On the connection between the magneto-elliptic and magneto-rotational instabilities

KRZYSZTOF A. MIZERSKI[1]†, AND WLADIMIR LYRA[2,3,4]

[1]Department of Mechanics and Physics of Fluids, Institute of Fundamental and Technological Research, Polish Academy of Sciences, Pawinskiego 5B, 02-106, Poland

[2]Department of Astrophysics, American Museum of Natural History, 79th Street at Central Park West, New York NY 10024-5192, USA

[3]Jet Propulsion Laboratory, 4800 Oak Grove Drive, Pasadena CA 91109, USA

[4]NASA Carl Sagan Fellow

(Received 11 July 2011; Accepted 15 February 2012.)

It has been recently suggested that the magneto-rotational instability (MRI) is a limiting case of the magneto-elliptic instability (MEI). This limit is obtained for horizontal modes in the presence of rotation and an external vertical magnetic field, when the aspect ratio of the elliptic streamlines tends to infinite. In this paper we unveil the link between these previously unconnected mechanisms, explaining both the MEI and the MRI as different manifestations of the same *Magneto-Elliptic-Rotational Instability* (MERI). The growth rates are found and the influence of the magnetic and rotational effects is explained, in particular the effect of the magnetic field on the range of negative Rossby numbers at which the horizontal instability is excited. Furthermore, we show how the horizontal rotational MEI in the rotating shear flow limit links to the MRI by the use of the local shearing box model, typically used in the study of accretion discs. In such limit the growth rates of the two instability types coincide for any power-type background angular velocity radial profile with negative exponent corresponding to the value of the Rossby number of the rotating shear flow. The MRI requirement for instability is that the background angular velocity profile is a decreasing function of the distance from the centre of the disk which corresponds to the horizontal rotational MEI requirement of negative Rossby numbers. Finally a physical interpretation of the horizontal instability, based on a balance between the strain, the Lorentz force and the Coriolis force is given.

Key words:

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1. Introduction

The stability of vortices is a problem of paramount importance in fluid mechanics. Considering that turbulence consists of vorticity blobs, vortices are the fundamental unit of turbulent flow. Unveiling the mechanism that renders them unstable should therefore provide vital insights into the nature of turbulence itself. Phenomenologically, turbulence can be described as a series of bifurcations, starting with a primary instability that converts shear into vorticity, creating vortices. This is followed by another bifurcation, a secondary instability, to break these vortices into lesser vortical structures. These in turn

† Email address for correspondence: kamiz@ippt.gov.pl

shall experience a sequence of "inertial instabilities", leading to a cascade. Though the Kelvin-Helmholtz instability and the Rayleigh-Taylor instability are well established as examples of primary instabilities, the highly successful theory of the turbulent cascade put forth by Kolmogorov (1941) rested on a heuristic picture of secondary instability, established by early experiments (e.g., Taylor 1923). It was not until the works of Pierrehumbert (1986) and Bayly (1986) that the elliptic instability was introduced as a mechanism for the secondary instability. A fluid in rigid rotation supports a spectrum of stable inertial waves, the simplest case being circularly polarized transverse plane waves oscillating at twice the frequency of the base flow. Strain is introduced when the streamlines pass from circular to elliptical, and some modes find resonance with the strain field, leading to de-stabilization. The nonlinear evolution and saturation of the elliptic instability was studied numerically by Schaeffer & Le Dizès (2010) and experimentally by Eloy *et al.* (2003). Herreman *et al.* (2010) generalized the linear and weakly nonlinear theories for the elliptic instability (Waleffe 1989) to the magnetohydrodynamics case and conducted experiments to explain some aspects of the nonlinear transition process.

The seminal work of Pierrehumbert (1986) and Bayly (1986) was followed by studies that considered the incorporation of more physics into the problem. Miyakazi (1993) included background rotation, unveiling a strong two-dimensional mode, for which the wavevector and the vortex axis are parallel. For incompressible flow such modes are solenoidal, so the motion itself is horizontal to the vortex axis. This horizontal instability is thus the two-dimensional mode of what could be called an elliptic-rotational instability (ERI). This instability is not of resonant nature, but centrifugal, appearing as exponential growth of epicyclic disturbances. This behaviour invites a connection with the Rayleigh (centrifugal) instability, and indeed the mechanism is similar (Lesur & Papaloiozou 2009), suggesting that the Rayleigh instability is a limit of the ERI.

If the elliptic instability is such a fundamental process for understanding hydrodynamical turbulence, one can expect that the magneto-elliptic instability (MEI) should likewise provide a similar framework when it comes to MHD turbulence. In the geophysical context, Kerswell (1993) studied the influence of a toroidal magnetic field and stratification, both axial and radial, on the stability of elliptic flow in uniform rotation. In his configuration, axial stratification and the magnetic field were invariably stabilizing, though radial stratification enhanced the elliptical instability. Lebovitz & Zweibel (2004) studied the problem of stability of axially magnetized elliptic streamlines, finding that the MEI has new families of destabilizing resonances, corresponding to Alfvén waves. Just as Miyakazi (1993) extended the work of Bayly (1986) to include background rotation, the work of Lebovitz & Zwebeil (2004) was similarly extended by Mizerski & Bajer (2009), who analysed the joint effect of the magnetic and rotational effects on the stability of the Euler flow with elliptical streamlines, by the use of numerical and asymptotic analytical methods in the limit of small ellipticity of the flow. Their comprehensive study, based on the analytic technique developed by Lebovitz & Zweibel (2004) included both the modes excited via a resonance mechanism and the horizontal modes, that propagate along the magnetic field lines. The case of the horizontal instability was then further investigated in the non-magnetic case by Lesur & Papaloizou (2009) and in the magnetized case by Bajer & Mizerski (2011). However, the problem of the MEI in the presence of rotation is conceptually difficult and there are aspects which still require clarification and further insight is necessary.

