Synthesis for Optimal Two-Player Decentralized Control over TCP Erasure Channels with State Feedback

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Abstract

We propose a state-space synthesis for infinite horizon decentralized LQG control problems over TCP erasure channels with state feedback under sufficient conditions on convergence and stability. The decentralized system under consideration consists of two interconnected systems with unidirectional information communication from system 1 to system 2. The finite horizon counterpart of this problem was proposed previously, along with two modified Riccati recursions. When the mean packet drop rate $\mathbb{E}(N)$ satisfies two given conditions on convergence and stability, we show that the sequence of optimal finite horizon costs converges and construct an optimal infinite horizon synthesis.

1 Introduction

Recent advances in communication networks and VLSI have made it possible to provide services remotely. In fact, an increasing number of current services and future deployments, such as distributed database management, grid computing, and the smart grid, are now being built over the cloud, making it inevitable that the control, or management, of these services is also built on top of communication networks. While these interconnections facilitate the provision of ubiquitous services, the differences between a decentralized and a centralized system, such as the topology and the link conditions of a network, now pose a great challenge for system design.

In this paper, we consider a state feedback Linear Quadratic Gaussian (LQG) control problem with specific network topology and link conditions. For the topology, we consider a two-player/agent problem where player 1 observes system state x^1 and decides control action u^1 that affects system states x^1 and x^2. Player 2 observes system states x^1 and x^2 and decides control action u^2 that affects system state x^2. With only this sparsity, or topology, constraint, Swigart and Lall [4] [5] [6] showed that an explicit state space solution exists for the optimal decentralized controller in both the finite and infinite horizon cases.

For the link condition, we model the arrival of control packets through (controller) communication channels suffering from spatiotemporal Bernoulli distributed packet drops as in [2]. In practice, this phenomenon is generally caused by shadowing or fading in wireless communications. When packet drop acknowledgements are available to players, this information structure is referred to as a TCP protocol. An explicit state space solution for optimal LQG control over TCP erasure channels is proposed in [2]. Here we consider that there is **no** packet drop in (sensor) communication channels.

Here we further extend the boundary of decentralized optimal control over TCP erasure channels to the infinite horizon case as in Problem 3 in Section 2. Recently Chang and Lall [1] showed that there exists an explicit state space solution for optimal two-player decentralized finite horizon LQG control over TCP erasure channels. As opposed to the finite horizon problem, stability plays a critical role here. We pose the problems on stability in Section 2. In Section 4, we provide a general condition with three assumptions that characterizes an optimal infinite horizon policy. In the following sections, we show when these assumptions are met in our problem. We first repeat the finite horizon optimal solutions in Section 5 from [1]. We then propose conditions when the sequence of finite horizon decentralized optimal costs converge, with a similar approach to [2]. We finally give a synthesis for the controller and prove that it actually minimizes the infinite horizon cost.

2 Problem Formulation

We use the following notations. For system models, we use superscript to denote spatial subsystems and subscript to denote time indices. We use $x_{0:t}$ as an abbreviation of $(x_0, \dots, x_t)$. In a centralized system, for each nonnegative time index $t \in \mathbb{Z}_+$, $x_t \in \mathbb{R}^n$ denotes the state, and $u_t \in \mathbb{R}^m$ the control action. The system dynamics are as follows:

$$x_{t+1} = Ax_t + BN_t u_t + v_t, \tag{1}$$

where exogenous noise $v_t \sim \mathcal{N}(0, \Sigma_v)$ i.i.d. and the initial state $x_0 \sim \mathcal{N}(0, \Sigma_s)$, independent of v_t. The

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link condition of the actuator channels is modeled by $N_t = \text{diag}(\nu_t)$, where $\nu_t = (\nu_t^1, \nu_t^2, \cdots, \nu_t^m)$. $\nu_t^j \sim \text{Bernoulli}(\bar{\nu}^j)$ denotes the i.i.d. Bernoulli random binary variable modeling the random information drop on j-th actuator of the actuator channels at time t. We assume $0 < \bar{\nu}^j \leq 1$ for every j.

