



电动力学

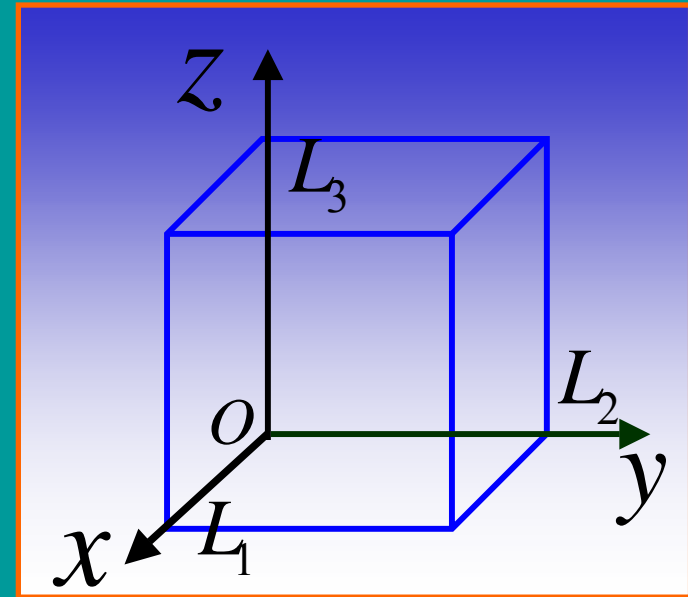
第二十讲

西安石油大学理学院
应用物理系



二. 谐振腔

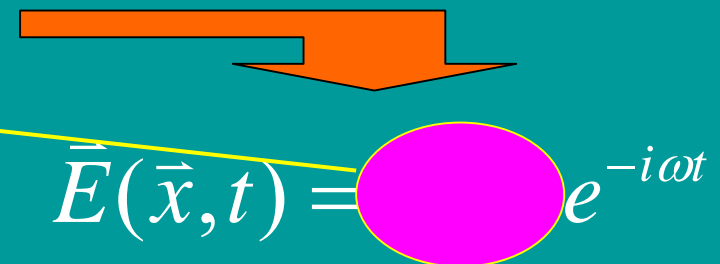
谐振腔：由六块金属壁面组成的腔体，它产生的电磁振荡通常在微波范围；由镜面构成的谐振腔可以产生频率更高的光波振荡



1. 矩形谐振腔中电磁波满足的方程

建立如图所示的直角坐标系，取腔的内表面分别为 $x=0$ 、 L_1 ， $y=0$ 、 L_2 ， $z=0$ 、 L_3 面，将腔内表面看作理想导体

$$\left\{ \begin{array}{l} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla \cdot \vec{E} = 0 \end{array} \right. \quad \begin{array}{l} E_t = 0 \\ \frac{\partial E_n}{\partial n} = 0 \end{array}$$

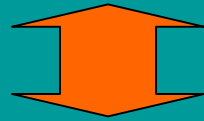


$$\vec{E}(\vec{x}, t) = \text{[pink circle]} e^{-i\omega t}$$



2. 谐振腔中电磁波的解——分离变量法

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$



$$(\nabla^2 E_x + k^2 E_x) \hat{e}_x + (\nabla^2 E_y + k^2 E_y) \hat{e}_y + (\nabla^2 E_z + k^2 E_z) \hat{e}_z = 0$$

设 $u(x,y,z)$ 为 \mathbf{E} 的任意一个直角分量



$$\nabla^2 u + k^2 u = 0$$

$$u(x,y,z) = X(x)Y(y)Z(z)$$



$$YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0$$



$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} + k^2 = 0$$

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon$$

X, Y, Z 相互无关

$$\begin{cases} \frac{d^2 X}{dx^2} + k_x^2 X = 0 \\ \frac{d^2 Y}{dy^2} + k_y^2 Y = 0 \\ \frac{d^2 Z}{dz^2} + k_z^2 Z = 0 \end{cases}$$

