proposition 1].

2.2. Factorizations

In the following we introduce a second parameter μ, but for notat dependence of the various quantities on this parameter.

Proposition 2.3. *If the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ a coefficients $b_n^{(\nu)}$ satisfy the relation*

$$b_n^{(n+\mu)} = 0,$$

entailing that for $\nu = n + \mu$, the recursion relation (2.1a) reads

$$p_{n+1}^{(n+\mu)}(x) = (x + a_n^{(n+\mu)}) p ...$$

then there holds the factorization

$$p_n^{(m+\mu)}(x) = \tilde{p}_{n-m}^{(-m)}(x) p_m^{(m+\mu)}(x),$$

with the "complementary" polynomials $\tilde{p}_n^{(-m)}(x)$ (of course of recursion relation analogous (but not identical) to (2.1):

$$\tilde{p}_{n+1}^{(-m)}(x) = (x + a_{n+m}^{(m+\mu)}) \tilde{p}_n^{(-m)}(x) + ...$$

$$\tilde{p}_{-1}^{(-m)}(x) = 0, \qquad \tilde{p}_0^{(-m)} ...$$

entailing

$$\tilde{p}_1^{(-m)}(x) = x + a_m^{(m+ ...}$$

$$\tilde{p}_2^{(-m)}(x) = (x + a_{m+1}^{(m+\mu)})(x + a_m^{(m+\mu)}) + b_m^{(...}$$

with

$$x_m^{(\pm)} = \frac{1}{2}\left\{-a_m^{(m+\mu)} - a_{m+1}^{(m+\mu)} \pm [(a_m^{(m+\mu)} - ...\right.$$

and so on.

This is a slight generalization (proven below, in Appendix A) of [

complementary polynomials $\tilde{p}_n^{(-m)}(x)$, being defined by three-terr orthogonal families, hence they should have to be eventually inves them the kind of findings reported in this paper.

The following two results are immediate consequences of Propositi

Corollary 2.4. *If (2.9) holds—entailing (2.10) and (2.11) with (2.12*

$$p_n^{(n-1+\mu)}(-a_{n-1}^{(n-1+\mu)}$$

and the polynomial $p_n^{(n-2+\mu)}(x)$ has the two zeros $x_{n-2}^{(\pm)}$ (see (2.12

$$p_n^{(n-2+\mu)}(x_{n-2}^{(\pm)}) =$$

The first of these results is a trivial consequence of (2.10); the moreover, that from the factorization formula (2.11), one can lik zeros of $p_n^{(n-4+\mu)}(x)$, by evaluation from (2.12) $\tilde{p}_3^{(-m)}(x)$ and $\tilde{}$ solvability of algebraic equations of degrees 3 and 4.

These findings often have a *Diophantine* connotation, due to the n terms of *integers*.

Corollary 2.5. *If (2.9) holds—entailing (2.10) and (2.11) with (2 satisfy the properties*

$$a_{n-m}^{(-m+\mu)}(\underline{\rho}) = a_n^{(m+\bar{\mu})}(\underline{\tilde{\rho}}), \qquad b_{n-m}^{(-m-}$$

then clearly

$$\tilde{p}_n^{(m)}(x;\underline{\rho}) = p_n^{(m+\bar{\mu})}($$

entailing that the factorization (2.11) takes the neat form

$$p_n^{(m+\mu)}(x;\underline{\rho}) = p_{n-m}^{(-m+\bar{\mu})}(x;\underline{\tilde{\rho}})\,p_m^{(m+\mu)}($$

Note that—for future convenience, see below—one has emphasi depend on additional parameters (indicated with the vector variab of course be independent of n, but they might depend on m).

The following remark is relevant when both Propositions 2.1 and 2.

Remark 2.6. As implied by (2.3b), the condition (2.9) can be enfor

$$\omega^{(\nu)} = A_{\nu-1}^{(\nu-1+\mu)} - A_{\nu}$$

entailing that the nonlinear recursion relation (2.3a) reads

$$[A_{n-1}^{(\nu)} - A_{n-1}^{(\nu-1)}][A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + A_{\nu-1}^{(\nu-1}$$
$$= [A_{n-1}^{(\nu-1)} - A_{n-1}^{(\nu-2)}][A_{n-1}^{(\nu-1)} - A_{n-2}^{(\nu-}$$

Corollaries 2.4 and 2.5 and Remark 2.6 are analogous to [5, Coroll

2.3. Complete Factorizations and Diophantine Findings

The *Diophantine* character of the findings reported below is due
zeros in terms of *integers* (see in particular the examples in Sectio

Proposition 2.7. *If the (monic, orthogonal) polynomials* $p_n^{(\nu)}(x)$ *are*
with coefficients $a_n^{(\nu)}$ *and* $b_n^{(\nu)}$ *satisfying the requirements sufficie*
(namely (2.3), with (2.2) and (2.9), or just with (2.18)), then

$$p_n^{(n+\mu)}(x) = \prod_{m=1}^{n} [x - x_m^{(1}$$

with the expressions (2.6b) of the zeros $x_m^{(1,\nu)}$ *and the standard c*
when its lower limit exceeds its upper limit. Note that these n *zer*
particular,

$$p_0^{(\mu)}(x) = 1, \qquad p_1^{(1+\mu)}(x) = x - x_1^{(1,1+\mu)}, \qquad p_2^{(2+\mu}$$

and so on.

These findings correspond to [6, Proposition 2.2 (first part)].

The following results are immediate consequences of Proposition 2

Corollary 2.8. *If Proposition 2.7 holds, then also the polynom*
$p_n^{(n+\mu)}(x)$ *, see (2.19)) can be written in the following completely f*

$$p_n^{(n-1+\mu)}(x) = [x + a_{n-1}^{(n-1)}] \prod_{m=1}^{n-1}$$

$$p_n^{(n-2+\mu)}(x) = [x - x_m^{(+)}][x - x_m^{(-)}]$$

Analogously, complete factorizations can clearly be written for the
last part of Corollary 2.4.

And of course the factorization (2.11) together with (2.19a) en
polynomial $p_n^{(m+\mu)}(x)$ *with* $m = 1, \ldots, n$ *features the* m *zeros* $x_\ell^{(1,\ell+\mu}$

$$p_n^{(m+\mu)}(x_\ell^{(1,\ell+\mu)}) = 0, \quad \ell = 1, \ldots$$

Proposition 2.9. *Assume that, for the class of polynomials* $p_n^{(\nu)}($
moreover that, for some value of the parameter μ *(and of cou*
coefficients $c_n^{(2n+\mu)}$ *vanish (see (2.7a) and (2.7c)),*

$$c_n^{(2n+\mu)} = b_{n-1}^{(2n+\mu-2)} + g_n^{(2n+\mu)}$$

then the polynomials $p_n^{(2n+\mu)}(x)$ *factorize as follows:*

$$p_n^{(2n+\mu)}(x) = \prod_{m=1}^{n} [x - x_m^{(2$$

entailing

$$p_0^{(\mu)}(x) = 1, \quad p_1^{(2+\mu)}(x) = x - x_1^{(2,2+\mu)}, \quad p_2^{(4+\mu}$$

and so on.