In the two-player decentralized setting, we consider two interconnected systems with unidirectional information communication from system 1 to system 2. The system state $x_t = (x_t^1, x_t^2)$ consists of the substates of two subsystems with $x_t^1 \in \mathbb{R}^{n_1}$, $x_t^2 \in \mathbb{R}^{n_2}$, and $n = n_1 + n_2$. The control action $u_t = (u_t^1, u_t^2)$ consists of the control actions of two players with $u_t^1 \in \mathbb{R}^{m_1}$, $u_t^2 \in \mathbb{R}^{m_2}$, and $m = m_1 + m_2$. The explicit system dynamics are as follows:

$$\begin{bmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{bmatrix} = \begin{bmatrix} A^{11} & 0 \\ A^{21} & A^{22} \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} + \begin{bmatrix} B^{11} & 0 \\ B^{21} & B^{22} \end{bmatrix} \begin{bmatrix} N_t^1 & 0 \\ 0 & N_t^2 \end{bmatrix} \begin{bmatrix} u_t^1 \\ u_t^2 \end{bmatrix} + \begin{bmatrix} v_t^1 \\ v_t^2 \end{bmatrix}, \quad (2)$$

where the exogenous noise $v_t = (v_t^1, v_t^2)$ with

$$\begin{bmatrix} v_t^1 \\ v_t^2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_v^1 & 0 \\ 0 & \Sigma_v^2 \end{bmatrix} \right),$$

i.i.d. for all $t \in \mathbb{Z}_+$. The link condition of the actuator channel i is modeled by $N_t^i = \text{diag}(\nu_t^{i1}, \ldots, \nu_t^{im_i})$. For convenience, we let

$$A = \begin{bmatrix} A^{11} & 0 \\ A^{21} & A^{22} \end{bmatrix}, \quad B = \begin{bmatrix} B^{11} & 0 \\ B^{21} & B^{22} \end{bmatrix}, \text{ and}$$

$$N_t = \begin{bmatrix} N_t^1 & 0 \\ 0 & N_t^2 \end{bmatrix}, \quad \nu_t = (\nu_t^{11}, \ldots, \nu_t^{1m_1}, \nu_t^{21}, \ldots, \nu_t^{2m_2}).$$

where $\nu_t^{11} = \nu_t^1, \nu_t^{12} = \nu_t^2, \ldots, \nu_t^{21} = \nu_t^{(m_1+1)}$, and so on.

Suppose $m \in \mathbb{Z}_+$ and $J \subset \{1, \ldots, m\}$. Define the diagonal matrix $N_J \in \mathbb{R}^{m \times m}$ by

$$(N_J)_{ii} = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

For any $J \subset \{1, \ldots, m\}$, suppose N_J is an instance of packet drop N_t. We then have

$$\Pr(N_J) = \left(\prod_{j \in J} \bar{\nu}^j \right) \left(\prod_{j \notin J} (1 - \bar{\nu}^j) \right).$$

Since N_t is i.i.d. over time t, we drop the subscript t when appropriate. Similar notation is adopted when N_J is an instance of packet drop rates N_t^i with $i \in \{1, 2\}$.

Definition 1. *For any linear operator $f : \mathbb{R}^{l \times l} \mapsto \mathbb{R}^{p \times q}$, define the operator $\mathbb{E}_N : (\mathbb{R}^{l \times l} \mapsto \mathbb{R}^{p \times q}) \mapsto \mathbb{R}^{p \times q}$ by $\mathbb{E}_N(f(N)) = \sum_{J \in 2^{\mathfrak{I}}} \Pr(N_J) f(N_J)$.*

We list several similar problems for optimal control over TCP erasure channels over finite/infinite horizon and centralized/decentralized framework.

Problem 2. *(Finite Horizon Centralized/Decentralized Problems) The objective of* ***finite*** *horizon problems is to find the minimum cost $\min_\mu \mathcal{J}_T(\mu) =$*

$$\min_\mu \mathbb{E} \left\{ \sum_{t=0}^{T-1} \left(x_t^T Q x_t + u_t^T N_t R N_t u_t \right) + x_T^T Q_T x_T \right\} \quad (3)$$

over the set of policies μ, where $Q \geq 0$, $Q_T \geq 0$, $R > 0$. In (3), the expectation is taken over x_0, $v_{0:T-1}$, $N_{0:T-1}$.

- *In centralized problems, addressed in [2], the system dynamics follow (1), $\mu = \{\mu_t \,|\, t \in \{0, 1, \ldots, T-1\}\}$, $u_t = \mu_t(I_t)$ and*

$$I_t = \{x_{0:t}, u_{0:t-1}, N_{0:t-1}\}. \quad (4)$$

- *In decentralized problems, addressed in [1], the system dynamics follow (2), $\mu = \{\mu_t^i \,|\, i \in \{1, 2\}, t \in \{0, 1, \ldots, T-1\}\}$, $u_t^1 = \mu_t^1(I_t^1)$, $u_t^2 = \mu_t^2(I_t^2)$, and*

$$\begin{aligned} I_t^1 &= \{x_{0:t}^1, u_{0:t-1}^1, N_{0:t-1}^1\}, \\ I_t^2 &= \{x_{0:t}, u_{0:t-1}^2, N_{0:t-1}\}. \end{aligned} \quad (5)$$

Problem 3. *(Infinite Horizon Cetralized/Decentralized Problems) The objective of* ***infinite*** *horizon problems is to find the minimum cost*

$$\min_\mu \mathcal{J}_\infty(\mu) = \min_\mu \left(\lim_{T \to \infty} \frac{1}{T+1} \mathcal{J}_T \right)(\mu) \quad (6)$$

over the set of policies μ, where $Q \geq 0$, $Q_T \geq 0$, $R > 0$.