解出 X, Y, Z 后, 即可求得 u 的通解为

$$u(x, y, z) = (C_1 \cos k_x x + D_1 \sin k_x x)(C_2 \cos k_y y + D_2 \sin k_y y)(C_3 \cos k_z z + D_3 \sin k_z z)$$

3. 由边界条件确定系数

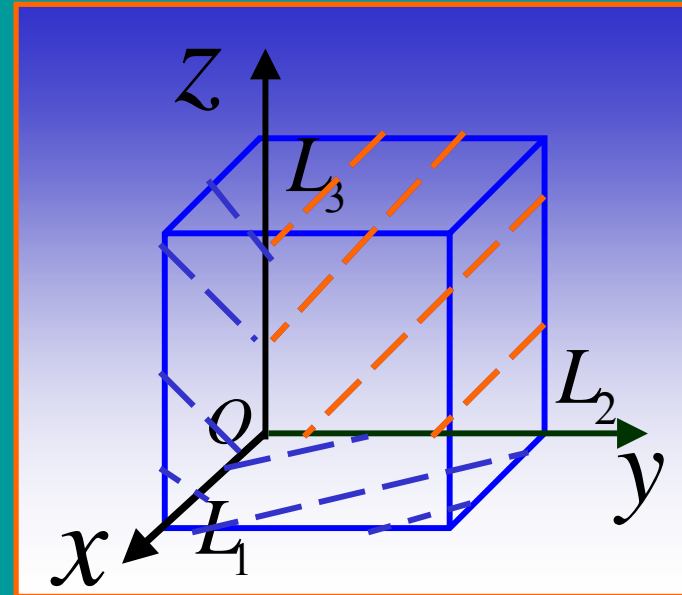
假定 $u(x, y, z) = E_x$

$$E_t = 0 \quad \frac{\partial E_n}{\partial n} = 0$$



$$E_t = 0 \quad \frac{\partial E_n}{\partial n} = 0$$

$$E_x(x, y, z) = (C_1 \cos k_x x + D_1 \sin k_x x) \\ (C_2 \cos k_y y + D_2 \sin k_y y) \\ (C_3 \cos k_z z + D_3 \sin k_z z)$$



1) $x=0, y=0, z=0$ 面, 即 $(y-z, z-x, x-y)$ 面

■ $x=0$ ($y-z$) 面 $E_x \perp y-z$ 面 $\longrightarrow \frac{\partial E_x}{\partial x} \Big|_{x=0} = 0$

$$\left. \begin{array}{l} (-C_1 k_x \sin k_x x + D_1 k_x \cos k_x x) \\ (C_2 \cos k_y y + D_2 \sin k_y y) \\ (C_3 \cos k_z z + D_3 \sin k_z z) \end{array} \right|_{x=0} = 0 \longrightarrow D_1 = 0$$



■ $y=0$ (z-x)面和 $z=0$ (x-y)面

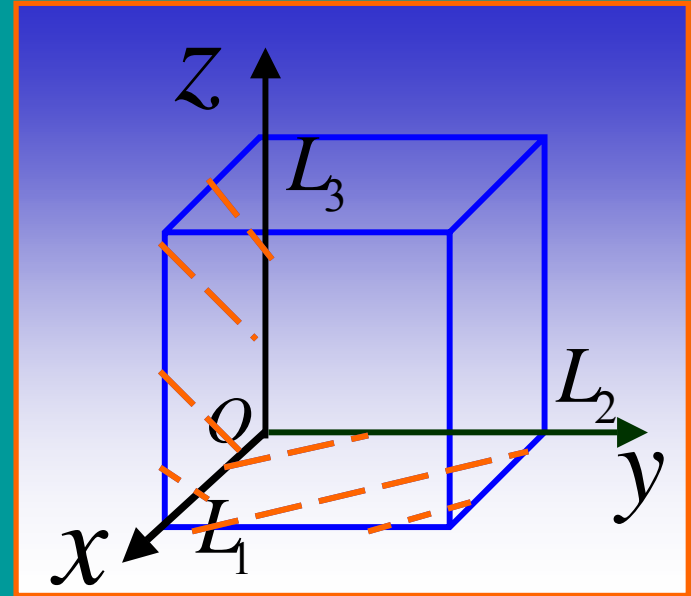
$$E_x \Big|_{z-x \text{面}} \longrightarrow E_x \Big|_{y=0} = 0$$

