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Rossby Wave Frequency Change Induced by Small-Scale Topography

Jacques Vanneste

Department of Mathematics and Statistics, University of Edinburgh, Edinburgh, United Kingdom

ABSTRACT

A homogenization technique is used to study the change in the frequency of planetary Rossby waves that results from their interaction with a small-scale two-dimensional topography. The frequency change is computed explicitly for a topography consisting of a random distribution of well-separated cylindrical seamounts; it corresponds to a phase-speed increase (decrease) when the flat-bottom Rossby wave frequency is larger (smaller) than a typical topographic frequency. The topography is also shown to lead to a finite damping of the Rossby waves, even in the limit of infinitesimally small Ekman friction.

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1. Introduction

Recently, the widely studied effect of bottom topography on oceanic Rossby waves has received a great deal of attention. This renewed interest follows the observation of baroclinic Rossby waves by satellite altimeter, which indicated a significant influence of topographic features on wave propagation (Chelton and Schlax 1996). In particular, it has been suggested that topographic effects may cause the observed enhancement of the phase speed of the waves compared with that predicted by the simple, flat-bottom theory [although mean-flow effects have a primary importance; see Killworth et al. (1997)].

Recent studies have been devoted to slowly varying topography (Killworth and Blundell 1999), ridges (Tailleux and McWilliams 2000), as well as to more general situations (Reznik and Tsybaneva 1999; Bobrovich and Tsybanev 1999), and focused primarily on one-dimensional topographies. By contrast, two-dimensional topographies are considered in Vanneste (2000). This paper investigates (barotropic) quasigeostrophic motion over small-scale, periodic topography: using a multiple-scale (or homogenization) technique, the large-scale motion is shown to evolve according to an averaged quasigeostrophic equation in which the effect of topography is represented by a time-convolution term. Although Vanneste (2000) concentrates mainly on the enhancement of dissipation that is caused by the small-scale topography, it is clear from the

averaged equation that topography affects wave propagation in a more complex way and, in particular, that it perturbs the frequency of planetary Rossby waves. The aim of the present note is to demonstrate this explicitly.

To this end, we investigate the impact of small-scale topography on Rossby wave propagation using a highly idealized model in which the topography consists of cylindrical seamounts separated by distances large compared to their radii. Admittedly, the model and asymptotic regime considered are not very realistic; however, they allow the rigorous derivation of simple analytical results in a problem whose general treatment is fairly difficult. Since the purpose of this note is mainly illustrative, further assumptions are made in order to achieve maximum simplicity: the model is barotropic and neglects viscous dissipation and large-scale variations of the topography. These assumptions can easily be relaxed; in particular, it would be straightforward to examine the effect of baroclinicity by applying our method to a multilayer model.

In the plethoric literature on topographic effects, a variety of asymptotic regimes have been investigated [see Reznik and Tsybaneva (1999) for a discussion]. Here, we consider a small-scale topography that is steep, that is, such that the associated potential-vorticity gradient is much larger than potential-vorticity gradient associated with the β effect. This assumption ensures that the topography has an effect on the large-scale flow of the same order as the β effect and thus modifies the Rossby wave dispersion relation at leading order (Vanneste 2000). In contrast, most earlier studies (Thomson 1975; Prahalad and Sengupta 1986) assume a much shallower topography; this allows standard techniques for the study of waves in random media to be employed [see, e.g., Mysak (1978) and references therein] but only leads to small changes in the dispersion relation that affect wave propagation only for distances and times much longer than the wavelength and period. Notable exceptions are the studies of localization (Sengupta et al. 1992; Klyatskin 1996), as well as the recent papers by Reznik and Tsybaneva (1999) and Bobrovich and Tsybanev (1999). However, the high anisotropy of the (one-dimensional) topography considered in these papers makes their theory and results markedly different from those presented here.

The plan of this note is as follows. In section 2, the dispersion relation for large-scale Rossby waves in the presence of topography is derived using a homogenization approach.[1] The particular case of well-separated cylindrical seamounts is examined in section 3 where a dispersion relation valid for random distributions of seamounts is obtained perturbatively. The cases of seamounts with fixed height and with normally distributed heights is analyzed in detail in section 4. A similar qualitative conclusion is drawn in both cases: in addition to enhancing the dissipation of the Rossby waves, the topography induces a change in their frequency. In absolute value this change is a decrease (increase) when the flat-bottom, Rossby wave frequency is smaller (larger) than a suitably defined topographic frequency. Some remarks conclude the note in section 5.