One such insight may come from astrophysics, which also provides the motivation for the present study. When modeling the problem of magnetized vortices in protoplanetary disks, Lyra & Klahr (2011) found that the horizontal magneto-elliptic mode, when taken to the limit of infinite ellipticity, yields the same growth rates and range of unstable wavenumbers as the well known magneto-rotational instability (MRI, Velikhov 1959, Chandrasekhar 1960). The MRI is an instability of magnetized Euler flow in circular differential rotation, which renders the motion unstable when the angular velocity decreases outwards. Since Keplerian flow is an Euler flow in circular differential rotation, and magnetic fields are ubiquitous in astrophysics, the MRI is the best candidate to generate turbulence in accretion disks (Balbus & Hawley 1991, 1998, Armitage 1998, Hawley 2000, Fromang & Nelson 2006, Lyra et al. 2008, Flock et al. 2011).

The result of Lyra & Klahr (2011) tantalizingly suggests that the MRI is a limiting case of the MEI. In this paper we explore this interesting prospect in more detail, unveiling the connection between these two previously unconnected instabilities. In a way, the present study is an attempt at unification, explaining both instabilities as different manifestations of the same *Magneto-Elliptic-Rotational Instability* (MERI).

The paper is structured as follows. In the next section we briefly review the horizontal rotational MEI presenting a somewhat different approach from Bajer & Mizerski (2011) and show that the external magnetic field has a profound effect on the range of values of the Rossby number for which the horizontal rotational MEI is excited. In Sect. 3 we further develop the idea presented in Lyra & Klahr (2011) and demonstrate, by the use of the so-called local shearing box model (cf. Hawley et al. 1995) how the horizontal rotational MEI, in the limit of rotating shear flow can be related to the MRI. Next we present a physical interpretation of the horizontal instability based on a balance between strain generated by the basic flow and the Coriolis and Lorentz forces. Conclusions are presented in Sect. 5.

2. The calculation of growth rates of the horizontal rotational MEI

Let us consider the Euler flow with elliptic streamlines,

$$\mathbf{u}_0 = \gamma\left[-\left(1+\varepsilon\right)y,\, \left(1-\varepsilon\right)x\right] \tag{2.1}$$

rotating with the angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{e}}_z$, where $2\gamma\hat{\mathbf{e}}_z$ is its uniform vorticity and $0 < \varepsilon < 1$ is the strain. We now briefly review the stability properties of the flow (2.1) in the presence of uniform vertical field $\mathbf{B_0} = B_0\hat{\mathbf{e}}_z$ with respect to the horizontal modes, which are defined by the fact that they propagate only in the vertical direction, i.e. the wave vector of the perturbation has only the 'z'-component. The solenoidal constraints for the velocity and magnetic fields then imply that the perturbations have only horizontal components. Thus the velocity and magnetic field perturbations have the form

$$\mathbf{v} = \hat{\mathbf{v}}\mathrm{e}^{\mathrm{i}kz} \qquad \text{and} \qquad \mathbf{b} = \hat{\mathbf{b}}\mathrm{e}^{\mathrm{i}kz} \tag{2.2}$$

respectively. The equations to solve are the Euler and induction equations

$$\partial_t v_i + v_j\partial v_i = -\frac{1}{\rho}\partial_i P - 2O_{ij}v_j + \frac{1}{\rho\mu_0}B_k\partial_k B_i, \tag{2.3}$$

$$\partial_t B_i + v_j\partial_j B_i = B_j\partial_j B_i, \tag{2.4}$$

where the magnetic pressure and centrifugal potential were incorporated in the pressure P and $O_{ik} = \varepsilon_{ijk}\Omega_j$. We apply the scaling $b_i \rightarrow b_i/\sqrt{\mu_0\rho}$ so that, in principle, we are solving for the Alfvén velocity and after linearisation the evolution equations for the

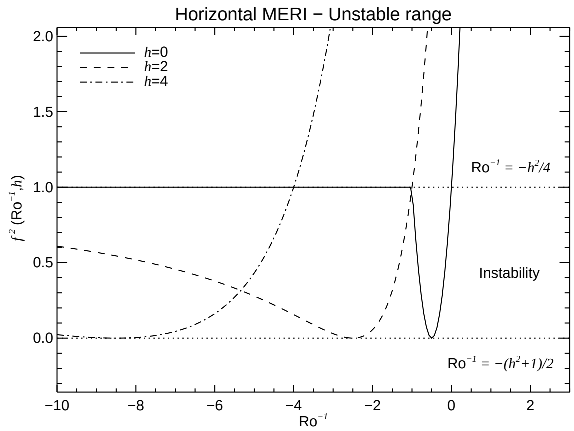

FIGURE 1. The square of the function ψ_+ as in Eq. (2.7), that defines the unstable horizontal modes of the magneto-elliptic-rotational instability. Instability exists when $|\psi_+| < \varepsilon$, where $0 \leqslant \varepsilon \leqslant 1$ is a measure of eccentricity. The instability exists only for negative Rossby numbers, i.e., anticyclonic rotation. The unstable range of wavenumbers is $\mathrm{Ro}^{-1} < -h^2/4$ and unbounded.

horizontal rotational MEI modes yield

$$\begin{bmatrix} \dot{\hat{v}}_x \\ \dot{\hat{v}}_y \\ \dot{\hat{b}}_x \\ \dot{\hat{b}}_y \end{bmatrix} = \begin{bmatrix} 0 & 1+\varepsilon+2\mathrm{Ro}^{-1} & ih & 0 \\ -(1-\varepsilon)-2\mathrm{Ro}^{-1} & 0 & 0 & ih \\ ih & 0 & 0 & -(1+\varepsilon) \\ 0 & ih & 1-\varepsilon & 0 \end{bmatrix} \begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{b}_x \\ \hat{b}_y \end{bmatrix} \tag{2.5}$$

where $\tau = t/\gamma$ and the upper dot denotes a derivative with respect to τ, $\mathrm{Ro} = \gamma/\Omega$ is the Rossby number and $h = kB_0/\gamma\sqrt{\mu_0\rho}$. The eigenvalues of the matrix in (2.5) which we denote by σ_j, $j = 1, 2, 3, 4$ are the complex growth rates of the perturbations. They are

$$\sigma_1/\gamma = \sqrt{\varepsilon^2 - \psi_+^2}, \qquad \sigma_2/\gamma = -\sqrt{\varepsilon^2 - \psi_+^2}, \tag{2.6a}$$

$$\sigma_3/\gamma = \sqrt{\varepsilon^2 - \psi_-^2}, \qquad \sigma_4/\gamma = -\sqrt{\varepsilon^2 - \psi_-^2}, \tag{2.6b}$$

where

$$\psi_\pm = \mathrm{Ro}^{-1} \pm \sqrt{\left(\mathrm{Ro}^{-1}+1\right)^2 + h^2}. \tag{2.7}$$