Likewise, if for all nonnegative integer values of n, the following tw

$$d_n^{(3n+\mu)} = b_{n-1}^{(3n+\mu-3)} + g_n^{(3n+\mu)} g_{n-1}^{(3n+\mu-2)} + g_n^{(3n+\mu-1}$$

$$e_n^{(3n+\mu)} = 0, \quad \text{that is}, \; g_n^{(3n+\mu)} = 0 \quad \text{or} \quad g_{n-1}^{(3n+}$$

then the polynomials $p_n^{(3n+\mu)}(x)$ factorize as follows:

$$p_n^{(3n+\mu)}(x) = \prod_{m=1}^{n} [x - x_m^{(3$$

entailing

$$p_0^{(\mu)}(x) = 1, \quad p_1^{(3+\mu)}(x) = x - x_1^{(3,3+\mu)}, \quad p_2^{(6+\mu}$$

and so on.

Here of course the n (n-independent!) zeros $x_m^{(2,2m+\mu)}$, respecti
(2.8b).

These findings correspond to [6, Proposition 2].

3. Results for the Polynomials of the Askey Scheme

In this section, we apply to the polynomials of the Askey scheme [class of polynomials (including the classical polynomials) may b functions, Rodriguez-type formulas, their connections with hyperg our machinery, as outlined in the preceding section, we introduc satisfy:

$$p_{n+1}(x;\underline{\eta}) = [x + a_n(\underline{\eta})] p_n(x;\underline{\eta})$$

with the "initial" assignments

$$p_{-1}(x;\underline{\eta}) = 0, \quad p_0(x$$

clearly entailing

$$p_1(x;\underline{\eta}) = x + a_0(\underline{\eta}), \quad p_2(x;\underline{\eta}) = [x + a$$

and so on. Here the components of the vector $\underline{\eta}$ denote the *a* polynomials.

Let us emphasize that in this manner we introduced the *monic* (o below we generally also report the relation of this version to the m

To apply our machinery we must identify, among the parame parameter ν playing a special role in our approach. This can be g class of polynomials, see below). Once this identification (i.e., the relations (3.1) coincide with the relations (2.1) via the self-evident

$$p_n^{(\nu)}(x) \equiv p_n(x;\underline{\eta}(\nu)), \qquad a_n^{(\nu)} \equiv a_n(\underline{\eta}($$

Before proceeding with the report of our results, let us also emph feature symmetries regarding the dependence on their parameters of some of them—obviously *all* the properties of these polynor symmetry properties; but it would be a waste of space to report duplications are hereafter omitted (except that sometimes results trivially related via such symmetries: when this happens this fact notation of [9]—up to obvious changes made whenever necessar notation. When we obtain a result that we deem interesting but is we identify it as *new* (although given the very large literature on such a result has not been already published; indeed we will be gr is indeed the case and will let us know). And let us reiterate that such results, this investigation cannot be considered "exhaustive via assignments of the ν-dependence $\underline{\eta}(\nu)$ different from those con

3.1. Wilson

The monic Wilson polynomials (see [9], and note the notational re with $\alpha,\beta,\gamma,\delta$)

$$p_n(x;\alpha,\beta,\gamma,\delta) \equiv p_n($$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{\eta}) = \alpha^2 - \tilde{A}_n - \tilde{C}_n, \qquad b_n($$

where

$$\tilde{A}_n = \frac{(n+\alpha+\beta)(n+\alpha+\gamma)(n+\alpha+}{(2n-1+\alpha+\sigma)(2n}$$

$$\tilde{C}_n = \frac{n(n-1+\beta+\gamma)(n-1+\beta+}{(2n-2+\alpha+\sigma)(2n-}$$

$$\sigma \equiv \beta+\gamma+\delta, \qquad \rho \equiv \beta\gamma+\beta\delta+$$

The standard version of these polynomials reads (see [9]):

$$W_n(x;\alpha,\beta,\gamma,\delta) = (-1)^n(n-1+\alpha+\beta+$$

Let us also recall that these polynomials $p_n(x;\alpha,\beta,\gamma,\delta)$ are inva $\alpha,\beta,\gamma,\delta$.

As for the identification of the parameter ν (see (3.2)), two possibil

3.1.1. First Assignment

$$\alpha = -\nu.$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = [6(2n-2-\nu+\sigma)]^{-1} n\{4-5\sigma+$$
$$+[-10+9c$$
$$+(8-4\sigma+$$

$$\omega^{(\nu)} = -\nu^2,$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Wilson polynomials (3.3):

$$p_n^{(\nu)}(x) = p_n(x; -\nu, \beta,$$

Hence, with this identification, Proposition 2.1 becomes applicat Wilson polynomials satisfy the second recursion relation (2.4a) witl

$$g_n^{(\nu)} = \frac{n(n-1+\beta+\gamma)(n-1+\beta+}{(2n-2-\nu+\sigma)(2n-}$$

Note that this finding is obtained without requiring any limitatio $p_n(x; \alpha, \beta, \gamma, \delta)$.

It is, moreover, plain that with the assignment

$$\nu = n-1+\beta, \quad \text{namely, } \alpha =$$

the factorizations implied by Proposition 2.3, and the properties $\mu = \beta - 1$. These are *new* findings. As for the additional findings ent 3.1.3. And Proposition 2.7 becomes as well applicable, entailing (*n*

$$p_n(x; -n+1-\beta, \beta, \gamma, \delta) = \prod_{m=1}^{n} [x$$

while Corollary 2.8 entails even more general properties, such as (

$$p_n[-(\ell-1+\beta)^2; -m+1-\beta, \beta, \gamma, \delta] = 0,$$

Remark 3.1. A look at the formulas (3.3) suggests other possibl such as $\nu = n-2+\sigma$, namely, $\alpha = 2-n-\sigma$. However, these assignm because for this to happen, it is *not* sufficient that the numerator required that the denominator in that expression *never* vanish. In the parameter ν in terms of n that satisfy these requirements.

3.1.2. Second Assignment

$$\alpha = -\frac{\nu}{2}, \quad \beta = \frac{1-}{2}$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = [6(4n-3-2\nu+2\gamma+2\delta)]^{-1} n\{3-4\gamma-4\delta+6\gamma\delta$$
$$-[11-12\gamma-12\delta+12$$
$$+4(3+2\nu-2\gamma-2\delta)n^2$$

$$\omega^{(\nu)} = -\frac{\nu^2}{4},$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Wilson polynomials (3.3):

$$p_n^{(\nu)}(x) = p_n\left(x; -\frac{\nu}{2}, \frac{1-}{2}\right.$$

Hence, with this identification, Proposition 2.1 becomes applicat Wilson polynomials satisfy the second recursion relation (2.4a) witl

$$g_n^{(\nu)} = \frac{n(n-1+\gamma+\delta)(2n-1-\nu+}{(4n-3-2\nu+2\gamma+2\delta)(4n-}$$

Note that this assignment entails now the (single) restriction polynomials $p_n(x;\alpha,\beta,\gamma,\delta)$.