$$\begin{aligned} & (C_1 \cos k_x x + D_1 \sin k_x x) \\ \longrightarrow & (C_2 \cos k_y y + D_2 \sin k_y y) \\ & (C_3 \cos k_z z + D_3 \sin k_z z) \Big|_{y=0} = 0 \end{aligned}$$

$$\longrightarrow C_2 = 0$$

$$E_x \Big|_{x-y \text{面}} \longrightarrow E_x \Big|_{z=0} = 0 \longrightarrow \begin{aligned} & (C_1 \cos k_x x + D_1 \sin k_x x) \\ & (C_2 \cos k_y y + D_2 \sin k_y y) \\ & (C_3 \cos k_z z + D_3 \sin k_z z) \Big|_{z=0} = 0 \end{aligned}$$

$$\longrightarrow C_3 = 0$$



$$E_t = 0 \quad \frac{\partial E_n}{\partial n} = 0$$



$$D_1=0$$

$$C_2=0$$

$$C_3=0$$

$$E_x(x, y, z) = (C_1 \cos k_x x + D_1 \sin k_x x)(C_2 \cos k_y y + D_2 \sin k_y y)(C_3 \cos k_z z + D_3 \sin k_z z)$$



$$E_x = A_1 \cos k_x x \sin k_y y \sin k_z z \quad (A_1 = C_1 D_2 D_3)$$

2) $x=L_1$ 面, $y=L_2$ 面, $z=L_3$ 面

■ $x=L_1$ 面

$E_x \perp$ 该面



$$\left. \frac{\partial E_x}{\partial x} \right|_{x=L_1} = 0$$



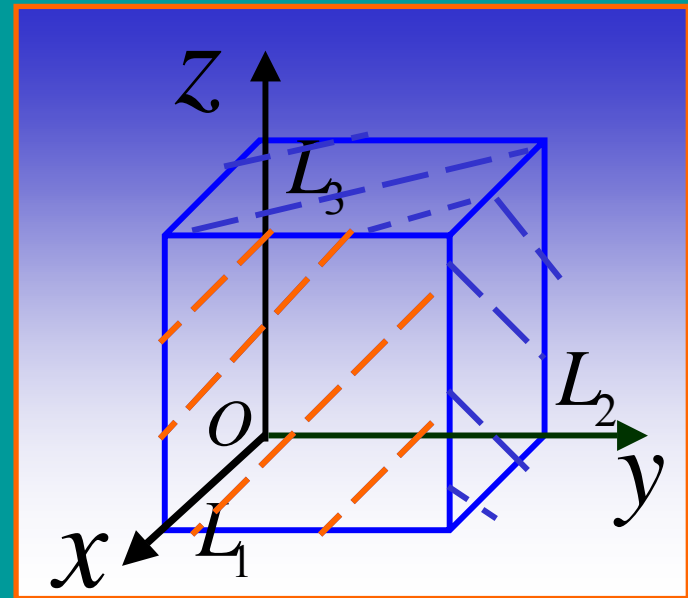
$$-A_1 k_x \sin k_x x \sin k_y y \sin k_z z \Big|_{x=L_1} = 0$$