2. Homogenization and dispersion relation

Under the quasigeostrophic scaling, the barotropic wave dynamics is governed by the linearized potential vorticity equation

$$\partial_t(\nabla^2\psi - \lambda^2\psi) + \beta\partial_{x_1}\psi + \frac{f}{H}\epsilon_{ij}\partial_{x_i}\psi\partial_{x_j}h$$

$$+ r\nabla^2\psi = 0 \qquad\qquad (2.1)$$

in which ψ is the streamfunction, λ the inverse of the radius of deformation, H the ocean's average depth, h the height of the bottom topography, and r the Ekman friction coefficient. In this equation, x_1 and x_2 are the usual zonal and meridional coordinates; the Jacobian $\partial(\psi, h)$ describing the topographic effect is conveniently written using the permutation symbol ϵ_{ij} and an implicit summation.

The essential assumption we make is that the topography varies on a scale much smaller than the typical scale of the waves; formally we write

$$h = h(\varepsilon^{-1}\mathbf{x}) = h(\xi), \qquad \varepsilon \ll 1,$$

where $\xi = \varepsilon^{-1}\mathbf{x}$ is a fast spatial variable. The topography is taken as a periodic or random function of ξ with zero average: $\langle h\rangle = 0$, for some suitable spatial and ensemble average $\langle \cdot \cdot \rangle$. We also suppose that h has no large-scale variation, although this restriction is easily relaxed.

We seek a solution of (2.1) in the form of a normal mode,

$$\psi = \mathrm{Re}[\hat{\psi}(\mathbf{x}, \xi)e^{-i\omega t}],$$

where ω is the frequency. Following the homogenization procedure ([Bensoussan et al. 1989](#)), we perform the substitution $\partial_{x_i} \rightarrow \varepsilon^{-1}\partial_{\xi_i} + \partial_{x_i}$ in [(2.1)](#) and expand $\hat{\psi}$ according to

$$\hat{\psi} = \hat{\psi}^{(0)} + \varepsilon\hat{\psi}^{(1)} + \varepsilon^2\hat{\psi}^{(2)} + \cdots .$$

This leads to a sequence of differential equations for the $\hat{\psi}_i$. The first one, obtained at $O(\varepsilon^{-2})$, is given by

$$u\nabla^2_\xi \hat{\psi}^{(0)} + \frac{f}{H}\epsilon_{ij}\partial_{\xi_i}\hat{\psi}^{(0)}\partial_{\xi_j}h = 0,$$

where $u = -i\omega + r$ and $\nabla^2_\xi = \partial_{\xi_i}\partial_{\xi_i}$. A solution is simply $\hat{\psi}^{(0)} = \hat{\psi}^{(0)}(\mathbf{x})$; that is, the leading-order streamfunction is independent of the fast spatial variable. At order $O(\varepsilon^{-1})$, one finds

$$u\nabla^2_\xi \hat{\psi}^{(1)} + \frac{f}{H}\epsilon_{ij}\partial_{\xi_i}\hat{\psi}^{(1)}\partial_{\xi_j}h = -\frac{f}{H}\epsilon_{ij}\partial_{x_i}\hat{\psi}^{(0)}\partial_{\xi_j}h.$$

Since this equation is linear and $\partial_{x_i}\hat{\psi}^{(0)}$ is independent of ξ, its solution can be written

$$\hat{\psi}^{(1)} = w_i(\xi)\partial_{x_i}\hat{\psi}^{(0)} + \Phi(\mathbf{x}) \tag{2.2}$$

for some undetermined (and irrelevant) function $\Phi(\mathbf{x})$ and for w_i, $i = 1, 2$, satisfying

$$u\nabla^2_\xi w_i + \frac{f}{H}\epsilon_{kj}\partial_{\xi_k}w_i\partial_{\xi_j}h = -\frac{f}{H}\epsilon_{ij}\partial_{\xi_j}h \tag{2.3}$$