It is, moreover, plain that with the assignments

$$\nu = n-\frac{1}{2}, \quad \text{hence } \alpha = -\frac{n}{2}+\frac{1}{4}$$

$$\nu = n-2+2\delta, \qquad \gamma = \delta-\frac{1}{2}, \qquad \alpha = -\frac{n}{2}+$$

respectively,

$$\nu = n-1+2\delta, \qquad \gamma = \delta+\frac{1}{2}, \qquad \alpha = -\frac{n}{2}+$$

the factorizations implied by Proposition 2.3 and the properties $\mu = -1/2$, $\mu = -2+2\delta$, respectively, $\mu = -1+2\delta$. These are *new* f Corollary 2.5, they are reported in Section 3.1.3. And Propositi *Diophantine* factorizations

$$p_n\left(x; -\frac{n}{2}+\frac{1}{4}, -\frac{n}{2}+\frac{3}{4}, \gamma, \delta\right) = \prod_{m=}^{n}$$

$$p_n\left(x; -\frac{n}{2}+1-\delta, -\frac{n}{2}+\frac{3}{2}-\delta, \delta-\frac{1}{2}, \delta\right) =$$

respectively,

$$p_n\left(x; -\frac{n}{2}+\frac{1}{2}-\delta, -\frac{n}{2}+1-\delta, \delta+\frac{1}{2}, \delta\right) =$$

(A referee pointed out that (3.17a) is not new, as one can eval which is indeed the case in (3.17a); and, moreover, that the two left-hand sides are identical as a consequence of the symm transformation $\delta \Rightarrow \delta + 1/2$.)

And Corollary 2.8 entails even more general properties, such as (n

$$p_n\Big[-\Big(\frac{2\ell-1}{4}\Big)^2;-\frac{m}{2}+\frac{1}{4},-\frac{m}{2}+\frac{3}{4},\gamma,\delta\Big]=0,$$

$$p_n\Big[-\Big(\frac{\ell-2+2\delta}{2}\Big)^2;-\frac{m}{2}+1-\delta,-\frac{m}{2}+\frac{3}{2}-\delta,\delta-\frac{1}{2}.$$

respectively,

$$p_n\Big[-\Big(\frac{\ell-1+2\delta}{2}\Big)^2;-\frac{m}{2}+\frac{1}{2}-\delta,-\frac{m}{2}+1-\delta,\delta+\frac{1}{2}$$

Moreover, with the assignments

$$\nu=2n-2+2\delta,\qquad \alpha=-n+1-\delta,$$

respectively,

$$\nu=2n-1+2\delta,\qquad \alpha=-n+\frac{1}{2}-\delta$$

Proposition 2.9 becomes applicable, entailing (*new* findings) the *Di*

$$p_n\Big(x;-n+1-\delta,-n+\frac{3}{2}-\delta,\gamma,\delta\Big)=$$

respectively,

$$p_n\Big(x;-n+\frac{1}{2}-\delta,-n+1-\delta,\gamma,\delta\Big)=$$

obviously implying the relation

$$p_n\Big(x;-n+1-\delta,-n+\frac{3}{2}-\delta,\gamma,\delta\Big)=p_n\Big(x;-$$

3.1.3. Factorizations

The following *new* relations among monic Wilson polynomials are i

$$p_n(x;-m+1-\beta,\beta,\gamma,\delta)$$
$$=p_{n-m}(x;m+\beta,\gamma,1-\beta,\delta)p_m(x;-m+$$

$$p_n\Big(x;-\frac{m}{2}+1-\delta,-\frac{m}{2}+\frac{3}{2}-\delta,\delta-\frac{1}{2},\delta\Big)$$
$$=p_{n-m}\Big(x;\frac{m}{2}-\frac{1}{2}+\delta,\frac{m}{2}+\delta,1-\delta,-\delta+\frac{3}{2}\Big)p_m$$

Note that the polynomials appearing as second factors in the factorizable, see (3.10) and (3.17b) (we will not repeat this remar

3.2. Racah

The monic Racah polynomials (see [9])

$$p_n(x;\alpha,\beta,\gamma,\delta)\equiv p_n($$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{n}) = \tilde{A}_n + \tilde{C}_n, \qquad b_n(\underline{n})$$

where

$$\tilde{A}_n = \frac{(n+1+\alpha)(n+1+\alpha+\beta)(n+1}{(2n+1+\alpha+\beta)(2n+}$$

$$\tilde{C}_n = \frac{n(n+\alpha+\beta-\gamma)(n+\alpha}{(2n+\alpha+\beta)(2n+1}$$

The standard version of these polynomials reads (see [9])

$$R_n(x;\alpha,\beta,\gamma,\delta) = \frac{(n+\alpha+\beta+1)_{}}{(\alpha+1)_n(\beta+\delta+1)_n(}$$

Note, however, that in the following we do *not* require the para restrictions $\alpha = -N$, $\beta+\delta = -N$, or $\gamma = -N$, with N a positive integer a for the standard Racah polynomials [9].

Let us recall that these polynomials are invariant under various res

$$\begin{aligned} p_n(x;\alpha,\beta,\gamma,\delta) &= p_n(x;\alpha,\beta,\beta+ \\ &= p_n(x;\beta+\delta,\alpha \\ &= p_n(x;\gamma,\alpha+\beta \end{aligned}$$

Let us now identify the parameter ν as follows (see (3.2)):

$$\alpha = -\nu.$$

With this assignment, one can set, consistently with our previous t

$$\begin{aligned} A_n^{(\nu)} &= [6(2n-\nu+\beta)]^{-1} n\{\beta(2+3\gamma+3\delta) - [2 \\ &+ [4+6(\gamma+\delta)+3(\beta \\ &+4(-\nu+\beta)n^2 + 2n^3 \end{aligned}$$

$$\omega^{(\nu)} = (\nu-1)(\nu+\gamma$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Racah polynomials (3.22):

$$p_n^{(\nu)}(x) = p_n(x;-\nu,\beta$$

Hence, with this identification, Proposition 2.1 becomes applicat Racah polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = -\frac{n(n+\beta)(n+\beta+}{(2n-\nu+\beta)(2n+}$$

Note that this finding is obtained without requiring any limitatio $p_n(x;\alpha,\beta,\gamma,\delta)$.

It is, moreover, plain that with the assignments

$$\nu = n, \quad \text{hence } \alpha =$$

$$\nu = n - \delta, \quad \text{hence } \alpha =$$

respectively,

$$\nu = n + \beta - \gamma, \quad \text{hence } \alpha =$$

the factorizations implied by Proposition 2.3 and the properties $\mu = 0$, $\mu = -\delta$, respectively, $\mu = \beta - \gamma$. These are *new* findings. As fo they are reported in Section 3.2.1. And Proposition 2.7 become *Diophantine* factorizations

$$p_n(x; -n, \beta, \gamma, \delta) = \prod_{m=1}^{n} [x - (m$$

$$p_n(x; -n + \delta, \beta, \gamma, \delta) = \prod_{m=1}^{n} [x - (n$$

respectively,

$$p_n(x; -n - \beta + \gamma, \beta, \gamma, \delta) = \prod_{m=1}^{n} [x - (m$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n[(\ell - 1)(\ell + \gamma + \delta); -m, \beta, \gamma, \delta] = 0, \quad \ell$$

$$p_n[(\ell + \gamma)(\ell - \delta - 1); -m + \delta, \beta, \gamma, \delta] = 0,$$

respectively,

$$p_n[(\ell - 1 + \beta - \gamma)(\ell + \beta + \delta); -m - \beta + \gamma, \beta, \gamma, \delta] =$$