$$\sin k_x L_1 = 0$$



$$k_x L_1 = m\pi \\ m = 0, 1, 2, \dots$$



$$E_t = 0$$

$$\frac{\partial E_n}{\partial n} = 0$$



$$E_x = A_1 \cos k_x x \sin k_y y \sin k_z z \quad (A_1 = C_1 D_2 D_3)$$

■ $y=L_2$ 面和 $z=L_3$ 面

$$E_x \Big|_{L_2 \text{面}} \longrightarrow E_x \Big|_{y=L_2} = 0$$

$$\longrightarrow A_1 \cos k_x x \sin k_y y \sin k_z z \Big|_{y=L_2} = 0$$

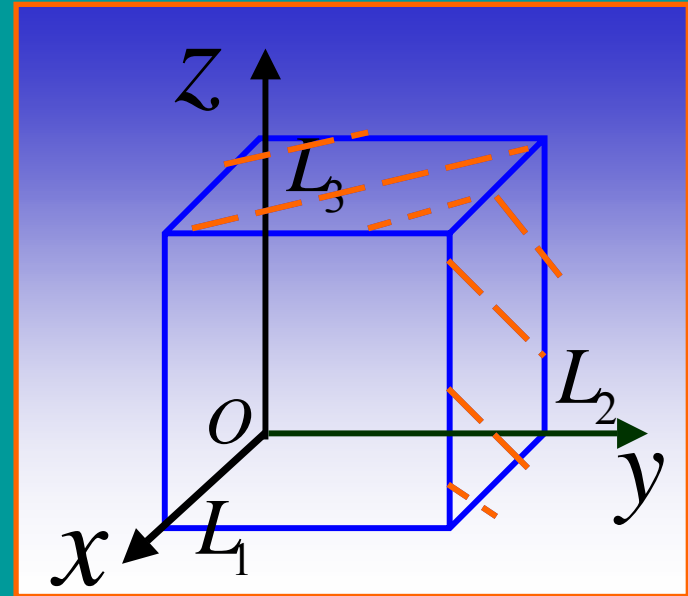
$$\longrightarrow \sin k_y L_2 = 0$$

$$\longrightarrow k_y L_2 = n\pi \quad n = 0, 1, 2, \dots$$

$$E_x \Big|_{L_3 \text{面}} \longrightarrow E_x \Big|_{z=L_3} = 0$$

$$\longrightarrow A_1 \cos k_x x \sin k_y y \sin k_z z \Big|_{z=L_3} = 0 \longrightarrow \sin k_z L_3 = 0$$

$$\longrightarrow k_z L_3 = p\pi \quad p = 0, 1, 2, \dots$$



$$E_t = 0 \quad \frac{\partial E_n}{\partial n} = 0$$



$$k_x = \frac{m\pi}{L_1}, \quad k_y = \frac{n\pi}{L_2}, \quad k_z = \frac{p\pi}{L_3}$$
$$m, n, p = 0, 1, 2, \dots$$

$$\begin{cases} E_x = A_1 \cos k_x x \sin k_y y \sin k_z z \\ E_y = A_2 \sin k_x x \cos k_y y \sin k_z z \\ E_z = A_3 \sin k_x x \sin k_y y \cos k_z z \end{cases}$$

其中 A_1, A_2, A_3 为任意常数

$$\nabla \cdot \vec{E} = 0$$



$$k_x A_1 + k_y A_2 + k_z A_3 = 0$$



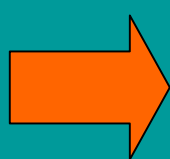
谐振腔内电磁场的本征振荡为

$$\left\{ \begin{array}{l} E_x = A_1 \cos \frac{m\pi}{L_1} x \sin \frac{n\pi}{L_2} y \sin \frac{p\pi}{L_3} z \\ E_y = A_2 \sin \frac{m\pi}{L_1} x \cos \frac{n\pi}{L_2} y \sin \frac{p\pi}{L_3} z \\ E_z = A_3 \sin \frac{m\pi}{L_1} x \sin \frac{n\pi}{L_2} y \cos \frac{p\pi}{L_3} z \\ \frac{m}{L_1} A_1 + \frac{n}{L_2} A_2 + \frac{p}{L_3} A_3 = 0 \end{array} \right.$$