Note that for all $h \geqslant 0$ and $\mathrm{Ro} \in \mathbb{R}$

$$|\psi_-| \geqslant 1 \geqslant \varepsilon, \tag{2.8}$$

which means that σ_3 and σ_4 are imaginary, thus produce only oscillations. The basic state is unstable with respect to perturbations propagating along the 'z' axis if and only if there exist Ro and h such that

$$|\psi_+| < 1 \qquad \text{and} \qquad \varepsilon \geqslant |\psi_+|. \tag{2.9}$$

The function $\psi_+\left(\mathrm{Ro}^{-1}; h\right)$, plotted in Fig. 1, possesses the following properties

$$\psi_+ \geqslant 1 \;;\; \text{for} \;\; \mathrm{Ro}^{-1} \geqslant -\frac{h^2}{4} \tag{2.10a}$$

$$\psi_+ = 1 \;;\; \text{for} \;\; \mathrm{Ro}^{-1} = -h^2/4 \tag{2.10b}$$

$$0 \leqslant |\psi_+| < 1 \;;\; \text{for} \;\; \mathrm{Ro}^{-1} < -\frac{h^2}{4} \tag{2.10c}$$

$$\psi_+ = 0 \;;\; \text{for} \;\; \mathrm{Ro}^{-1} = -(h^2+1)/2 \tag{2.10d}$$

and

$$\lim_{\mathrm{Ro}^{-1}\to-\infty} \psi_+\left(\mathrm{Ro}^{-1};h\right) = \tag{2.11}$$
$$\lim_{\mathrm{Ro}^{-1}\to-\infty} \mathrm{Ro}^{-1}\left\{1-\left[1+\mathrm{Ro}+\mathcal{O}\left(\mathrm{Ro}^2\right)\right]\right\} = -1.$$

Therefore the instability is only possible if $\mathrm{Ro}^{-1} < -h^2/4$ since only then, for a certain ellipticity $\varepsilon \geqslant |\psi_+|$ the growth rate σ_1 can be real and positive. Moreover, equation (2.12), in particular, means that as Ro^{-1} tends to $-\infty$, for some $\mathrm{Ro}^{-1} < -(h^2+1)/2$ the function $\psi_+^2(\mathrm{Ro}^{-1};h)$ must have an inflection point. The limit $h \to 0$ is singular in the sense that the two curves $\psi_+^2(\mathrm{Ro}^{-1};h)$ and $\psi_-^2(\mathrm{Ro}^{-1};h)$ merge into a parabola (and a straight line parallel to the Ro^{-1} axis of constant value 1), touching each other at the inflection point of the function $\psi_+^2(\mathrm{Ro}^{-1};h)$ at $\mathrm{Ro}^{-1} = -1$, $\psi_+^2 = \psi_-^2 = 1$. This means that for $h = 0$ the horizontal instability is possible only for a bounded set of values $\mathrm{Ro}^{-1} \in (-1,0)$. The situation significantly changes immediately after switching on even a very weak magnetic field since then the domain in which the horizontal instability is possible becomes unbounded, i.e. $\mathrm{Ro}^{-1} \in (-\infty, -h^2/4)$. As the instability exists only for negative Rossby numbers, only anticyclonic vortices are unstable.

As Lyra & Klahr (2011) have pointed out, there is instability in the limit of a vanishing vortex in a shearing box (cf. Hawley et al. 1995). In that case, the Rossby number remains finite owing to the vorticity of the Keplerian shear, and the case is equivalent to a Kida vortex (Kida 1981) of infinite aspect ratio

$$\varepsilon = 1 \qquad \text{and} \qquad \mathrm{Ro} = -\frac{3}{4} \tag{2.12}$$

In the limit $\varepsilon = 1$ only the horizontal instability persists (the purely horizontal modes are the most unstable ones and the modes destabilised via resonances are suppressed; cf. Mizerski & Bajer 2009, 2011), for which the dispersion relation is (cf. equation (2.6a))

$$\frac{\sigma^2}{\gamma^2} = \varepsilon^2 - \left[\mathrm{Ro}^{-1} + \sqrt{\left(\mathrm{Ro}^{-1}+1\right)^2 + \frac{k^2 v_A^2}{\gamma^2}}\right]^2, \tag{2.13}$$

where $v_A = B_0/\sqrt{\mu_0\rho}$ is the Alfvén speed. For $\mathrm{Ro} < 0$, also substituting $\gamma = \mathrm{Ro}\,\Omega$ and $q = kv_A/\Omega$, we obtain

$$\frac{\sigma^2}{\gamma^2} = \varepsilon^2 - \mathrm{Ro}^{-2}\left[\sqrt{(\mathrm{Ro}+1)^2 + q^2} - 1\right]^2, \tag{2.14}$$

and applying the limits defined by (2.12)

$$\frac{\sigma^2}{\gamma^2} = \frac{8}{9}\left(\sqrt{16q^2+1} - 2q^2 - 1\right). \tag{2.15}$$

Thus the growth rate is real and positive for $0 < q < \sqrt{3}$, and the maximal value $\sigma = |\gamma| = (3/4)\,|\Omega|$ is achieved for $q = \sqrt{15}/4 \approx 0.9682$. This is precisely the same dispersion relation obtained for the MRI by Balbus & Hawley (1991, see also Hawley & Balbus 1991).