3.2.1. Factorizations

The following *new* relations among Racah polynomials are implied

$$p_n(x; -m, \beta, -1, 1) = p_{n-m}(x; m, \beta, -1, 1) p_m(x$$

$$p_n(x; -m + \delta, \beta, -\delta, \delta) = p_{n-m}(x; m - \delta, 2\delta + \beta, \delta, -\delta) p$$

$$\begin{aligned} &p_n(x; -m - \beta + \gamma, \beta, \gamma, c - \gamma) \\ &\quad = p_{n-m}(x; m + \beta - \gamma + c, -\beta + 2\gamma - c, \gamma, c - \gamma) p_m(x; \end{aligned}$$

$$p_n(x; \alpha, -m, \gamma, \delta) = p_{n-m}(x; \alpha, m, \delta, \gamma) p_m(x;$$

$$p_n(x, \alpha, -m - \alpha + \eta, \eta, \delta) = p_{n-m}(x, \eta, m, \eta + \delta - \alpha, \alpha) p_m$$

3.3. Continuous Dual Hahn (CDH)

In this section (some results of which were already reported in (CDH) polynomials $p_n(x; a, \beta, \gamma)$ (see [9], and note the notational with a, β, γ),

$$p_n(x;\alpha,\beta,\gamma) \equiv p_n(x$$

defined by the three-term recursion relations (3.1) with

$$a_n(\underline{n}) = \alpha^2 - (n+\alpha+\beta)(n+\alpha+\gamma)$$

$$b_n(\underline{n}) = -n(n-1+\alpha+\beta)(n-1+\alpha$$

The standard version of these polynomials reads (see [9])

$$S_n(x;\alpha,\beta,\gamma) = (-1)^n p_n($$

Let us recall that these polynomials $p_n(x;\alpha,\beta,\gamma)$ are invariant unde

Let us now proceed and provide two identifications of the paramet

3.3.1. First Assignment

$$\alpha = -\nu.$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = n\left[-\frac{5}{6} + \beta + \gamma - \beta\gamma + (\beta+\gamma-1)\nu + \right.$$

$$\omega^{(\nu)} = -\nu^2,$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized CDH polynomials (3.32):

$$p_n^{(\nu)}(x) = p_n(x;-\nu,$$

Hence, with this identification, Proposition 2.1 becomes applicable, polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = n(n-1+\beta+$$

Note that this finding is obtained without requiring any limitati $p_n(x;\alpha,\beta,\gamma)$.

It is, moreover, plain that with the assignment

$$\nu = n-1+\beta, \quad \text{hence } \alpha =$$

the factorizations implied by Proposition 2.3 and the properties $\mu = -1+\beta$. These are *new* findings. As for the additional finding Section 3.3.3. And Proposition 2.7 becomes as well applicable, ent

$$p_n(x;-n+1-\beta,\beta,\gamma) = \prod_{m=1}^{n}[x$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n[-(\ell-1+\beta)^2;-m+1-\beta,\beta,\gamma] = 0,$$

Likewise, with the assignment

$$\nu = 2n + \beta, \qquad \alpha = -2n - \beta$$

Proposition 2.9 becomes applicable, entailing (*new* finding) the *Dio*

$$p_n\left(x; -2n - \beta, \beta, \frac{1}{2}\right) = \prod_{m=1}^{n} [x +$$

3.3.2. Second Assignment

$$\alpha = -\frac{1}{2}\nu + c, \qquad \beta = -\frac{1}{2}($$

where c is an a priori arbitrary parameter.

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = n\left[-\frac{4}{3} + \frac{3}{2}\gamma + \frac{5}{2}c - c^2 - 2\gamma c + \left(-\frac{5}{4} + \gamma + c\right)\right.$$

$$\omega^{(\nu)} = -\frac{1}{4}(1 - 2c +$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized CDH polynomials (3.32):

$$p_n^{(\nu)}(x) = p_n\left(x; c - \frac{\nu}{2}, c -\right.$$

Hence, with this identification, Proposition 2.1 becomes applicable, polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = n\left(n - 1 - \frac{\nu}{2} +\right.$$

Note that this assignment entails the (single) limitation $\beta = \alpha - 1/2$

It is, moreover, plain that with the assignment

$$\nu = n + 2c - \frac{3}{2}, \quad \text{hence } \alpha = -\frac{n}{2} +$$

the factorizations implied by Proposition 2.3 and the properties $\mu = 2c - 3/2$. These are *new* findings. As for the additional findin Section 3.3.3. And Proposition 2.7 becomes as well applicable, ent

$$p_n\left(x; -\frac{n}{2} + \frac{3}{4}, -\frac{n}{2} + \frac{1}{4}, \gamma\right) = \prod_{m=1}^{n}$$

And Corollary 2.8 entails even more general properties, such as (*n*

$$p_n\left[-\left(\frac{2\ell - 1}{4}\right)^2; -\frac{m}{2} + \frac{3}{4}, -\frac{m}{2} + \frac{1}{4}, \gamma\right] = 0,$$

Likewise with the assignments

$$\nu = 2(n - 1 + c + \gamma), \quad \text{hence } \alpha = -n +$$

respectively,

$$v = 2\left(n - \frac{3}{2} + c + \gamma\right), \quad \text{hence } \alpha = -n +$$

Proposition 2.9 becomes applicable, entailing (*new* findings) the *Di*

$$p_n\left(x; -n + 1 - \gamma, -n + \frac{1}{2} - \gamma, \gamma\right) = \prod_{m=}$$

respectively,

$$p_n\left(x; -n + \frac{3}{2} - \gamma, -n + 1 - \gamma, \gamma\right) = \prod_{m=}$$

Note that the right-hand sides of the last two formulas coincide; coincide as well.

3.3.3. Factorizations

The following *new* relations among continuous dual Hahn polynor 2.5:

$$p_n(x; -m + 1 - \beta, \beta, \gamma) = p_{n-m}(x; m + \beta, 1 - \beta, \gamma) p_m($$

3.4. Continuous Hahn (CH)

The monic continuous Hahn (CDH) polynomials $p_n(x; \alpha, \beta, \gamma, \delta)$ (see parameters a, b, c, d used there with $\alpha, \beta, \gamma, \delta$),

$$p_n(x; \alpha, \beta, \gamma, \delta) \equiv p_n($$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{\eta}) = -i(\alpha + \tilde{A}_n + \tilde{C}_n), \qquad b_r$$

where

$$\tilde{A}_n = -\frac{(n - 1 + \alpha + \beta + \gamma + \delta)(n +}{(2n - 1 + \alpha + \beta + \gamma + \delta)(2r}$$

$$\tilde{C}_n = \frac{n(n - 1 + \beta + \gamma)(n -}{(2n + \alpha + \beta + \gamma + \delta - 1)(2n +}$$

The standard version of these polynomials reads (see [9])

$$S_n(x; \alpha, \beta, \gamma, \delta) = (-1)^n p_n($$

Let us recall that these polynomials are symmetrical under the exc

$$p_n(x; \alpha, \beta, \gamma, \delta) = p_n(x; \beta, \alpha, \gamma, \delta) = p_n(x; \alpha$$

Let us now proceed and provide two identifications of the paramete

3.4.1. First Assignment

$$\alpha = -\nu.$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = in\frac{-\beta + \gamma + \delta - 2\gamma\delta + (1 - 2\beta}{2(2 - \beta - \gamma - \delta -}$$

$$\omega^{(\nu)} = -i\ \nu,$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized CH polynomials (3.53):

$$p_n^{(\nu)}(x) = p_n(x; -\nu, \beta,$$

Hence, with this identification, Proposition 2.1 becomes applicable polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = \frac{in(n - 1 + \beta + \gamma)(n -}{(2n - 2 - \nu + \beta + \gamma + \delta)(2n -}$$