谐振频率或称本征频率

$$\left. \begin{aligned} k^2 &= k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon \\ k_x &= \frac{m\pi}{L_1} \quad k_y = \frac{n\pi}{L_2} \quad k_z = \frac{p\pi}{L_3} \end{aligned} \right\}$$


$$\omega_{mnp} = \frac{\pi}{\sqrt{\mu \varepsilon}} \sqrt{\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2 + \left(\frac{p}{L_3}\right)^2}$$

4. 讨论

- 1) 给定一组(m,n,p), 上述解就代表谐振腔中的一种谐振波型或称本征振荡。腔内可能存在多种谐振波型的迭加。只有当激励信号频率 $\omega = \omega_{mnp}$ 时, 谐振腔才处于谐振态



- 2) 对每一组(m,n,p)值，有两个独立的偏振波形
- 3) 由谐振腔的本征频率计算式可知，有可能存在不同的(m,n,p)组合，其频率相等，这称为“模式简并”。
- 4) 在(m,n,p)组合中，不存在其中两个为零的波形，否则 $\mathbf{E}=0$
- 5) 最低频率谐振波。例如，若 $L_1 > L_2 > L_3$ ，则最低频率的谐振波模为(1, 1, 0)。其谐振频率和谐振波长

$$\omega_{110} = \frac{\pi}{\sqrt{\mu\varepsilon}} \sqrt{\frac{1}{L_1^2} + \frac{1}{L_2^2}} \quad f_{110} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\frac{1}{L_1^2} + \frac{1}{L_2^2}}$$

$$\lambda_{110} = \frac{2L_1L_2}{\sqrt{L_1^2 + L_2^2}}$$



$$\left\{ \begin{array}{l} E_x = A_1 \cos \frac{m\pi}{L_1} x \sin \frac{n\pi}{L_2} y \sin \frac{p\pi}{L_3} z \\ E_y = A_2 \sin \frac{m\pi}{L_1} x \cos \frac{n\pi}{L_2} y \sin \frac{p\pi}{L_3} z \\ E_z = A_3 \sin \frac{m\pi}{L_1} x \sin \frac{n\pi}{L_2} y \cos \frac{p\pi}{L_3} z \end{array} \right.$$

$$\left. \begin{array}{l} m=1 \\ m=1 \\ p=0 \end{array} \right\}$$

波形

$$\left\{ \begin{array}{l} E_x = E_y = 0 \\ E_z = A_3 \sin \frac{\pi}{L_1} x \sin \frac{\pi}{L_2} y \end{array} \right.$$



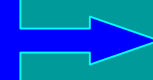
$$\vec{E} = E_z \vec{e}_z$$

$$\vec{k} = \frac{m\pi}{L_1} \hat{e}_x + \frac{n\pi}{L_2} \hat{e}_y + \frac{p\pi}{L_3} \hat{e}_z$$



$$\vec{k} = \frac{\pi}{L_1} \hat{e}_x + \frac{\pi}{L_2} \hat{e}_y$$

横电波



$$\vec{k} \cdot \vec{E} = 0$$



§ 5 波导管

- ❖ 随着近代无线电技术的发展，高频电磁波得到了广泛的应用，这就需要研究高频电磁能量的传输问题
- ❖ 研究表明，随着电磁波频率的提高，电磁能量的传输方法发生了明显的变化
 - 直流电：必须将正负极与用电器连通，采用双线制
 - 交流电：存在多种输电线路
 - 1) 最简单的是双线制
 - 2) 随着频率的升高，平行双线演化为同轴电缆