- *In centralized problems, addressed in [2], the system dynamics follow (1), $\mu = \{\mu_0, \mu_1, \ldots\}$; the control actions follow $u_t = \mu_t(I_t)$; the information states I_t are defined as (4).*
- *In decentralized problems, addressed in this paper, the system dynamics follow (2), $\mu = (\mu^1, \mu^2)$ with $\mu^1 = \{\mu_0^1, \mu_1^1, \ldots\}$ and $\mu^2 = \{\mu_0^2, \mu_1^2, \ldots\}$; the control actions follow $u_t^1 = \mu_t^1(I_t^1)$ and $u_t^2 = \mu_t^2(I_t^2)$; the information states I_t^1 and I_t^2 are defined as (5).*

In this paper, we extend the finite horizon solution to the infinite horizon problem by taking the horizon length $T \to \infty$. A similar approach can be found in [2] when the centralized Problem 2 is extended to the centralized Problem 3.

Regarding the issue of stability, one should note that the synthesis problem in (6) may be infeasible, for both the centralized and decentralized case. For example, A system with unstable A and $N_t = 0$ for all t would result in an infinite cost, regardless of the controller chosen. On the other hand, the synthesis problem reduces to a regular centralized LQG problem or the decentralized LQG problem proposed by [6], when $N_t = I$ for all t and proper reachability and stabilizability constraints hold, which is always feasible.

The definition for stability is not clear in the case when the packet drop is random. Consider a linear dynamical system as in (1) and suppose the controller is given by the linear dynamics

$$\begin{aligned} &\xi_{t+1} = A_t^K(N_t^1)\xi_t + B_t^K(N_t^1)x_t^1, \\ &u_t = C^K\xi_t + D^K x_t^1, \end{aligned} \tag{7}$$

where $\xi_0 = 0$, A_t^K and B_t^K depend on N_t^1 through observations. The superscript K here denotes that A_t^K, B_t^K, C_t^K, and D_t^K are state-space realizations of the controller. The closed-loop map is given by

$$\begin{bmatrix} x_{t+1} \\ \xi_{t+1} \end{bmatrix} = A_{cl}(N_t)\begin{bmatrix} x_t \\ \xi_t \end{bmatrix} + v_{cl} \ , \tag{8}$$

where

$$A_{cl}(N_t) = \begin{bmatrix} A + BN_t[D^K \ 0] & BN_tC^K \\ [B_t^K(N_t^1) \ 0] & A_t^K(N_t^1) \end{bmatrix}, \quad v_{cl} = \begin{bmatrix} v_t \\ 0 \end{bmatrix}. \tag{9}$$

Here $\mathbb{E}(x_t) = 0$ and $\mathbb{E}(\xi_t) = 0$ because $\mathbb{E}(x_0) = 0$ and $\xi_0 = 0$. Note that for certain sample-path of $\{N_t\}$, the closed-loop state may not be bounded, even when $A_{cl}(N_t)$ is stable for all N_t.

An infinite horizon policy is feasible when the closed-loop map is stable, as follows.

Definition 4. *A closed-loop map $z_{t+1} = A_{cl}(N_t)z_t + v_{cl}$ with $\mathbb{E}(z_t) = 0$ for all $t \in \mathbb{Z}_+$ is stable if $\lim_{t\to\infty} \operatorname{cov}(z_t)$ is bounded.*

3 Main Results

We first define an operator to simplify the notation in the main Theorems.

Definition 5. *Define the modified algebraic Riccati operator as*

$$\begin{aligned} &\mathcal{R}(P, A, B, Q, R, \mathbb{E}(N)) = Q + A^TPA - A^TPB \\ &\quad \times \mathbb{E}_N(N)\{\mathbb{E}_N(N(R + B^TPB)N)\}^{-1}\mathbb{E}_N(N)B^TPA. \end{aligned}$$

and the modified Riccati recursion as

$$P_{t+1} = \mathcal{R}(P_t, A, B, Q, R, \mathbb{E}(N)),$$

where N is a diagonal random matrix with N_{ii} Bernoulli distributed with mean $\mathbb{E}_N(N)_{ii}$.