Note that this assignment entails no restriction on the 4 parameter

It is, moreover, plain that with the assignment

$$\nu = n - 1 + \gamma, \quad \text{hence } \alpha =$$

the factorizations implied by Proposition 2.3 and the properties $\mu = -1 + \gamma$. These are *new* findings. And Proposition 2.7 become *Diophantine* factorization

$$p_n(x; -n + 1 - \gamma, \beta, \gamma, \delta) = \prod_{m=1}^{n} [$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n[-i(\ell - 1 + \gamma); -m + 1 - \gamma, \beta, \gamma, \delta] = 0,$$

3.4.2. Second Assignment

Analogous results also obtain from the assignment

$$\gamma = -\nu.$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = -\frac{in[n(\alpha + \beta - \delta + \nu) + (2\delta - 1)}{2(2n - 2 + \alpha + \beta}$$

$$\omega^{(\nu)} = i\nu,$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized CH polynomials (3.53):

$$p_n^{(\nu)}(x) = p_n(x; \alpha, \beta, -$$

Hence, with this identification, Proposition 2.1 becomes applicable polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = \frac{in(n-1+\alpha+\beta)(n-}{(2n-2-\nu+\alpha+\beta+\delta)(2n\cdot}$$

Note that this assignment entails no restriction on the 4 parameter

It is, moreover, plain that with the assignment

$$\nu = n-1+\alpha, \quad \text{hence } y =$$

the factorizations implied by Proposition 2.3, and the properties $\mu = -1+\alpha$. These are *new* findings. And Proposition 2.7 become *Diophantine* factorization

$$p_n(x;\alpha,\beta,-n+1-\alpha,\delta) = \prod_{m=1}^{n} [$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n[i(\ell-1+\alpha);\alpha,\beta,-m+1-\alpha,\delta] = 0,$$

3.5. Hahn

In this subsection, we introduce a somewhat generalized version (generalized) monic Hahn polynomials $p_n(x;\alpha,\beta,y)$ (see [9], and with the arbitrary parameter y: hence the standard Hahn polyno *integer* and $n = 1,2,\ldots,N$),

$$p_n(x;\alpha,\beta,y) \equiv p_n(x$$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{n}) = -(\tilde{A}_n + \tilde{C}_n), \qquad b_n(\underline{n}$$

where

$$\tilde{A}_n = \frac{(n+1+\alpha)(n+1+\alpha+}{(2n+1+\alpha+\beta)(2n+}$$

$$\tilde{C}_n = \frac{n(n+1+\alpha+\beta+y}{(2n+\alpha+\beta)(2n+1}$$

The standard version of these polynomials reads (see [9])

$$Q_n(x;\alpha,\beta,y) = \frac{(n+1+\alpha+\beta)_{}}{(1+\alpha)_n(-y)_n}$$

Let us now proceed and provide three identifications of the parame

3.5.1. First Assignment

$$\alpha = -\nu.$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = \frac{n[\beta+(1+2y)\nu-(\beta}{2(2n-\nu+}$$

$$\omega^{(\nu)} = \nu - 1,$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Hahn polynomials (3.69):

$$p_n^{(\nu)}(x) = p_n(x; -\nu,$$

Hence, with this identification, Proposition 2.1 becomes applicable, polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = -\frac{n(n+\beta)(n-}{(2n-\nu+\beta)(2n+}$$

Note that this assignment entails no restriction on the 3 parameter

It is, moreover, plain that with the assignments

$$\nu = n$$

respectively,

$$\nu = n + 1 + \beta + \gamma$$

the factorizations implied by Proposition 2.3 and the properties $\mu = 1 + \beta + \gamma$. These are *new* findings. And Proposition 2.7 becom *Diophantine* factorizations

$$p_n(x; n, \beta, \gamma) = \prod_{m=1}^{n} (x -$$

respectively,

$$p_n(x; n+1+\beta+\gamma, \beta, \gamma) = \prod_{m=1}^{n}$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n(\ell - 1; m, \beta, \gamma) = 0, \quad \ell = 1, \ldots$$

respectively,

$$p_n(\ell + \beta + \gamma; m + 1 + \beta + \gamma, \beta, \gamma) = 0, \quad \ell$$

3.5.2. Second Assignment

$$\beta = -\nu + \gamma + c,$$

where c is an arbitrary parameter.

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = -n\frac{a - \gamma - c + \nu + 2a\gamma + (-}{2(a + \gamma + c -}$$

$$\omega^{(\nu)} = 1 - \nu + 2\gamma -$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Hahn polynomials (3.69):

$$p_n^{(\nu)}(x) = p_n(x;\ \alpha, -\nu +$$

Hence, with this identification, Proposition 2.1 becomes applicable, polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = \frac{n(n+\alpha)(n-1}{(2n-\nu+\alpha+\gamma+c)(2n+}$$

Note that this assignment entails no restriction on the 3 parameter

It is, moreover, plain that with the assignments

$$\nu = n + \gamma + c, \quad \text{hence}$$

respectively,

$$\nu = n + 1 + \alpha + 2\gamma + c, \quad \text{hence } \beta$$

the factorizations implied by Proposition 2.3 and the properties $\mu = \gamma + c$, respectively, $\mu = 1 + \alpha + 2\gamma + c$. These are *new* findings. entailing (*new* findings) the *Diophantine* factorizations

$$p_n(x; \alpha, -n, \gamma) = \prod_{m=1}^{n} (x +$$

respectively,

$$p_n(x; \alpha, -n - 1 - \alpha - \gamma, \gamma) = \prod_{m}$$

And Corollary 2.8 entails even more general properties, such as (*n*

$$p_n(-\ell + 1 + \gamma; \alpha, -m, \gamma) = 0, \quad \ell = 1,$$

respectively,

$$p_n(-\ell - \alpha; \alpha, -m - 1 - \alpha - \gamma, \gamma) = 0, \quad \ell =$$

3.5.3. Third Assignment

$$\beta = -\nu + c, \qquad \gamma =$$

where c is an arbitrary parameter.