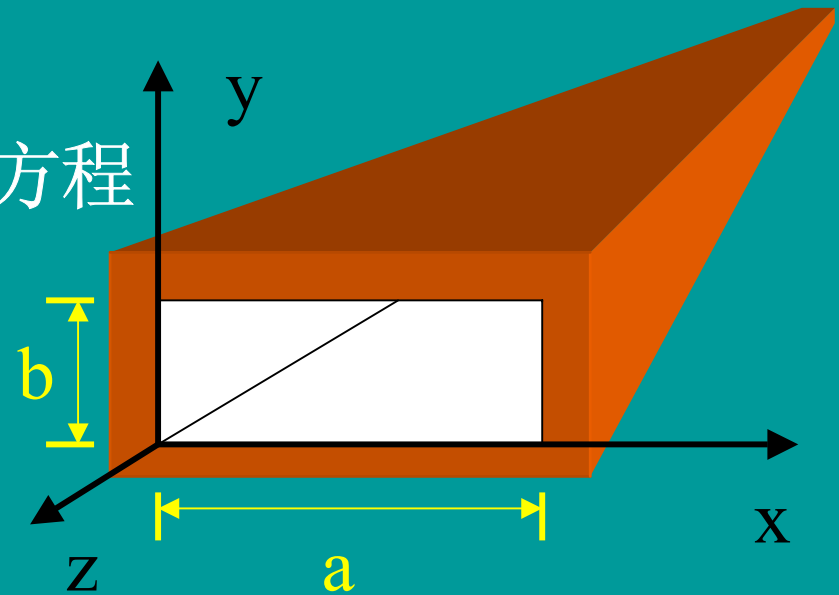


- 3) 频率继续提高，到微波频段，同轴电缆演化为波导技术
 - 4) 频率再提高，到光波频段，金属波导管演化为光缆
- ❖ 波导管是一根中空的金属管，用于微波传输的波导即是一根中空的金属管



1. 矩形波导中电磁波满足的方程

建立直角坐标系。波导中的电磁波沿z轴方向，波导内壁为x=0、a面及y=0、b面



$$\begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 & E_t = 0 \\ \nabla \cdot \vec{E} = 0 & \frac{\partial E_n}{\partial n} = 0 \end{cases}$$

传播因子

电磁波沿z轴传播，波导中电磁波解的形式为

$$\vec{E}(\vec{x}, t) = \vec{E}(x, y) e^{i(k_z z - \omega t)} = \vec{E}(x, y) e^{ik_z z} e^{-i\omega t}$$

$$= \text{[Red Circle]} e^{-i\omega t}$$

$$\vec{E}(\vec{x}) = \vec{E}(x, y) e^{ik_z z}$$



2. 波导中电磁波的解

$$\left. \begin{aligned} \nabla^2 \vec{E} + k^2 \vec{E} &= 0 \\ \vec{E}(\vec{x}) &= \vec{E}(x, y) e^{ik_z z} \end{aligned} \right\}$$

令 $u(x, y)$ 为 \mathbf{E} 的某一直角分量。则每一直角分量均满足

$$k^2 - k_z^2 = k_x^2 + k_y^2$$

→
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \vec{E}(x, y) + (k^2 - k_z^2) \vec{E}(x, y) = 0$$

$$\vec{E}(x, y) = E_x(x, y) \hat{e}_x + E_y(x, y) \hat{e}_y + E_z(x, y) \hat{e}_z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y) + (k^2 - k_z^2) u(x, y) = 0$$