We are now ready to state the main Theorems, to be proved later. In the first Theorem, we provide a sufficient condition for the modified Riccati recursions to converge for all initial conditions.

Theorem 6. *(Convergence) If there exists two pairs (K, P) and (V, Y) such that $P > 0$, $Y > 0$, and*

$$\begin{aligned} P > \mathbb{E}_N(&(A - BNK)^TP(A - BNK) \\ &+ Q + K^TNRNK), \end{aligned} \tag{10}$$

$$\begin{aligned} Y > \mathbb{E}_{N^2}(&(A^{22} - B^{22}N^2V)^TY(A^{22} - B^{22}N^2V) \\ &+ Q^{22} + V^TN^2R^{22}N^2V), \end{aligned} \tag{11}$$

then for any initial condition $P_0 \geq 0$ and $Y_0 \geq 0$, the modified Riccati recursions

$$P_{t+1} = \mathcal{R}(P_t, A, B, Q, R, \mathbb{E}_N(N)), \tag{12}$$

$$Y_{t+1} = \mathcal{R}(Y_t, A^{22}, B^{22}, Q^{22}, R^{22}, \mathbb{E}_{N^2}(N^2)), \tag{13}$$

converge to unique positive semidfinite fixed points P_∞ and Y_∞ of the modified Riccati equations, respectively.

Next we give a sufficient condition for stability.

Theorem 7. *(Stability) Consider a linear dynamical system as in (1) with the controller as in (7). The closed-loop system (8) is stable if there exists*

$$S > 0, \quad S > \mathbb{E}_N\left\{A_{cl}SA_{cl}^T\right\}. \tag{14}$$

In the third theorem, we provide a controller synthesis for the decentralized system given the sufficient conditions in Theorem 6 and Theorem 7 hold.

Theorem 8. *(Synthesis) Consider the decentralized system in Problem 3. Suppose the packet drop rates $\mathbb{E}_N(N)$ are such that the MAREs (10) and (11) hold, then P_t and Y_t in the recursion (12) and (13) converge to P_∞ and Y_∞, respectively. With these P_∞ and Y_∞, let the decentralized controller gains be*

$$K_\infty = \left(\mathbb{E}_N(N(R + B^TP_\infty B)N)\right)^{-1}\mathbb{E}_N(N)B^TP_\infty A, \tag{15}$$

$$\begin{aligned} J_\infty = &\left(\mathbb{E}_{N^2}(N^2(R^{22} + B^{22,T}Y_\infty B^{22})N^2)\right)^{-1} \\ &\times \mathbb{E}_{N^2}(N^2)B^{22^T}Y_\infty A^{22}. \end{aligned} \tag{16}$$

A controller synthesis is given by (7) where

$$A_t^K(N_t^1) = A^{22} - B^{21}N_t^1K_\infty^{12} - B^{22}\mathbb{E}(N^2)K_\infty^{22}, \tag{17}$$

$$B_t^K(N_t^1) = A^{21} - B^{21}N_t^1K_\infty^{11} - B^{22}\mathbb{E}(N^2)K_\infty^{21}, \tag{18}$$

$$C^K = \begin{bmatrix} -K_\infty^{12} \\ J_\infty - K_\infty^{22} \end{bmatrix}, \tag{19}$$

$$D^K = \begin{bmatrix} -K_\infty^{11} & 0 \\ -K_\infty^{21} & -J_\infty \end{bmatrix}. \tag{20}$$

Suppose the closed-loop map satisfies (14), then the controller is optimal, with the optimal cost being

$$\min_\mu \mathcal{J}_\infty(\mu) = \operatorname{trace}\left(\begin{bmatrix} P_\infty^{11} & P_\infty^{12} \\ P_\infty^{21} & Y_\infty \end{bmatrix}\Sigma_v\right). \tag{21}$$

The optimal controllers are as follows. For controller 1,

$$\begin{aligned} \xi_{t+1} &= A_t^K(N_t^1)\xi_t + B_t^K(N_t^1)x_t^1, \\ u_t^1 &= -K_\infty^{11}x_t^1 - K_\infty^{12}\xi_t, \end{aligned}$$

and for controller 2,

$$\begin{aligned} \xi_{t+1} &= A_t^K(N_t^1)\xi_t + B_t^K(N_t^1)x_t^1, \\ u_t^2 &= -K_\infty^{21}x_t^1 - K_\infty^{22}\xi_t - J_\infty(x_t^2 - \xi_t), \end{aligned}$$

where $\xi_0 = 0$ and N_t^1 is known to controllers.