3. Why do we see the MRI in a rotating shear flow?

The MRI was originally formulated for axisymmetric flows with circular streamlines and the condition for MRI to operate is that the angular velocity decreases with radius. However, as it turns out, the MRI also seems to operate in non-axisymmetric flows and appears as a limiting case of the MEI in the presence of rotation, in the limit when the latter becomes a rotating shear flow (Lyra & Klahr 2011). Let us now assume that the Euler flow with elliptic streamlines given in (2.1), rotates with the angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{e}}_z$ about an axis parallel the 'z' axis and located at distance R from the centre of the elliptical vortex. Let us also consider the inertial (non-rotating) system of reference (x',y',z'), with the origin $x' = 0$, $y' = 0$ at the axis of rotation (see Fig. 2). In this frame of reference the vector of position of the centre of the elliptical vortex is

$$\mathbf{R}' = R\left[-\sin\Omega t,\, \cos\Omega t\right], \tag{3.1}$$

where the prime indicates representation in the non-rotating frame and

$$\frac{\mathrm{d}\mathbf{R}'}{\mathrm{d}t} = \mathbf{\Omega}\times\mathbf{R}', \tag{3.2}$$

where the time derivative is taken within the non-rotating frame, i.e. holding the unit vectors $\hat{\mathbf{e}}'_x$ and $\hat{\mathbf{e}}'_y$ constant. It was chosen that

$$\mathbf{R}'\left(t=0\right) = R\hat{\mathbf{e}}'_y. \tag{3.3}$$

Using the transformations

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 0 \\ R \end{bmatrix}, \qquad \begin{aligned} \hat{\mathbf{e}}_x &= \cos\Omega t\hat{\mathbf{e}}'_x + \sin\Omega t\hat{\mathbf{e}}'_y \\ \hat{\mathbf{e}}_y &= -\sin\Omega t\hat{\mathbf{e}}'_x + \cos\Omega t\hat{\mathbf{e}}'_y \end{aligned} \tag{3.4}$$

and

$$\mathbf{u}'_0 = \mathbf{u}_0 + \mathbf{\Omega}\times\mathbf{r}' + \frac{\mathrm{d}\mathbf{R}'}{\mathrm{d}t} - \mathbf{\Omega}\times\mathbf{R}' = \mathbf{u} + \mathbf{\Omega}\times\mathbf{r}' \tag{3.5}$$

where $\mathbf{r}' = [x', y', z']^T$, we obtain the following expression for the basic elliptic velocity field in the inertial frame

$$u'_{0x} = -\gamma\left[y' + \varepsilon\left(y'\cos 2\Omega t - x'\sin 2\Omega t\right)\right] + \gamma\left(1+\varepsilon\right)R\cos\Omega t - \Omega y', \tag{3.6}$$

$$u'_{0y} = \gamma\left[x' - \varepsilon\left(x'\cos 2\Omega t + y'\sin 2\Omega t\right)\right] + \gamma\left(1+\varepsilon\right)R\sin\Omega t + \Omega x'. \tag{3.7}$$

Using the cylindrical coordinates (r',φ',z) we may write the angular velocity of the flow in the inertial frame as

$$\frac{u'_{0\varphi}}{r'} = \frac{u'_{0y}x' - u'_{0x}y'}{r'^2} = \Omega\left\{1 + \mathrm{Ro}\left[1 - \varepsilon\cos\left(2\varphi' - 2\Omega t\right)\right] - \mathrm{Ro}\left(1+\varepsilon\right)\frac{R}{r'}\sin\left(\varphi' - \Omega t\right)\right\}, \tag{3.8}$$

which, clearly, is a function of r' (as long as $R \neq 0$) and φ' (in the above $\mathrm{Ro} = \gamma/\Omega$ is the Rossby number). Lyra & Klahr (2011) have shown numerically that in the rotating shear flow limit defined in (2.12) the rotational MEI growth rate (i.e. the horizontal instability growth rate) profile with the parameter q indeed matches the MRI profile. In such limit the expression (3.8) yields

$$\frac{u'_{0\varphi}}{r'} = \Omega\left\{\frac{1}{4} + \frac{3}{4}\cos\left(2\varphi' - 2\Omega t\right) + \frac{3}{2}\frac{R}{r'}\sin\left(\varphi' - \Omega t\right)\right\}, \tag{3.9}$$

therefore the angular velocity in the inertial frame, in the rotating shear flow limit, is depended on the distance from the origin as long as $R \neq 0$. Moreover, the flow in the

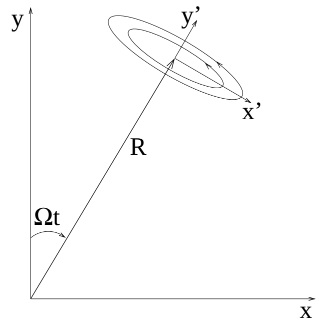

FIGURE 2. A schematic picture of the rotating elliptic flow.

inertial frame is not axially symmetric, but nevertheless the resemblance to the MRI is observed. Furthermore the linear stability analysis which leads to the expressions for the growth rate (2.13) and thus (2.14) is independent of the value of R which means that even for $R=0$, i.e. when the angular velocity is independent of r' we obtain the MRI-like results in the rotating shear flow limit of the rotational MEI. Since the Rossby number must be negative for the instability to settle in, it seems that it is crucial that the MRI requirement for the angular velocity to be a decreasing function of r' must be met in the neighborhood of the centre of the rotating elliptic vortex, i.e. for $x=0$, $y=0$. In principle for a given value of the Rossby number the angular velocity (3.9) is a decreasing function of r' only in a half-plane and an increasing function of r' in the other half-plane, depending on the signs of Ro and $\sin(\varphi'-\Omega t)$. However, as said, the instability occurs only for $\mathrm{Ro}<0$, in which case the angular velocity decreases with r' for $\Omega t<\varphi'<\pi+\Omega t$, and the steepest decrease is observed at $\varphi'=\pi/2+\Omega t$, which is the azimuthal coordinate of the centre of the elliptic vortex.
The explanation is based on the local shearing sheet model for accretion disks, since in this model the shearing flow is simply a local representation of the background rotation of the axisymmetric vortex with angular velocity decreasing with r'. In other words for an axisymmetric vortex with the angular velocity depended on r' as $\Omega(r')\sim 1/r'^s$ and thus gravity as $g\sim r'^n$, with $n=-2s+1$, in the vicinity of $r'=R$ we get

$$\Omega(r)\sim\frac{1}{(R+r)^s}\approx\frac{1}{R^s}-sr\frac{1}{R^{s+1}}, \tag{3.10}$$

and from (3.8), in the vicinity of $\mathbf{r}'=\mathbf{R}'$, i.e. for $r'=R+r$ and $\varphi'=\pi/2+\Omega t+\varphi$, we obtain

$$\frac{u'_{0\varphi}}{r'}=\Omega\left\{1+\mathrm{Ro}+\mathrm{Ro}\cos(2\varphi)-2\mathrm{Ro}\cos\varphi+2\mathrm{Ro}\frac{r}{R}\cos\varphi\right\}. \tag{3.11}$$