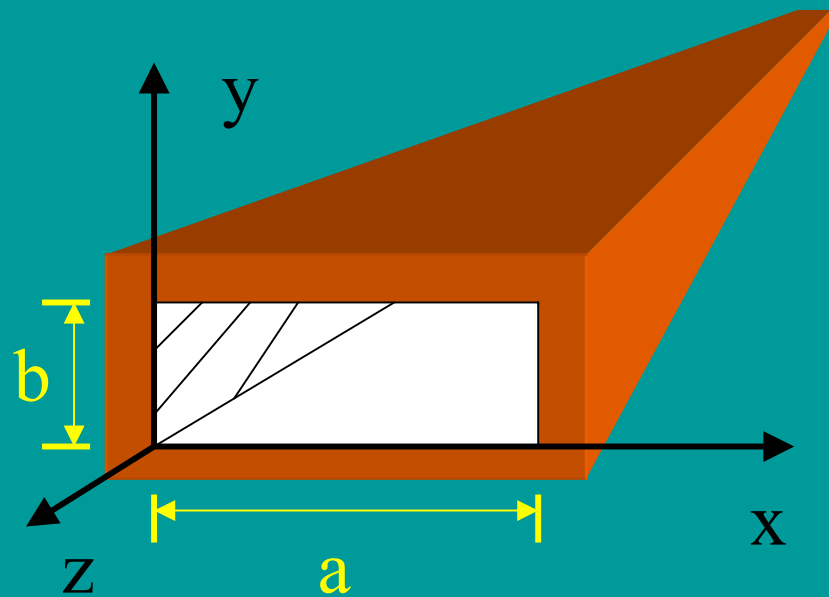
可以用分离变量法得到 $\vec{E}(x, y)$ 各直角分量的通解

$$\begin{cases} E_x(x, y) = (C_{1x} \cos k_x x + D_{1x} \sin k_x x)(C_{2x} \cos k_y y + D_{2x} \sin k_y y) \\ E_y(x, y) = (C_{1y} \cos k_x x + D_{1y} \sin k_x x)(C_{2y} \cos k_y y + D_{2y} \sin k_y y) \\ E_z(x, y) = (C_{1z} \cos k_x x + D_{1z} \sin k_x x)(C_{2z} \cos k_y y + D_{2z} \sin k_y y) \end{cases}$$

由 $x=0$ 面上的边界条件

$$E_y = E_z = 0 \quad \frac{\partial E_x}{\partial x} = 0$$

→ $C_{1y} = C_{1z} = D_{1x} = 0$



$$E_t = 0 \quad \frac{\partial E_n}{\partial n} = 0$$



可以用分离变量法得到 $\vec{E}(x, y)$ 各直角分量的通解

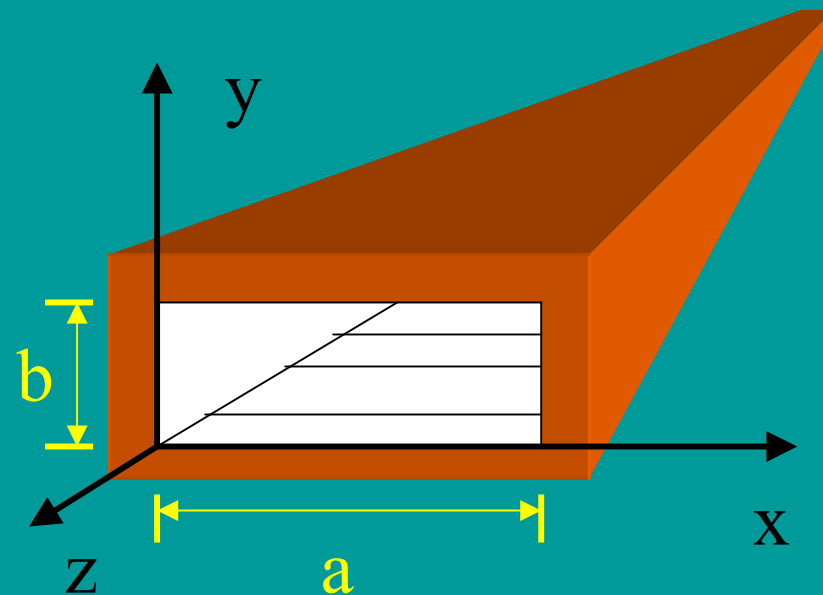
$$\begin{cases} E_x(x, y) = (C_{1x} \cos k_x x + D_{1x} \sin k_x x)(C_{2x} \cos k_y y + D_{2x} \sin k_y y) \\ E_y(x, y) = (C_{1y} \cos k_x x + D_{1y} \sin k_x x)(C_{2y} \cos k_y y + D_{2y} \sin k_y y) \\ E_z(x, y) = (C_{1z} \cos k_x x + D_{1z} \sin k_x x)(C_{2z} \cos k_y y + D_{2z} \sin k_y y) \end{cases}$$