and $\langle w_i \rangle = 0$. At order $O(1)$ a solvability condition must be imposed for the determination of $\hat{\psi}^{(2)}$; using [(2.2)](#), this condition is written as a function of $\hat{\psi}^{(0)}$ only as

$$-i\omega(\nabla^2\hat{\psi}^{(0)} - \lambda^2\hat{\psi}^{(0)}) + \beta\partial_{x_1}\hat{\psi}^{(0)} + r\nabla^2\hat{\psi}^{(0)}$$

$$+ \frac{f}{H}\langle\epsilon_{ik}\partial_{x_k}hw_j\rangle\partial^2_{x_ix_j}\hat{\psi}^{(0)} = 0.$$

This equation is the eigenvalue problem from which the frequency ω of the Rossby waves can be determined. It does not depend explicitly on the spatial coordinates, so that in an infinite domain solutions of the form $\hat{\psi}^{(0)} \sim \exp(i\mathbf{k}\cdot\mathbf{x})$, for some wavevector $\mathbf{k} = (k_1, k_2)$, can be introduced. This leads to the dispersion relation

$$\omega + \frac{\beta k_1}{k^2 + \lambda^2} + \frac{irk^2}{k^2 + \lambda^2}$$

$$+ is_{ij}(\omega)\frac{k_ik_j}{k^2 + \lambda^2} = 0, \tag{2.4}$$

where $k = |\mathbf{k}|$ and

$$s_{ij}(\omega) := \frac{f}{2H}\langle\epsilon_{ik}\partial_{x_k}hw_j + \epsilon_{jk}\partial_{x_k}hw_i\rangle \tag{2.5}$$

is a symmetric 2×2 tensor. The first three terms of (2.4) are usual, but the fourth term is new. It results from the interaction between small-scale topography and large-scale flow and depends on ω through $s_{ij}(\omega)$. In principle, w_i is first derived by solving (2.3), then $s_{ij}(\omega)$ is computed from (2.5) and substituted into (2.4) to provide an implicit equation for the frequency ω.

In general (2.3) needs to be solved numerically and the dependence of w_i on ω cannot be expressed in closed form; thus, the solution of (2.4) for ω must rely on an extensive iterative procedure that requires the solution of the partial differential equation (2.3) at each iteration. Analytical progress can nevertheless be made by considering topographies consisting of well-separated, isolated features.

3. Random array of cylindrical seamounts

Difficulties in solving (2.3) for w_i arise because the coefficients in this equation depend on space through h. The problem is significantly simplified for a piecewise topography: in this case, w_i is harmonic (i.e., $\nabla_\xi^2 w_i = 0$) everywhere except on the boundaries of the topography where it satisfies a jump condition. However, even in the simplest settings, for example, a topography consisting of a periodic array of seamounts, it is not possible to derive exact analytical expressions for w_i. [The amount of calculation involved is best illustrated by considering that involved in well-studied analogous problems, namely the determination of an effective conductivity in a periodic medium with cylindrical or spherical inclusions (e.g., Perrins et al. 1979; Sang and Acrivos 1983), or the study of potential flows past an array of cylinders or spheres (Hasimoto 1959).]

Here, since our objective is mainly to gain qualitative insight into the Rossby wave frequency change due to topography, we concentrate on an asymptotic limit that allows the derivation of simple analytical results. The limit we consider is that of well-separated seamounts; more precisely, we study cylindrical seamounts with radius a whose centers are separated by a distance d and we assume that $a/d \ll 1$. In this limit, studied by Maxwell for the conductivity problem, the interaction between seamounts is neglected so that (2.3) is solved (analytically) for an isolated seamount in an infinite domain. Interestingly, the limit allows a simple treatment of the random case in which the radius of the seamounts a, their height H_t, and the position of their centers are distributed randomly.