According to the shearing sheet approximation we now substitute in (3.11) for $\Omega=1/R^s$ and since $\cos\varphi\approx 1+\mathrm{O}(\varphi^2)$ the expression (3.11) up to order one in the distance from the point $\mathbf{r}'=\mathbf{R}'$ (thus in the region of validity of the shearing sheet approximation) takes the form

$$\frac{u'_{0\varphi}}{r'}\approx\frac{1}{R^s}+2\mathrm{Ro}\frac{r}{R^{s+1}}, \tag{3.12}$$

which agrees with (3.10) for

$$\mathrm{Ro}=-\frac{s}{2}=\frac{n-1}{4}. \tag{3.13}$$

The Kida (1981) solution for a stationary elliptic vortex embedded in a shear flow yields

$$\mathrm{Ro} = -\frac{s}{2}\frac{2}{\left(1+\varepsilon-\sqrt{1-\varepsilon^2}\right)}, \tag{3.14}$$

which in the shear flow limit $\varepsilon = 1$ agrees with (3.13). The horizontal rotational MEI in the presence of a magnetic field operates for all negative Rossby numbers and according to (3.13) this corresponds to $s > 0$, i.e. the angular velocity $\Omega(r')$ decreasing with r', which is the requirement for the MRI. It is now clear, that since the growth rate of the horizontal instability in a rotating shear flow (2.13) does not depend on position, and at the centre of the elliptical vortex $\mathbf{r}' = \mathbf{R}'$ the rotating shear flow models locally the flow of an axisymmetric disk with angular velocity decreasing with r', the growth rate must correspond to the growth rate of the MRI taken at $\mathbf{r}' = \mathbf{R}'$, since in the short-wavelength limit analysed by Balbus & Hawley (1991) the MRI criterion is local. Nevertheless it remains an interesting feature of the rotating shear flow limit that despite the fact that the flow is not axially symmetric and for $R = 0$ can even be independent of r' the stability characteristics are the same as of the classic axially symmetric, short-wavelength MRI.

It is also important to realise that, in general, the horizontal instability persists even without the presence of the magnetic field. The MRI operates only if the magnetic field is present. Furthermore, as already mentioned, even the r'-dependence of the angular velocity is not necessary for the development of the horizontal instability, since if $R = 0$ the expression (3.8) becomes independent of r' and the growth rate (2.13) is not affected. Thus, in principle, it is the lack of axial symmetry that is crucial for the development of the horizontal rotational MEI. Nevertheless there exists a common physical ground for both the instability types, i.e. horizontal rotational MEI and MRI, since the rotating shear flow limit of the rotational MEI, even though non-axially symmetric, exhibits the properties of the MRI. The common feature of both instability types is the shear which is present in the elliptic flows as well as in axisymmetric flows with angular velocity dependent on radius and this feature accompanied by the rotational effects (local in MRI and global in rotational MEI) is behind the physical interpretation of the common limit for both instabilities discussed in here. This point is elaborated in the next Section.

We have, therefore, demonstrated the direct link between the megneto-elliptic and magneto-rotational instabilities and in the non-magnetic case between the horizontal elliptic and centrifugal instabilities and the physical conditions under which previously unrelated instability types can be linked were explained in detail. The dispersion relations for horizontal MEI and MRI, likewise for horizontal EI and the centrifugal instability are the same in the limit $\epsilon = 1$, which suggests, that the physics of the horizontal magneto-elliptic instability is similar to the physics of the MRI.

Let us make a final comment on the non-magnetic case. As mentioned the horizontal instability is present even if the magnetic field is not. The axisymmetric flows in the non-magnetic case are only subject to the centrifugal instability for which the Rayleigh's criterion says that instability can occur if the angular momentum is a decreasing function of r' in some region. According to (3.10), (3.12) and (3.13) the angular momentum of an axisymmetric vortex decreases with r' if the Rossby number for the rotating shear flow satisfies $\mathrm{Ro} < -1$. This is an exact instability condition for the horizontal modes in the non-magnetic case (cf. Mizerski & Bajer 2009) for a rotating shear flow, i.e. for $\varepsilon = 1$, since in such case the growth rate of the horizontal modes is (cf. (2.13) and (3.13))

$$\sigma^2 = 2\,(s-2)\,\Omega^2\,(R)\,. \tag{3.15}$$

On the other hand, the classic Rayleigh's criterion, formulated for systems with impermeable boundaries at $r' = R_1$ and $r' = R_2$, for horizontal modes and $\Omega(r') \sim 1/r'^s$

yields

$$\sigma^2 = -2\frac{s-2}{s-1}\frac{R_2^{2-2s} - R_1^{2-2s}}{R_2^2 - R_1^2}. \qquad (3.16)$$

If the Rayleigh's criterion is made local, by assuming a narrow gap limit, i.e. $R_2 = R_1 + \delta r$ and $\delta r / R_1 \ll 1$ the above expression (3.16) reduces to

$$\sigma^2 = 2\,(s-2)\,\Omega^2\,(R_1)\,, \qquad (3.17)$$

which agrees with (3.15).

It seems in place to briefly comment on the case when the applied field in a disc is azimuthal, since the numerical and experimental works of Hollerbach & Rüdiger (2005) and Stefani *et al.* (2006) suggest a great importance of the azimuthal field for the reduction of the instability thereshold. For the rotating shear flow $\mathbf{u}_0 = -2\gamma y \hat{\mathbf{e}}_x$ modeling locally the flow in a disc one could take $\mathbf{B_0} = (B_0 + J_0 y)\hat{\mathbf{e}}_x$. Such stability problem with perturbations in the form of horizontal modes takes the form

$$\frac{\sigma}{\gamma}\begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{b}_x \end{bmatrix} = \begin{bmatrix} 0 & 2\left(1+\mathrm{Ro}^{-1}\right) & 0 \\ -2\mathrm{Ro}^{-1} & 0 & -\Gamma \\ 0 & -\Gamma & 0 \end{bmatrix}\begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{b}_x \end{bmatrix}, \qquad (3.18)$$