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = -\frac{n[\nu + \alpha - c + 2\alpha\nu +}{2(2n - \nu + \alpha}$$

$$\omega^{(\nu)} = \nu,$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Hahn polynomials (3.69):

$$p_n^{(\nu)}(x) = p_n(x; \alpha, -\nu$$

Hence, with this identification, Proposition 2.1 becomes applicable, polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = -\frac{n(n+\alpha)(n+1}{(2n-\nu+\alpha+c)(2n+}$$

Note that this assignment entails no restriction on the 4 parameter

It is, moreover, plain that with the assignment

$$\nu = n + c, \qquad \beta = -n,$$

the factorizations implied by Proposition 2.3 and the properties im These are *new* findings. And Proposition 2.7 becomes as well a factorization

$$p_n(x;\alpha,-n,n+c) = \prod_{m=1}^{n}($$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n(\ell+c;\alpha,-m,m+c) = 0, \quad \ell = 1,$$

3.6. Dual Hahn

In this subsection, we introduce a somewhat generalized versior These (generalized) monic dual Hahn polynomials $p_n(x;\gamma,\delta,\eta)$ (parameter N with the arbitrary parameter η: hence the standard H a *positive integer* and $n = 1,2,\ldots,N$),

$$p_n(x;\gamma,\delta,\eta) \equiv p_n(x$$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{\eta}) = \tilde{A}_n + \tilde{C}_n, \qquad b_n(\underline{\eta})$$

where

$$\tilde{A}_n = (n+1+\gamma)(n-\eta), \qquad \tilde{C}_n =$$

The standard version of these polynomials reads (see [9])

$$R_n(x;\gamma,\delta,\eta) = \frac{1}{(1+\gamma)_n(-\eta)_n}$$

Let us now proceed and provide two identifications of the paramete

3.6.1. First Assignment

$$\eta = \nu.$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = n\Big[\frac{1}{3} + \frac{-\gamma+\delta}{2} - \gamma\nu - \big(1+\nu$$

$$\omega^{(\nu)} = \nu(1+\nu+\gamma+$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized dual Hahn polynomials (3.92):

$$p_n^{(\nu)}(x) = p_n(x; \gamma, \delta$$

Hence, with this identification, Proposition 2.1 becomes applicable Hahn polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = -n(n+\gamma)$$

Note that this assignment entails no restriction on the 3 parameter

It is, moreover, plain that with the assignments

$$\nu = n - 1, \quad \text{hence } \eta =$$

(which is, however, incompatible with the requirement characteriz N a *positive integer* and $n = 1, 2, \ldots, N$), respectively,

$$\nu = n - 1 - \delta, \quad \text{hence } \eta =$$

the factorizations implied by Proposition 2.3 and the properties $\mu = -1$, respectively, $\mu = -1 - \delta$. These are *new* findings. As for the are reported in Section 3.6.3. And Proposition 2.7 becomes *Diophantine* factorizations

$$p_n(x; \gamma, \delta, n-1) = \prod_{m=1}^{n} [x - (m$$

respectively,

$$p_n(x; \gamma, \delta, n-1-\delta) = \prod_{m=1}^{n} [x - (n$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n[(\ell - 1)(\ell + \gamma + \delta); \gamma, \delta, m-1] = 0, \quad \ell$$

respectively,

$$p_n[(\ell + \gamma)(\ell - 1 - \delta); \gamma, \delta, m - 1 - \delta] = 0,$$

While for

$$\nu = 2n, \quad \text{hence } \eta = 2n, \text{ and r}$$

respectively,

$$\nu = 2n - \delta, \quad \text{hence } \eta = 2n - \delta, \text{ an}$$

Proposition 2.9 becomes applicable, entailing (*new* findings) the *Di*

$$p_n(x; \gamma, \gamma, 2n) = \prod_{m=1}^{n} [x - 2(2n$$

respectively,

$$p_n(x; \gamma, -\gamma, 2n+\gamma) = \prod_{m=1}^{n} [x - (2m$$

3.6.2. Second Assignment

$$\gamma = -\nu, \qquad \delta = \nu +$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = n\Big[\frac{1}{3} + (1+\eta)\nu + \frac{1}{2}c - \big(1+\nu$$

$$\omega^{(\nu)} = (\nu-1)(\nu+$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Dual Hahn polynomials (3.92):

$$p_n^{(\nu)}(x) = p_n(x;-\nu,\nu$$

Hence, with this identification, Proposition 2.1 becomes applicable Hahn polynomials satisfy the second recursion relation (2.4a) with

$$g_n^{(\nu)} = -n(n-1-$$

Note that this assignment entails no restriction on the 3 parameter

It is, moreover, plain that with the assignments

$$\nu = n, \quad \text{hence } y = -n,$$

respectively,

$$\nu = n - 1 - \eta - c, \quad \text{hence } y = -n + 1 +$$

the factorizations implied by Proposition 2.3 and the properties im respectively, $\mu = -1-\eta-c$. These are *new* findings. As for the add reported in Section 3.6.3. And Proposition 2.7 becomes as well a factorizations

$$p_n(x;-n,n+c,\eta) = \prod_{m=1}^{n}[x-($$

respectively,

$$p_n(x;-n+1+\eta+c,n-1-\eta,\eta) = \prod_{m=1}^{n}[x-$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n[(\ell-1)(\ell+c);-m,m+c,\eta] = 0, \quad \ell$$

respectively,

$$p_n[(\ell-1-\eta)(\ell-2-\eta-c);-m+1+\eta+c,-m-1-$$

While for

$$\nu = 2n-\eta, \quad \text{hence } y = -2n+\eta, \text{ and moreo}$$

respectively,

$$\nu = 2n+1, \quad \text{hence } y = -2n-1, \text{ and moreover } c =$$

Proposition 2.9 becomes applicable, entailing (*new* findings) the *Di*

$$p_n(x;-2n+\eta,2n-\eta,\eta) = \prod_{m=1}^{n}[x-(2n$$

respectively,

$$p_n(x;-2n-1,2n-1-2\eta,\eta)=\prod_{m=1}^{n}[x\cdot$$

3.6.3. Factorizations

The following *new* relations among dual Hahn polynomials are impl

$$p_n(x;\gamma,-\gamma,m-1)=p_{n-m}(x;\gamma,-\gamma,-m-1)p_m($$

$$p_n(x;\gamma,\delta,m-1-\delta)=p_{n-m}(x;\delta,\gamma,-m-1-\gamma)p_m$$

$$p_n(x;-m,m,\eta)=p_{n-m}(x;m,-m,\eta)p_m(x;$$

$$p_n(x;-m+1+\eta+c,m-1-\eta,\eta)$$
$$=p_{n-m}(x;m-1-\eta,-m+1+\eta+c,-\eta-c-2)p_m(x;-$$

3.7. Shifted Meixner-Pollaczek (SMP)

In this subsection, we introduce and treat a modified version of th
The standard (monic) Meixner-Pollaczek (MP) polynomials $p_n(x;\alpha,$

$$p_n(x;\alpha,\lambda)\equiv p_n(x;$$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{\eta})=\alpha(n+\lambda)=\frac{n}{t}$$

$$b_n(\underline{\eta})=-\frac{1}{4}(1+\alpha^2)n(n-1+2\lambda)$$

The standard version of these polynomials reads (see [9])

$$P_n^{(\lambda)}(x;\tan\phi)=\frac{(2\sin\phi)^n}{n!}p_n(x;c$$

However, we have not found any assignment of the parameters α
machinery. We, therefore, consider the (monic) "shifted Meixner-I

$$p_n(x;\alpha,\lambda,\beta)=p_n(x+$$

defined of course via the three-term recursion relation (3.1) with

$$a_n(\underline{\eta})=\alpha(n+\lambda)+\beta=\frac{n}{t}$$

$$b_n(\underline{\eta})=-\frac{1}{4}(1+\alpha^2)n(n-1+2\lambda)$$

Then, with the assignment

$$\lambda=-\frac{1}{2}(\nu+c),\qquad\beta=-$$

(entailing no restriction on the parameters a,λ,β, in as much as t
set, consistently with our previous treatment,

$$A_n^{(\nu)}=\frac{1}{2}n(an-a\nu-i\nu-iC-ac-a),$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by
with the normalized shifted Meixner-Pollaczek polynomials:

$$p_n^{(\nu)}(x)=p_n\Bigl(x;a,-\frac{1}{2}(\nu+c)$$

Hence, with this identification, Proposition 2.1 becomes applicabl
shifted Meixner-Pollaczek polynomials satisfy the second recursion

$$g_n^{(\nu)}=-\frac{1}{2}n(a+i$$

It is, moreover, plain that with the assignment

$$\nu=n-1-c \quad \text{hence, } \lambda=-\frac{1}{2}(n-1),$$

the factorizations implied by Proposition 2.3, and the properties
$\mu=-1-c$. And Proposition 2.7 becomes as well applicable, entailin

$$p_n\Bigl(x;a,-\frac{1}{2}(n-1),-\frac{1}{2}i(n-1-c+C)\Bigr)=$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n\Bigl(i\Bigl(l-1+\frac{C-c}{2}\Bigr);a,-\frac{1}{2}(m-1),-\frac{1}{2}i(m-1-c+$$