The height field associated with an isolated cylindrical seamount centered at the origin is given in polar coordinates (ρ, θ) by

$$h = H_t[1 - \Upsilon(\rho - a)], \quad (3.1)$$

where $\Upsilon(\cdot)$ is the Heavisde function. (Of course, H_t can be negative in which case the topographic feature is a valley rather than a mountain—the analysis below encompasses this case, although for simplicity we systematically refer to the topographic features as seamounts). Introducing this expression into the first component of (2.3) and using polar coordinates leads to the following equations for w_1:

$$\nabla_\xi^2 w_1 = 0 \quad \text{for } \rho \neq a \quad (3.2)$$

$$u[\partial_\rho w_1]_-^+ + \frac{fH_t}{aH} \partial_\theta w_1 - \frac{fH_t}{H} \sin\theta = 0 \quad \text{for } \rho = a, \quad (3.3)$$

where $[\cdot]_-^+$ denotes the jump across $\rho = a$. The second component of (2.3) leads to similar equations for w_2; however, w_2 is most easily derived directly from w_1 by rotating θ by $\pi/2$. A continuous solution of (3.2) is given by

$$w_1 = \begin{cases} A\dfrac{\rho}{a}\cos\theta + B\dfrac{\rho}{a}\sin\theta & \text{for } \rho \leq a \\[2ex] A\dfrac{a}{\rho}\cos\theta + B\dfrac{a}{\rho}\sin\theta & \text{for } \rho \geq a, \end{cases}$$

where A and B are two arbitrary constants. Imposing the jump condition (3.3) provides

$$A = \frac{}{1 + (u/\omega_t)^2} \quad \text{and} \quad B = \frac{}{1 + (u/\omega_t)^2},$$

where $\omega_t = fH_t/(2H)$ is the frequency of the first free topographic mode supported by an isolated cylindrical seamount ([Jansons and Johnson 1988](#)). On the boundary of the topography $\rho = a$ we therefore have

$$w_1 = \frac{-a}{1 + (u/\omega_t)^2}\left(\cos\theta + \frac{u}{\omega_t}\sin\theta\right), \qquad (3.4a)$$

$$w_2 = \frac{a}{1 + (u/\omega_t)^2}\left(\frac{u}{\omega_t}\cos\theta - \sin\theta\right). \qquad (3.4b)$$

We now compute the tensor $s_{ij}(\omega)$ defined by [(2.5)](#), which appears in the dispersion relation [(2.4)](#). In the random context, the average is both a spatial and ensemble average. Assuming that the distribution of seamounts is homogeneous, for any field g the average is written as

$$\langle g \rangle = n \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} \int_0^{\infty} gP(a, H_t)\rho \, d\rho \, d\theta \, da \, dH_t,$$

where n is the density of seamounts, that is, the number of seamounts per unit area, and $P(a, H_t)$ is the probability density function for the seamount radius and height. The field g in the right-hand side is that associated with a single seamount centered at the origin.

Using this average, along with [(3.1)](#) and [(3.4)](#), we find from [(2.5)](#),

$$s_{ij}(\omega) = \delta_{ij}2\pi nu \int_0^{\infty} \int_0^{\infty} \frac{a^2}{1 + (u/\omega_t)^2}P(a, H_t) \, da \, dH_t.$$

As is to be expected from the isotropy of the problem, $s_{ij}(\omega)$ is proportional to the identity tensor δ_{ij}. Since ω_t does not depend on a, the expression for $s_{ij}(\omega)$ simplifies according to

$$s_{ij}(\omega) = \delta_{ij}2\alpha u \int_{-\infty}^{\infty} \frac{Q(H_t)}{1 + (u/\omega_t)^2} \, dH_t,$$

where

$$Q(H_t) = C^{-1} \int_0^{\infty} \pi a^2 P(a, H_t) \, da.$$

In the above expression, C is a normalization constant, interpreted as the average area of the seamounts; $\alpha = nC$ is the area fraction occupied by the seamounts; and $Q(H_t)$ is a probability density function such that $Q(H_t) \, dH_t$ is the probability for the elevation of an arbitrary point of the ocean floor to be between H_t and $H_t + dH_t$.