with $\hat{b}_y = 0$ and $\Gamma = J_0/(\gamma\sqrt{\mu_0\rho})$. Solving for the eigenvalue one obtaines

$$\frac{\sigma^2}{\gamma^2} = -4Ro^{-1}\left(1 + Ro^{-1}\right) + \Gamma^2, \qquad (3.19)$$

which leads to a conclusion that the instability is possible if either $Ro < -2/(1+\sqrt{1+\Gamma^2})$ or $Ro > 2/(\sqrt{1+\Gamma^2} - 1)$ is satisfied. Thus a new unstable branch appears for large and positive Rossby numbers, which has the property, that for weak field gradients, $\Gamma \ll 1$ only strongly shearing discs are affected, $Ro = -s/2 \gg 1$. In principle the correspondence between the stability characteristics of the rotating shear flow with horizontal field and the local stability characteristics of a disc with azimuthal field should exist, however, the horizontal modes anaysed here are independent of y, thus axisymmetric in the global sense and as argued by Terquem & Papaloizou (1996) the most unstable modes in the case of azimuthal fields are the non-axisymmetric ones. Therefore the dynamics of discs with azimuthal basic fields should be dominated by the growth of non-axisymmetric modes not analysed in here.

4. Physical interpretation of the horizontal rotational MEI

The essence of the rotational MEI is the departure from axial symmetry, which imposes strain on the perturbations. The typical model based on consideration of radial perturbations which preserve angular momentum, useful for physical interpretation of the MRI and the centrifugal instability is no longer applicable when the vortex is elliptical. A similarly simple physical model of the rotational MEI would have to take into account not only the elliptical shape of the vortex but also the action of the Coriolis force, which is essential for the instability mechanism. We present a somewhat simpler approach and provide the physical explanation of the horizontal rotational MEI based on the balance of forces acting on a general horizontal perturbation, highlighting the importance of strain. Let us write down the equations governing the evolution of the horizontal modes, for which the wave vector is vertical, $\mathbf{k} = k\hat{e}_z$, and thus, because of the divergence free constraint, the perturbations posses only the horizontal components.

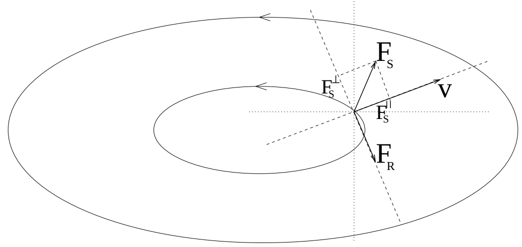

FIGURE 3. A sketch of the forces acting on fluid element in the presence of horizontal perturbation.

The Navier-Stokes equation in the rotating frame and in the non-magnetic case is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{u}_0 = -2\boldsymbol{\Omega} \times \mathbf{v}, \tag{4.1}$$

where $\mathbf{u}_0$ is the basic elliptic flow defined in (2.1) and $\mathbf{v}$ is the horizontal perturbation in the rotating frame. Note that the term describing advection of the perturbation by the elliptic flow, $(\mathbf{u}_0 \cdot \nabla)\mathbf{v}$ and the pressure terms vanish in the case of purely horizontal perturbations and the only terms that remain are the terms describing the stretching of the perturbation by the basic flow and the Coriolis force. Introducing $\mathbf{v} = \hat{\mathbf{v}}\mathrm{e}^{\mathrm{i}kz}$ and $\tau = \gamma t$ we can rewrite the equation (4.1) in the following form,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \begin{bmatrix} \hat{v}_x \\ \hat{v}_y \end{bmatrix} = \left(2\mathrm{Ro}^{-1} + 1\right) \begin{bmatrix} \hat{v}_y \\ -\hat{v}_x \end{bmatrix} + \varepsilon \begin{bmatrix} \hat{v}_y \\ \hat{v}_x \end{bmatrix}, \tag{4.2}$$

where $\mathrm{Ro} = \gamma/\Omega$ is the Rossby number. The first term on the right hand side of (4.2) corresponds only to local rotation and the second one to the local rotation and strain (in the following we will refer to the first term as the 'pure rotation term' and to the second one as the 'straining term'). We can easily solve the above equations (4.2) to obtain

$$\hat{v}_x = \frac{1}{2}\left(\chi^{-1}\hat{v}_{y0} + \hat{v}_{x0}\right)\mathrm{e}^{\sigma\tau/\gamma} + \frac{1}{2}\left(-\chi^{-1}\hat{v}_{y0} + \hat{v}_{x0}\right)\mathrm{e}^{-\sigma\tau/\gamma}, \tag{4.3}$$

$$\hat{v}_y = \frac{1}{2}\left(\hat{v}_{y0} + \chi\hat{v}_{x0}\right)\mathrm{e}^{\sigma\tau/\gamma} - \frac{1}{2}\left(-\hat{v}_{y0} + \chi\hat{v}_{x0}\right)\mathrm{e}^{-\sigma\tau/\gamma}, \tag{4.4}$$

where we have assumed that $\varepsilon^2 > \left(1 + 2\mathrm{Ro}^{-1}\right)^2$ (thus the system is unstable) and

$$\chi = \sqrt{\frac{\varepsilon - \left(1 + 2\mathrm{Ro}^{-1}\right)}{\varepsilon + \left(1 + 2\mathrm{Ro}^{-1}\right)}}, \qquad \frac{\sigma}{\gamma} = \sqrt{\varepsilon^2 - \left(1 + 2\mathrm{Ro}^{-1}\right)^2}, \tag{4.5}$$

(see (2.13)). The constants $\hat{v}_{x0}$ and $\hat{v}_{y0}$ are the initial values of the components of the perturbation. It can easily be seen that in the case when no strain is present, i.e. $\varepsilon = 0$, the only term that remains in the equation (4.2) corresponds to local rotation and there is no possibility for instability (cf. the growth rate in (4.5)). If, on the other hand, the pure rotation term vanishes, i.e. $\mathrm{Ro} = -2$ and only the straining term remains, the destabilisation is maximal. The unstable mode, which dominates the dynamics at large τ has a direction defined by

$$\frac{\hat{v}_y^{(\mathrm{unstable})}}{\hat{v}_x^{(\mathrm{unstable})}} = \chi. \tag{4.6}$$