3.8. Meixner

In this section (some results of which were already reported in
$p_n(x;\beta,c)$ (see [9]),

$$p_n(x;\beta,c)\equiv p_n(x;$$

defined by the three-term recursion relations (3.1) with

$$a_n(\underline{n})=\frac{\beta c+(1+c)n}{c-1}, \qquad b_n(\underline{n})$$

The standard version of these polynomials reads (see [9]):

$$M_n(x;\beta,c)=\frac{1}{(\beta)_n}\Bigl(\frac{c-1}{c}\Bigr)^n$$

We now identify the parameter ν via the assignment

$$\beta=-\nu.$$

One can then set, consistently with our previous treatment,

$$A_n^{(\nu)}=\frac{n[1+c+2c\nu-(1+c)n}{2(1-c)}$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Meixner polynomials (3.25):

$$p_n^{(\nu)}(x) = p_n(x; -\nu,$$

Hence, with this identification, Proposition 2.1 becomes applicab Meixner polynomials satisfy the second recursion relation (2.4a) wi

$$g_n^{(\nu)} = \frac{cn}{1-c}.$$

Note that this assignment entails no restriction on the 2 parameter

It is, moreover, plain that with the assignment

$$\nu = n - 1, \quad \text{hence } \beta =$$

the factorizations implied by Proposition 2.3, and the properties $\mu = -1$. These are *new* findings. And Proposition 2.7 becomes *Diophantine* factorization

$$p_n(x; 1-n, c) = \prod_{m=1}^{n} (x$$

And Corollary 2.8 entails even more general properties, such as (*n*

$$p_n(\ell - 1; 1 - m, c) = 0, \quad \ell = 1, \ldots$$

Likewise for

$$\nu = 2n \quad \text{hence } \beta = -2n \text{ and m}$$

Proposition 2.9 becomes applicable, entailing (*new* finding) the *Dio*

$$p_n(x; -2n, -1) = \prod_{m=1}^{n} (x -$$

3.9. Krawtchouk

The monic Krawtchouk polynomials $p_n(x; \alpha, \beta)$ (see [9]: and note t N used there with the parameters α and β used here, implying *positive integer* these polynomials $p_n(x; \alpha, \beta)$ coincide with the stan

$$p_n(x; \alpha, \beta) \equiv p_n(x;$$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{n}) = -\alpha\beta + n(2\alpha - 1), \qquad b_n(\underline{n}) =$$

The standard version of these polynomials reads (see [9])

$$K_n(x; \alpha, \beta) = \frac{1}{\alpha^n (-\beta)_n} p_n$$

We now identify the parameter ν via the assignment

$$\beta = \nu.$$

One can then set, consistently with our previous treatment,

$$A_n^{(\nu)} = n\left[\frac{1}{2} - \alpha - \alpha\nu + \left(-\frac{1}{2} + \alpha\right)\right.$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by
with the normalized Krawtchouk polynomials (3.135):

$$p_n^{(\nu)}(x) = p_n(x;\alpha,$$

Hence, with this identification, Proposition 2.1 becomes applicab
Krawtchouk polynomials satisfy the second recursion relation (2.4a

$$g_n^{(\nu)} = -\alpha n.$$

Note that this assignment entails no restriction on the 2 parameter

It is, moreover, plain that with the assignment

$$\nu = n - 1, \quad \text{hence } \beta =$$

(which is, however, incompatible with the definition of the standa
with N a *positive integer*), the factorizations implied by Propositio
become applicable with $\mu = -1$. These are *new* findings. And Propos
finding) the *Diophantine* factorization

$$p_n(x;\alpha,n-1) = \prod_{m=1}^{n} (x-$$

And Corollary 2.8 entails even more general properties, such as (n

$$p_n(\ell - 1;\alpha,m-1) = 0, \quad \ell = 1,\ldots$$

Likewise for

$$\nu = 2n \quad \text{hence } \beta = 2n \text{ and m}$$

(which is also incompatible with the definition of the standard Krav
a *positive integer*), Proposition 2.9 becomes applicable, entailing (n

$$p_n(x;-2n,-1) = \prod_{m=1}^{n} (x-$$

3.10. Jacobi

In this section (most results of which were already reported in
$p_n(x;\alpha,\beta)$ (see [9]),

$$p_n(x;\alpha,\beta) \equiv p_n(x;$$

defined by the three-term recursion relations (3.1) with

$$a_n(\underline{\eta}) = \frac{(\alpha+\beta)(\alpha-}{(2n+\alpha+\beta)(2n+}$$

$$b_n(\underline{\eta}) = -\frac{4n(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+}$$

The standard version of these polynomials reads (see [9])

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!}$$

Let us recall that for the Jacobi polynomials there holds the symme

$$p_n(-x;\beta,\alpha) = p_n(x;$$

We now identify the parameter ν as follows:

$$\alpha = -\nu.$$

With this assignment one can set, consistently with our previous tr

$$A_n^{(\nu)} = -\frac{n(\nu+\beta)}{2n-\nu+\beta},$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with the normalized Jacobi polynomials (3.146):

$$p_n^{(\nu)}(x) = p_n(x;-\nu$$

Hence, with this identification, Proposition 2.1 becomes appl normalized Jacobi polynomials satisfy the second recursion relatior

$$g_n^{(\nu)} = -\frac{2n(n+\beta}{(2n-\nu+\beta)(2n-}$$

It is, moreover, plain that with the assignment

$$\nu = n, \quad \text{hence } \alpha =$$

the factorizations implied by Proposition 2.3, and the properties $\mu = 0$. These seem *new* findings. As for the additional findings ent 3.10.1. And Proposition 2.7 becomes as well applicable, entailing (

$$p_n(x;-n,\beta) = (x-$$

And Corollary 2.8 entails even more general properties, suc $p_n(x;-m,\beta)$, $m = 1,\ldots,n$, feature $x = 1$ as a zero of order m.