The assumption of well-separated seamounts adopted here implies that $\alpha \ll 1$ and that $s_{ij}(\omega)$ is valid to order $O(\alpha)$ only. Correction terms of order $O(\alpha^2)$ appear if the interactions between seamounts are accounted for (cf. [Jeffrey 1973](#)). The dispersion relation [(2.4)](#) is therefore

$$+ \; i\frac{2\alpha uk^2}{k^2 + \lambda^2}\int_{-\infty}^{\infty}\frac{Q(H_t)}{1 + (u/\omega_t)^2}\,dH_t = O(\alpha^2). \quad (3.5)$$

It can be solved consistently by a regular perturbation expansion: introducing

$$\omega = \omega^{(0)} + \alpha\omega^{(1)} + O(\alpha^2)$$

leads at leading order to

$$\omega^{(0)} = -\frac{\beta k_1}{k^2 + \lambda^2} - \frac{irk^2}{k^2 + \lambda^2} = \omega_r - \frac{irk^2}{k^2 + \lambda^2},$$

where $\omega_r = \mathrm{Re}(\omega^{(0)})$ is the familiar (flat-bottom) Rossby wave frequency, and at order $O(\alpha)$ to

$$\omega^{(1)} = -\frac{2(\omega_r + i\delta)k^2}{k^2 + \lambda^2}\int_{-\infty}^{\infty}\frac{Q(H_t)}{1 - [(\omega_r + i\delta)/\omega_t]^2}\,dH_t,$$

$$(3.6)$$

where $\delta = \lambda^2 r/(k^2 + \lambda^2) > 0$. The real part of $\omega^{(1)}$ represents the Rossby wave frequency shift induced by the small-scale topography, whereas the imaginary part of of of $\omega^{(1)}$ represents an additional damping. In the next section, we study these quantities for specific distributions of the seamount heights $Q(H_t)$.

4. Rossby wave frequency change

a. Single-height seamounts

Consider first an ensemble of seamounts with the same height H_t (but possibly various radii) so that (3.6) becomes

$$\omega^{(1)} = -\frac{2(\omega_r + i\delta)k^2}{(k^2 + \lambda^2)\{1 - [(\omega_r + i\delta)/\omega_t]^2\}}.$$

We are most interested in the limit of weak dissipation $\delta \ll 1$, in which case $\omega^{(1)}$ is real. The relative change in the Rossby wave frequency is (α times)

$$\frac{\omega^{(1)}}{\omega_r} = -\frac{2k^2}{(k^2 + \lambda^2)[1 - (\omega_r/\omega_t)^2]}. \qquad (4.1)$$

From this expression, we conclude that, in absolute value, the topography leads to an increase (decrease) in the Rossby wave frequency if the flat-bottom frequency ω_r is larger (smaller) than the topographic frequency ω_t. Note that the change does not depend on the sign of ω_t so that the sign of H_t is irrelevant.

Equation (4.1) becomes invalid for $|\omega_r| \approx |\omega_t|$ as a result of a resonance between Rossby and topographic waves. A nonzero damping $\delta \neq 0$ smoothes this resonance; it can easily be shown that the transition between the increase and decrease of the Rossby wave frequency then occurs for $|\omega_r| = (\omega^2_t - \delta^2)^{1/2}$. For nonzero but small δ [$O(\alpha^{1/2})$ or smaller], the expansion in powers of α for the computation of w_i breaks down near resonance: this indicates that the interactions between the various seamounts become crucial when the Rossby and topographic waves are resonant. However, resonance appears in rather contrived situations and is absent when a distribution of seamount heights is considered.

Figures 1 O= and 2 O= illustrate the discussion above: they displays ω_r and $\mathrm{Re}(\omega^{(1)})$ as functions of the wavenumber k_1. Without loss of generality, we have taken $\lambda = \beta = 1$ (corresponding to a choice of space and time units); we have also chosen $k_2 = 1$. Figure 1 O= is obtained for a height of the topography such that $\omega_t = 0.25$. The results demonstrate the

change in the sign of $\text{Re}(\omega^{(1)})$ that occurs for $|\omega_r| \approx |\omega_t| = 0.25$. They also show the resonance phenomenon that appears in the absence of Ekman friction, that is, for $r = 0$, and its smoothing for $r \neq 0$, here for $r = 0.15$. Figure 2 ⊙ is obtained for $\omega_t = 0.5$: this is greater than the maximum Rossby wave frequency, so there is no sign change for $\text{Re}(\omega^{(1)})$, which is always positive (corresponding to a slow down of the Rossby waves), and there is no resonance phenomenon. Note that the imaginary part of $\omega^{(1)}$ (not shown) is always negative, as is expected for an additional damping. This can be established directly from (3.6) or more generally from (2.4)–(2.5) for arbitrary topographies (Vanneste 2000).