In general the straining term possesses not only the component parallel to the perturbation but also the perpendicular one (see figure 2) and they are given by

$$F_S^{\parallel} = \frac{2\varepsilon \hat{v}_x \hat{v}_y}{v}, \tag{4.7}$$

$$F_S^{\perp} = \varepsilon \frac{\hat{v}_x^2 - \hat{v}_y^2}{v}, \tag{4.8}$$

where $v = \sqrt{\hat{v}_x^2 + \hat{v}_y^2}$ and the magnitude of the pure rotation term, which is always perpendicular to the velocity vector of the perturbation is

$$F_R^2 = \left(1 + 2\mathrm{Ro}^{-1}\right)^2 v^2. \tag{4.9}$$

The parallel component $F_S^{\parallel}$ can either amplify the perturbation or damp it depending on the sign of $\hat{v}_x \hat{v}_y$, hence if the two perpendicular forces $F_S^{\perp}$ and F_R can not be in balance for any $\hat{v}_x$ and $\hat{v}_y$, the perturbation will not grow for all times, but it will oscillate. The perpendicular forces can only be in balance if

$$\frac{\varepsilon^2}{\left(1 + 2\mathrm{Ro}^{-1}\right)^2} = \left(\frac{v^2}{\hat{v}_x^2 - \hat{v}_y^2}\right)^2 > 1, \tag{4.10}$$

and then the perturbation grows unboundedly, amplified by the parallel component $F_S^{\parallel}$ (cf. Fig. 3). The equation (4.10) is solved by any perturbation satisfying (4.6), which means that if the balance between the perpendicular forces can occur for some $\hat{v}_x^{(\mathrm{unstable})}$ and $\hat{v}_y^{(\mathrm{unstable})}$, any random perturbation will tend to the unstable mode defined by (4.6) by rotating the perturbation vector towards the unstable direction (and only one direction is stable, for which the perpendicular forces are also in balance but the parallel force acts in opposite direction to the direction of the perturbation velocity vector damping the perturbation, i.e. for $\hat{v}_y^{(\mathrm{stable})}/\hat{v}_x^{(\mathrm{stable})} = -\chi$). Furthermore, the equation (4.10) means that destabilisation can only occur if $\varepsilon^2 > \left(1 + 2\mathrm{Ro}^{-1}\right)^2$, which is the instability requirement obtained in (4.5).

In the rotating shear flow limit, $\varepsilon = 1$, which as shown earlier corresponds locally to the centrifugal instability with horizontal modes, the unstable direction is determined by $\chi = \sqrt{-1/(1 + \mathrm{Ro})}$. Using the transformations from the previous section (3.4) we can obtain the form of the unstable mode in the non-rotating frame, in the vicinity of $\mathbf{r}' = \mathbf{R}'$,

$$\hat{v}_r' \approx C\sqrt{\frac{-1}{1 + \mathrm{Ro}}}\mathrm{e}^{\sigma t}, \qquad \hat{v}_\varphi' \approx C\mathrm{e}^{\sigma t}, \tag{4.11}$$

where C is constant and the cylindrical coordinates where used. The unstable direction corresponds exactly to the unstable direction in the classical centrifugal instability, given by

$$\frac{\hat{v}_\varphi'}{\hat{v}_r'} = -\frac{1}{\sigma}\left(2\Omega\left(r'\right) + r'\frac{\mathrm{d}\Omega}{\mathrm{d}r'}\right) = -\frac{2 - s}{\sigma}\Omega\left(r'\right) = \sqrt{-\left(1 + \mathrm{Ro}\right)}, \tag{4.12}$$

where $\Omega(r') \sim 1/r'^s$ and the expression (3.17) were used.

In the magnetic case the Lorentz force enters the balance, which depends on the wavelength of the perturbations through the electric currents. Therefore the flows which are stable in the non-magnetic case may become unstable in the presence of magnetic field if the wavelength of the perturbation can be sufficiently adjusted. The Navier-Stokes and the induction equations for the horizontal rotational MEI take the form

$$\frac{\partial \mathbf{v}}{\partial t} = -\left(\mathbf{v}\cdot\nabla\right)\mathbf{u}_0 - 2\boldsymbol{\Omega}\times\mathbf{v} + \mathrm{i}h\mathbf{b}, \tag{4.13}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \left(\mathbf{b}\cdot\nabla\right)\mathbf{u}_0 + \mathrm{i}h\mathbf{v}, \tag{4.14}$$

For $\varepsilon = 1$ they correspond to the MRI equations in Balbus & Hawley (1991) with the radial wave vector (and thus the vertical components of the perturbations) set to zero (in their notation $k_R = 0$). Of course such restriction makes the considerations of Balbus & Hawley (1991) less general. However, it leads to the same stability criterion $\mathrm{d}\Omega/\mathrm{d}R > 0$.

As previously we can separate the total force on the right hand side of (4.13) into a component along the perturbation vector $\mathbf{v}$,

$$F_{total}^{\parallel} = \frac{\mathrm{i}h}{v}\mathbf{v}\cdot\mathbf{b} + \frac{2\varepsilon v_x v_y}{v}, \tag{4.15}$$

and a component perpendicular to $\mathbf{v}$

$$F_{total}^{\perp} = \frac{\mathrm{i}h}{v}\left(v_x b_y - v_y b_x\right) + \frac{v_x^2 - v_y^2}{v} - v - 2\mathrm{Ro}^{-1}v. \tag{4.16}$$

Next we find a general form of a j-th eigenvector of the matrix in (2.5) associated with the eigenvalue σ_j $(j = 1, 2, 3, 4)$,

$$\hat{v}_x^j = -2\mathrm{Ro}^{-1}h\sigma_j, \tag{4.17a}$$

$$\hat{v}_y^j = h\left[\left(\varepsilon+1\right)\left(\varepsilon - 1 - 2\mathrm{Ro}^{-1}\right) - h^2 - \sigma_j^2\right], \tag{4.17b}$$

$$\hat{b}_x^j = -i\left[\sigma_j^2\left(1+\varepsilon\right) + h^2\left(\varepsilon + 1 + 2\mathrm{Ro}^{-1}\right) - \left(\varepsilon+1\right)\left(\varepsilon^2 - \left(1+2\mathrm{Ro}^{-1}\right)^2\right)\right], \tag{4.17c}$$

$$\hat{b}_y^j = -i\sigma_j\left[\varepsilon^2 - \left(1+2\mathrm{Ro}^{-1}\right)^2 - h^2 - \sigma_j^2\right]. \tag{4.17d}$$