由 $y=0$ 面上的边界条件

$$E_x = E_z = 0 \quad \frac{\partial E_y}{\partial y} = 0$$



$$C_{2x} = C_{2z} = D_{2y} = 0$$



$$E_t = 0 \quad \frac{\partial E_n}{\partial n} = 0$$




E各直角分量的通解

$$\left\{ \begin{aligned} E_x(x, y) &= (C_{1x} \cos k_x x + D_{1x} \sin k_x x)(C_{2x} \cos k_y y + D_{2x} \sin k_y y) \\ E_y(x, y) &= (C_{1y} \cos k_x x + D_{1y} \sin k_x x)(C_{2y} \cos k_y y + D_{2y} \sin k_y y) \\ E_z(x, y) &= (C_{1z} \cos k_x x + D_{1z} \sin k_x x)(C_{2z} \cos k_y y + D_{2z} \sin k_y y) \end{aligned} \right.$$

$$C_{1y} = C_{1z} = D_{1x} = 0$$

$$C_{2x} = C_{2z} = D_{2y} = 0$$


$$\left\{ \begin{aligned} E_x(x, y) &= C_{1x} D_{2x} \cos k_x x \sin k_y y = A_1 \cos k_x x \sin k_y y \\ E_y(x, y) &= D_{1y} C_{2y} \sin k_x x \cos k_y y = A_2 \sin k_x x \cos k_y y \\ E_z(x, y) &= D_{1z} D_{2z} \sin k_x x \sin k_y y = A_3 \sin k_x x \sin k_y y \end{aligned} \right.$$



由 $x=a$, $y=b$ 面上的边界条件可求出 k_x 、 k_y 满足的关系

$$k_x = \frac{m\pi}{a} \quad k_y = \frac{n\pi}{b} \quad m, n \in \{0, 1, 2, \dots\}$$

电磁波的空间量的解

$$\vec{E}(\vec{x}) = E_x(\vec{x})\hat{e}_x + E_y(\vec{x})\hat{e}_y + E_z(\vec{x})\hat{e}_z \quad \longrightarrow \quad \nabla \cdot \vec{E} = 0$$

$$E_x(\vec{x}) = A_1 \cos k_x x \sin k_y y e^{ik_z z}$$

$$E_y(\vec{x}) = A_2 \sin k_x x \cos k_y y e^{ik_z z}$$

$$E_z(\vec{x}) = A_3 \sin k_x x \sin k_y y e^{ik_z z}$$

$$\begin{aligned} 0 &\leq x \leq a \\ 0 &\leq y \leq b \\ -\infty &< z < \infty \end{aligned}$$

$$\longrightarrow \quad \nabla \cdot \vec{E} = \frac{\partial E_x(\vec{x})}{\partial x} + \frac{\partial E_y(\vec{x})}{\partial y} + \frac{\partial E_z(\vec{x})}{\partial z} = 0 \quad \longrightarrow$$



$$A_1 k_x + A_2 k_y - i k_z A_3 = 0$$

由此就求得了矩形波导中所传播的电磁波的解

$$\vec{E}(\vec{x}, t) = \vec{E}(\vec{x}) e^{-i\omega t}$$

$$\vec{E}(\vec{x}) = E_x(\vec{x}) \hat{e}_x + E_y(\vec{x}) \hat{e}_y + E_z(\vec{x}) \hat{e}_z$$

$$E_x(\vec{x}) = A_1 \cos k_x x \sin k_y y e^{i k_z z}$$

$$E_y(\vec{x}) = A_2 \sin k_x x \cos k_y y e^{i k_z z}$$

$$E_z(\vec{x}) = A_3 \sin k_x x \sin k_y y e^{i k_z z}$$

$$k_x = \frac{m\pi}{a}$$

$$k_y = \frac{n\pi}{b}$$

$$m, n \in \{0, 1, 2, \dots\}$$

$$A_1 k_x + A_2 k_y - i k_z A_3 = 0$$

$$\vec{H} = -\frac{i}{\omega\mu} \nabla \times \vec{E}$$



作业： 11, 13