b. Random height seamounts

When the seamount heights are distributed according to a smooth probability density function $Q(H_t)$, there is no isolated resonance, and the frequency change $\omega^{(1)}$ is well defined even in the limit of vanishing Ekman damping $r \rightarrow 0^+$. We focus on this case and obtain an expression for $\omega^{(1)}$ by letting $\delta \rightarrow 0^+$ in (3.6).[2] It is given by

$$
\frac{\omega^{(1)}}{\omega_r} = \frac{-2k^2}{k^2 + \lambda^2} \left\{ \mathcal{P} \int_{-\infty}^{\infty} \frac{Q(H_t)}{1 - (\omega_r/\omega_t)^2} dH_t \right.
$$
$$
\left. + i \frac{\pi \omega_r H}{f} [Q(2H\omega_r/f) + Q(-2H\omega_r/f)] \right\},
$$

$$(4.2)$$

where $\mathcal{P}$ denotes the Cauchy principal value. It is concluded from this expression that $\text{Im}(\omega^{(1)}) < 0$; that is, the small-scale topography introduces a finite damping of the Rossby waves even though the Ekman friction for the large-scale flow is infinitesimal.

To illustrate formula (4.2), we consider a zero-mean Gaussian distribution for the seamount height; that is,

$$
Q(H_t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{H_t^2}{2\sigma^2}\right),
$$

where $\sigma > 0$ is the root-mean-square of the height. Substituting this expression in (4.2), we find after some manipulations

$$
\frac{\omega^{(1)}}{\omega_r} = \frac{2k^2}{k^2 + \lambda^2} \frac{\omega_r}{\omega_\sigma}
$$

$$
\times \left[I\left(\frac{|\omega_r|}{\omega_\sigma}\right) - i\sqrt{\pi} \exp\left(-\frac{\omega_r}{\omega_\sigma}\right)^2 \right], \quad (4.3)
$$

where $\omega_\sigma = (2)^{1/2}\sigma f/(2H) > 0$ is the frequency of topographic waves supported by seamounts with height $(2)^{1/2}\sigma$ and where the function I is defined as

$$
I(x) = \frac{1}{\sqrt{\pi}} \mathcal{P} \int_{-\infty}^{\infty} \frac{y^2}{y^2 - 1} e^{-x^2 y^2} dy
$$

$$
= \frac{1}{x} - 2 \int_0^x e^{y^2 - x^2} dy, \quad \text{for } x > 0.
$$

The second expression for $I(x)$ follows from the first one through a series of manipulations. It is useful for numerical purposes since the integral it contains is regular, and it can be used to write $I(x)$ in terms of an error function with imaginary

argument. A numerical calculation shows that $I(x)$ has a unique zero at $x_0 = 0.924 \cdots$, with $I(x) > 0$ for $x < x_0$ and $I(x) < 0$ for $x > x_0$. Therefore, in absolute value, the small-scale topography decreases (increases) the Rossby wave frequency if $|\omega_r| < x_0 \omega_\sigma$ ($|\omega_r| > x_0 \omega_\sigma$). This conclusion is analogous to that reached for a single-height topography, but now the topographic frequency with which ω_r should be compared is $x_0 \omega_\sigma$. An important point here is that, for the Gaussian distribution, x_0 is an order-one quantity so that $x_0 \omega_\sigma$ can genuinely be interpreted as a typical topographic frequency.

Figures 3 and 4 show the real and imaginary parts of $\omega^{(1)}$ calculated from (4.3) for $\lambda = \beta = k_2 = 1$. In Fig. 3 the topography is such that $\omega_\sigma = 0.25$; it follows that the transition in the sign of the (real) frequency change occurs for $\omega_r = 0.2355$. The damping introduced by the topography is seen to be significant since $|\text{Im}(\omega^{(1)})| > |\text{Re}(\omega^{(1)})|$ except for large values of k_1. Note that the frequency shift $\text{Re}(\omega^{(1)})$ decreases rather slowly for large k_1 (it is then proportional to $1/\omega_r$). In Fig. 4, $\omega_\sigma = 0.5$; since $|\omega_r| < x_0|\omega_\sigma|$ for all k_1, the frequency shift is always positive, corresponding to a slowdown of the Rossby wave.