Since for any of the four eigenmodes $\mathbf{v}^j\cdot\mathbf{b}^j = p_c(\sigma_j) = 0$, where $p_c(\cdot)$ is the characteristic polynomial of the matrix in (2.5) we get (for a single eigenmode)

$$F_{total}^{j\parallel} = -4\varepsilon\mathrm{Ro}^{-1}\left[\left(\varepsilon+1\right)\left(\varepsilon - 1 - 2\mathrm{Ro}^{-1}\right) - \sigma_j^2 - h^2\right]. \tag{4.18}$$

We now concentrate on the unstable mode with index 1. Inserting $\sigma_j = \sigma_1$ (cf. (2.6)) into the above expression one can easily show that $F_{total}^{1\parallel} > 0$ if $\sigma_1 > 0$, i.e. for $\varepsilon^2 > \psi_+^2$.

The calculation of the force component perpendicular to the perturbation velocity vector for the unstable mode 1 is slightly more involved, but it is straightforward to show that

$$F_{total}^{1\perp} = 0. \tag{4.19}$$

Thus the mechanism of the instability is essentially the same as in the non-magnetic case, i.e. for the instability to be excited there must exist a possibility for a balance of all the rotational components of the forces acting on the perturbation, namely strain, the Coriolis and Lorentz forces, so that the perturbation for which such balance occurs can be amplified constantly in time by the strain. If such a balance cannot occur, the perturbations are amplified and damped in finite periods of time, thus they oscillate and do not grow. The unstable direction can be obtained from (4.17a,b). It can be seen, that in the magnetic case the unstable direction depends on the wavelength of the perturbation. This is no surprise, since it was already mentioned that since the Lorentz force depends on the wavelength through the currents, the balance of the rotational components depends on the choice of k. This explains why an elliptic vortex, which is stable in the non-magnetic case, may become unstable when the magnetic field is turned on with respect to perturbations for which the wavelengths must satisfy a certain condition. For the case of rotating shear flow, i.e. $\varepsilon = 1$, a system with any negative Rossby number (Ro < 0) is unstable with respect to horizontal perturbations with the wave number k satisfying (cf. (2.13)) $k^2 < -4\mathrm{Ro}\Omega^2/v_A^2$. In a vertically bounded domain, such as an accretion disk, there is a lower bound on the allowed values of k, thus such conditions are, in principle, modified (cf. Bajer & Mizerski 2011).

5. Conclusions

We have studied the properties and physical meaning of the horizontal MERI, i.e. the instability of the Euler flow with elliptical streamlines in the presence of external magnetic field and background rotation with respect to horizontal perturbations propagating along the field lines. The presence of anticyclonic background rotation is necessary for excitation of the horizontal instability and we have demonstrated that adding the magnetic field changes the range of values of the Rossby number at which the instability develops from $\mathrm{Ro} < -1$ in the non-magnetic case to $-4\gamma^2/(k^2 v_A^2) < \mathrm{Ro} < 0$ when the magnetic field is switched on.

The main purpose of this paper is the unification of the two, previously unrelated instability types. By the use of the local shearing box approximation the direct correspondence in the stability characteristics was shown between the well known short-wavelength MRI and the horizontal rotational MEI in the limit of rotating shear flow. The physical conditions under which the direct link between the two instabilities exists and the correspondence of the parameters between both instability types were explained in detail. The exponent s defining the radial dependence of the angular velocity in the disc, $\Omega(r) \sim 1/r^s$, is related to the local Rossby number in the box by $s = -2\mathrm{Ro}$ and the MRI requirement for instability, namely $s > 0$ corresponds to the rotational MEI requirement, i.e. $\mathrm{Ro} < 0$. Moreover, the dispersion relation for rotating shear flow with axial magnetic field for the horizontal modes was shown to be exactly the same as for the short-wavelength MRI and the well-known feature of the MRI in Keplerian discs, that the instability occurs for $0 < kv_A/\Omega < \sqrt{3}$ with the maximal growth rate $\sigma = 3\Omega/4$ achieved for $kv_A/\Omega = \sqrt{15}/4$ is also observed in rotating shear flow. The non-magnetic case of the horizontal instability was shown to have the same stability characteristics in the limit of rotating shear flow as the local centrifugal instability, with the Rayleigh's criterion corresponding to the non-magnetic horizontal instability requirement $\mathrm{Ro} < -1$. Finally, we have also provided a physical interpretation of the horizontal instability, based on the balance between the strain generated by the basic elliptic flow and the Coriolis force. The unstable and stable directions of the perturbation velocity field as functions of Ro and ε were found and the correspondence to the unstable direction of the perturbations for localised centrifugal instability was shown. It was also demonstrated that instability can only occur, if there is a possibility for the Coriolis force to be balanced by a component of the strain to eliminate the rotational tendency in the perturbation velocity, which, if present for all times, leads to oscillations. Such balance defines the unstable/stable direction and can occur only if $\varepsilon^2 > \left(1 + 2\mathrm{Ro}^{-1}\right)^2$, which agrees with the results of the linear analysis.

The full magnetic problem, for which the instability mechanism is essentially the same, was also discussed, highlighting the effect of the Lorentz force, which significantly affects the force balance, making the unstable direction dependent on the wavelength of the perturbation. The fact that the Lorentz force can be adjusted by the choice of the perturbation wavelength results in a possibility of achieving the balance of the rotational components of all three forces, which leads to continuous amplification of the perturbation, for negative Rossby numbers even outside the non-magnetic interval $2/(\varepsilon - 1) < \mathrm{Ro} < -2/(\varepsilon + 1)$. In particular in the rotating shear flow limit, $\varepsilon = 1$ all systems with $\mathrm{Ro} < 0$ are unstable, hence the presence of magnetic field leads to destabilisation of all elliptical vortices with $\mathrm{Ro} < -1$.

WL gratefully acknowledges partial financial support of the US-American National Science Foundation (NSF) under grant no. AST10-09802.

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