3.10.1. Factorizations

The following (*not new*) relations among Jacobi polynomials are im (3.153), of which the following formula is a generalization, just as

$$p_n(x;-m,\beta) = p_{n-m}(x;m,\beta)p_m(x;-m,\beta) = (x-1$$

3.11. Laguerre

In this section (most results of which were already reported in [$p_n(x;\alpha)$ (see [9]),

$$p_n(x;\alpha) \equiv p_n(x;\underline{r}$$

defined by the three-term recursion relations (3.1) with

$$a_n(\underline{\eta}) = -(2n+1+\alpha), \qquad b_n($$

The standard version of these polynomials reads (see [9])

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}p_n($$

We now identify the parameter ν as follows:

$$\alpha = -\nu.$$

With this assignment, one can set, consistently with our previous t

$$A_n^{(\nu)} = -n(n-\nu), \qquad \alpha$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by
with the normalized Laguerre polynomials (3.155):

$$p_n^{(\nu)}(x) = p_n(x; -$$

Hence, with this identification, Proposition 2.1 becomes applicable
Laguerre polynomials satisfy the second recursion relation (2.4a) v

$$g_n^{(\nu)} = n.$$

It is, moreover, plain that with the assignment

$$\nu = n, \quad \text{hence } \alpha =$$

the factorizations implied by Proposition 2.3, and the properties
$\mu = 0$. These seem *new* findings. As for the additional findings ent
3.11.1. And Proposition 2.7 becomes as well applicable, entailing (

$$p_n(x; -n) = x^n.$$

And Corollary 2.8 entails even more general properties, such
$p_n(x; -m)$, $m = 1, \ldots, n$, feature $x = 0$ as a zero of order m, see (1.1

3.11.1. Factorizations

The following (*not new*) relations among Laguerre polynomials are
see (3.162), of which the following formula—already reported abov

$$p_n(x; -m) = p_{n-m}(x; m) p_m(x; -m) = x^m p_n$$

3.12. Modified Charlier

In this subsection, we introduce and treat a modified version o
standard (monic) Charlier polynomials $p_n(x; a)$ (see [9]),

$$p_n(x; a, \lambda) \equiv p_n(x;$$

are defined by the three-term recursion relations (3.1) with

$$a_n(\underline{n}) = -n - a, \qquad b_n($$

The standard version of these polynomials reads (see [9])

$$C_n(x; a) = (-a)^{-n} p_n($$

However, we have not found any assignment of the parameter
machinery. To nevertheless proceed, we introduce the class of (m
characterized by the three-term recursion relation (3.1) with

$$a_n(\underline{\eta}) = -\gamma(n+\alpha)+\beta, \qquad b_n$$

that obviously reduce to the monic Charlier polynomials for $\beta = 0$,

$$\beta = -\nu, \qquad \gamma = -$$

one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = \frac{1}{2}n(n-1-2\nu+2\alpha),$$

implying, via (2.2), (2.3), that the polynomials $p_n^{(\nu)}(x)$ defined by with these (monic) modified Charlier polynomials:

$$p_n^{(\nu)}(x) = p_n(x;\alpha,-\nu$$

Hence, with this identification, Proposition 2.1 becomes applic modified Charlier polynomials satisfy the second recursion relation

$$g_n^{(\nu)} = -n.$$

There does not seem to be any interesting results for the zeros of

4. Outlook

Other classes of orthogonal polynomials to which our machinery is in this paper, have been identified by finding explicit classes of c orthogonal polynomials via the three-term recursion relations (2. the validity of the various propositions reported above. Hence, for results to those reported above hold, namely an additional thr parameter ν, and possibly as well factorizations identifying *Dio* hopefully soon, in a subsequent paper, where we also elucidate reported above and the wealth of known results on *discrete integ* with the machinery reported above are as well under investigat recursion relations satisfied by the classes of orthogonal polynom (2.1) (for appropriate choices of the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$) polynomials—possibly including the identification of ODEs satisfied

Appendix

A. A proof

In this Appendix, for completeness, we provide a proof of the fact although this proof is actually quite analogous to that provided (fo proceed by induction, assuming that (2.11) holds up to n, and the in the right-hand side of the relation (2.1a) with $\nu = m$, we get

$$p_{n+1}^{(m+\mu)}(x) = [(x+a_n^{(m+\mu)})\tilde{p}_{n-m}^{(-m)}(x) + b_n^{(m+\mu)}\tilde{p}_{n-1-}^{(-m)}$$

and clearly by using the recursion relation (2.12a) the square bra replaced by $\tilde{p}_{n+1-m}^{(-m)}(x)$, yielding

$$p_{n+1}^{(m+\mu)}(x) = \tilde{p}_{n+1-m}^{(-m)}(x) p_m^{(m+\mu)}(x),$$

Note that for $m = n + 1$, this formula is an identity, since $\tilde{p}_0^{(-m)}(x)$
holds for $m = n$, provided that (2.9) holds, see (2.1a) with $m = n$ an

But this is just the formula (2.11) with n replaced by $n + 1$. Q. E. D

Remark A .1. The hypothesis (2.9) has been used above, in this p
the final formula, (A.2), for $m = n$. Hence one might wonder wheth
validity of the final formula (A.2) for $m = n$ seems to be implied by
invoke (2.9). But in fact, by setting $m = n$ in the basic recurrence
only hold provided (2.9) also holds.

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References

1. F. Calogero, L. Di Cerbo, and R. Droghei, "On isochronous I equilibrium configurations, behaviour in their neighbourhood *of Physics A: Mathematical and General*, vol. 39, no. 2, pp. :
2. F. Calogero, L. Di Cerbo, and R. Droghei, "On isochronous S configurations, behavior in their neighborhood, Diophantine 355, no. 4-5, pp. 262 – 270, 2006.
3. F. Calogero, *Isochronous Systems*, Oxford University Press,
4. M. Bruschi, F. Calogero, and R. Droghei, "Proof of certain D remarkable classes of orthogonal polynomials," *Journal of F* no. 14, pp. 3815 – 3829, 2007.
5. M. Bruschi, F. Calogero, and R. Droghei, "Tridiagonal matri relations: I," *Journal of Physics A: Mathematical and Theor*
6. M. Bruschi, F. Calogero, and R. Droghei, "Tridiagonal matri relations: II," *Journal of Physics A: Mathematical and Theor*
7. J. Favard, "Sur les polynômes de Tchebicheff," *Comptes R* 2052 – 2053, 1935.
8. M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomi* *Mathematics and Its Applications*, Cambridge University Pres
9. R. Koekoek and R. F. Swarttouw, "The Askey-scheme of hy analogue," Delft University of Technology, Delft, The Nethe http://aw.twi.tudelft.nl/~koekoek/askey.html.

10. A. Erdélyi, Ed., *Higher Transcendental Functions*, A. Erdélyi,

11. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series,* USA, 5th edition, 1994.

12. M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathem Mathematical Tables*, M. Abramowitz and I. A. Stegun, Eds., *Mathematics Series*, U.S. Government Printing Office, Washi

13. http://mathworld.wolfram.com/OrthogonalPolynomials.html

14. W. Magnus and F. Oberhettinger, *Formeln und Sätze für die* Springer, Berlin, Germany, 2nd edition, 1948.

15. G. Szëgo, *Orthogonal Polynomials*, vol. 23 of *AMS Colloquiu* Providence, RI, USA, 1939.

16. F. Marcellán and R. Álvarez-Nodarse, "On the "Favard the *Computational and Applied Mathematics*, vol. 127, no. 1-2,

17. D. J. Dickinson, H. O. Pollak, and G. H. Wannier, "On a clas set," *Pacific Journal of Mathematics*, vol. 6, pp. 239–247,

18. M. Bruschi, F. Calogero, and R. Droghei, "Polynomials defin a second recursion relation: connection with discrete integra factorizations," In preparation.