5. Concluding remarks

In this note, a homogenization technique is used to derive the dispersion relation for large-scale Rossby waves in the presence of a steep small-scale topography. The frequency change induced by topography is computed explicitly for sparsely distributed cylindrical seamounts. The assumption $\alpha \ll 1$ of a low density of seamounts allows the derivation of an analytical expression for the frequency change; it also implies that this change is small $[O(\alpha)]$. It should be emphasized, however, that the homogenization procedure, and in particular the dispersion relation (2.4), can be employed for dense topographies, although in this case numerical computations similar to those used in different contexts (e.g., Sang and Acrivos 1983) will be necessary to evaluate $s_{ij}(\omega)$ from (2.5) and to compute the frequency. Importantly, a dense topography will lead to a frequency change of the same order as the flat-bottom Rossby wave frequency itself.

Our approach relies on a spatial averaging procedure and thus centers on large-scale waves. Consequently, the topographic waves, whose (small) scale is fixed by that of the topography, do not appear explicitly in the dispersion relation (2.4)—only the coupling between topography, β effect and Ekman friction has an impact on the large scales. It is therefore not surprising that, even for dense topography, the large-scale restoring mechanism provided by the β effect (or equivalently by a large-scale topography) is required for the existence of waves.[3] This contrasts our study with the work of Jansons and Johnson (1988) who focused on small-scale topographic modes supported by arrays of seamounts.

The main qualitative result of our work concerns the sign of the Rossby wave frequency shift. A general rule appears to be that topography "pushes away" the Rossby wave frequency from a typical topographic frequency; that is, the difference between the Rossby wave frequency and the topographic wave frequency increases as a consequence of the interaction between the two types of waves. This result, which is consistent with the general picture of linear wave interaction in stable systems (Craik 1985), is likely to hold in more general situations, for example, for baroclinic flows. Of course, the question of what the relevant topographic frequency is precisely for realistic topographies remains open. However, in realistic situations, it is likely that this topographic frequency is larger than the Rossby wave frequency and one may expect small-scale topography to cause a slowdown of the waves' phase speed, which should be contrasted with the phase speed increase observed by Chelton and Schlax (1996).

Another result of this work is the finiteness of the damping induced by topography in the presence of infinitesimal Ekman friction. This phenomenon, which is associated with the possible resonance between topographic and Rossby waves, is analogous to the finite damping that appears for infinitesimally damped harmonic oscillators forced by a continuous spectrum of frequencies (Landau and Lifschitz 1976, sec. 26): topographic waves play the role of oscillators, while a Rossby wave provides the external forcing. The analogy has a twist, however, since in the Rossby wave case the forcing has a single frequency whereas the continuous frequency spectrum is associated with the oscillators because of the continuous distribution of seamount heights.

REFERENCES

Bensoussan, A., J. L. Lions, and G. C. Papanicolaou, 1989: *Asymptotic Analysis of Periodic Structures.* Kluwer, 700 pp..

Bobrovich, A. V., and T. B. Tsybanev, 1999: Planetary waves in a stratified ocean of variable depth. Part 2: Continuously stratified ocean. *J. Fluid Mech.,* **388,** 147–169..

Chelton, D. B., and M. G. Schlax, 1996: Global observation of oceanic Rossby waves. *Science,* **272,** 234–238..

Craik, A. D. D., 1985: *Wave Interactions and Fluid Flows.* Cambridge University Press, 322 pp..

Hasimoto, H., 1959: On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres. *J. Fluid Mech.,* **5,** 317–328..

Jansons, K. M., and E. R. Johnson, 1988: Topographic Rossby waves above a random array of seamountains. *J. Fluid Mech.,* **191,** 373–388..

Jeffrey, D. J., 1973: Conduction through a random suspension of spheres. *Proc. Roy. Soc. London,* **335A,** 355–367..

Killworth, P. D., and J. R. Blundell, 1999: The effect of bottom topography on the speed of long extratropical planetary waves. *J. Phys. Oceanogr,* **29,** 2689–2710..

—— ,D. B. Chelton, and R. A. De Szoeke, 1997: The speed of observed and theoretical long extratropical planetary waves. *J. Phys. Oceanogr.,* **27,** 1946–1966.. Find this article online

Klyatskin, V. I., 1996: Localization of Rossby waves over a random cylindrical topography of the ocean bottom. *Izv. Atmos. Ocean. Phys.,* **32,** 757–765..

Landau, L. D., and E. M. Lifschitz, 1976: *Mechanics.* 3d ed. Pergamon, 224 pp..

Mysak, L. A., 1978: Wave propagation in random media, with oceanic applications. *Rev. Geophys. Space Phys.,* **16,** 233–261..

Perrins, W. T., D. R. McKenzie, and R. C. McPhedran, 1979: Transport properties of regular arrays of cylinders. *Proc. Roy. Soc. London,* **369A,** 207–225..

Prahalad, Y. S., and D. Sengupta, 1986: Barotropic planetary waves on a random bottom. *Wave Motion,* **8,** 407–414..

Reznik, G. M., and T. B. Tsybaneva, 1999: Planetary waves in a stratified ocean of variable depth. Part 1: Two-layer model. *J. Fluid Mech.,* **388,** 115–145..

Sang, A. S., and A. Acrivos, 1983: The effective conductivity of a periodic array of spheres. *Proc. Roy. Soc. London,* **386A,** 263–275..

Sengupta, D., L. I. Piterbarg, and G. M. Reznik, 1992: Localization of topographic Rossby waves over random relief. *Dyn. Atmos. Oceans,* **17,** 1–21..

Tailleux, R., and J. C. McWilliams, 2000: Acceleration, creation, and depletion of wind-driven, baroclinic Rossby waves over an ocean ridge. *J. Phys. Oceanogr.,* in press..

Thomson, R. E., 1975: The propagation of planetary waves over random topography. *J. Fluid Mech.,* **70,** 267–285..

Vanneste, J., 2000: Enhanced dissipation for quasi-geostrophic motion over small-scale topography. *J. Fluid Mech.,* **407,** 105–122..
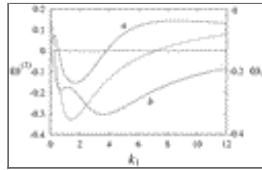
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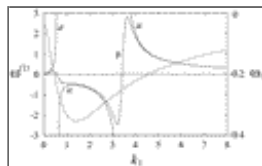
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Fig. 1. Leading-order Rossby wave frequency ω_r (dotted curve) and frequency shift $\mathrm{Re}(\omega^{(1)})$ (solid curves) as functions of k_1 for $\lambda = \beta = k_2 = 1$ and $\omega_t = 0.25$. The Ekman friction coefficient is $r = 0$ (*a*) and 0.15 (*b*).
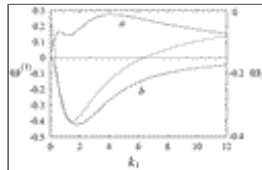
Fig. 2. Same as Fig. 1 ⬤= but with $\omega_t = 0.5$. The Ekman friction coefficient is $r = 0$ (*a*) and 0.5 (*b*).

Fig. 3. Leading-order Rossby wave frequency ω_r (dotted curve), frequency shift $\mathrm{Re}(\omega^{(1)})$ (solid curve *a*) and damping rate Im $(\omega^{(1)})$ (solid curve *b*) as functions of k_1 for $\lambda = \beta = k_2 = 1$. The seamount heights are distributed according to a zero-mean Gaussian distribution with $\omega_\sigma = 0.25$, and the limit of vanishingly small Ekman friction is considered.

Fig. 4. Same as Fig. 3 ⬤= but with $\omega_\sigma = 0.5$.

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[1] The dispersion relation could also be deduced from the more general result of Vanneste (2000).

[2] The limiting process implicitly assumes that $\alpha \ll \delta \ll 1$.

[3] This appears clearly from the dispersion relation (2.4): if β is set to zero, then the solutions will have $\mathrm{Re}(\omega) = 0$ since $s_{ij}(\omega)$ becomes real.

Corresponding author address: Dr. Jacques Vanneste, Department of Mathematics and Statistics, University of Edinburgh, James Clerk Maxwell Building, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom.

E-mail: vanneste@maths.ed.ac.uk

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