Theorem 8 is a natural extension to [6]. When no packets are lost, $N_t = I$ for all t, thereby reducing this problem and solution to those in [6]. On the other hand, when there is only one player, the problem and solution here reduces to those when packets never drop in sensor communication channels, in [2]. The reachability and stabilizability constraints in [6] and [2] are not required here as Theorem 6 is a stronger condition than those constraints. This can be shown by taking the expectation in (10) into two terms as a function of $\mathbb{E}_N(N)$ and $\mathrm{cov}(N)$ and applying properties of discrete time Lyapunov equations, and similarly for (11).

In the centralized case, [2] and [3] proposed Numerical methods to calculate (10) and (12) using semidefinite programming. The same approach can be applied here to calculate (11), (13), and (14) numerically.

4 Synthesis from DP Solutions

Here we provide a general framework that characterizes an optimal infinite horizon set of policies.

Lemma 9. *Suppose $g_k : \mathbb{S} \mapsto \mathbb{R}$, $\hat{w} \in \mathbb{S}$, $\hat{f} : \mathbb{S} \mapsto \mathbb{R}$, and*

1. *$m_k \leq g_k(y)$ for all $y \in \mathbb{S}$.*
2. *$\lim_{k\to\infty} m_k = \hat{f}(\hat{w})$.*
3. *$\lim_{k\to\infty} g_k(y) = \hat{f}(y)$ for all $y \in \mathbb{S}$.*

Then $\hat{w}$ minimizes $\hat{f}$.

Proof. We will show that $\hat{f}(\hat{w}) \leq \hat{f}(y)$ for all $y \in \mathbb{S}$. Suppose not, then there exists some z such that $\hat{f}(\hat{w}) > \hat{f}(z)$. Let $h_k = m_k - g_k(z)$. Then $\lim_{k\to\infty} h_k > 0$. However, $h_k \leq 0$ for all k. This is a contradiction. ∎

Corollary 10. *Suppose $\mathbb{S}$ and $\mathbb{H}^k$ with $k \in \mathbb{Z}_+$ are sets. Let $\hat{w} \in \mathbb{S}$, and $w_0, w_1, \dots$ be a sequence with $w_k \in \mathbb{H}^k$. Let $\mathcal{Q}_k : \mathbb{S} \mapsto \mathbb{H}^k$, $\hat{f} : \mathbb{S} \mapsto \mathbb{R}$, and $f_k : \mathbb{H}^k \mapsto \mathbb{R}$ be any functions. Suppose further that*

1. *w_k minimizes f_k.*
2. *$\lim_{k\to\infty} f_k(w_k) = \hat{f}(\hat{w})$.*
3. *for all $y \in \mathbb{S}$, $\lim_{k\to\infty} f_k(\mathcal{Q}_k y) = \hat{f}(y)$.*

Then $\hat{w}$ minimizes $\hat{f}$.

Proof. This is evident from Lemma 9 with $m_k = f_k(w_k)$ and $g_k = f_k \circ \mathcal{Q}_k$, where $\circ$ denotes function composition. ∎

In our particular problem, f_k represents the finite horizon cost function divided by its horizon, i.e. $\mathcal{J}_k/(k+1)$, w_k represents the set of optimal finite horizon policies μ with horizon $k+1$, which will be given in Theorem 11, and $\hat{f}$ represents the infinite horizon cost function $\mathcal{J}_\infty$. $\mathbb{S}$ and $\mathbb{H}^k$ represent the set of feasible infinite horizon policies and finite horizon policies with horizon $k+1$. As we will see next, we let the function $\mathcal{Q}_k$ be a natural projection from $\mathbb{S}$ to $\mathbb{H}^k$ by taking the first k actions and dropping the rest.

Note that in Corollary 10, any infinite horizon policy $\hat{w}$ that satisfies assumption 2 is optimal. It might not be unique or linear, and we did not provide an explicit form for any $\hat{w}$. We will construct a $\hat{w}$ by taking the limit of the sequence of the set of optimal finite horizon policies w_k, when it exists, in Section 6. In the following sections, we verify whether and when the assumptions in Corollary 10 hold for our specific problem. Assumptions 1, 2, and 3 will be considered in Sections 5, 7, and 7 respectively.

5 Finite Horizon Solutions

In this section, we repeat the result from [1], which provides optimal policies for finite horizon decentralized control over TCP erasure channels with state feedback.

Theorem 11. *Let $P_t \in \mathbb{R}^{n\times n}$, $Y_t \in \mathbb{R}^{n_2\times n_2}$, and $r_t \in \mathbb{R}$ satisfy the following recursions*

$$P_t = \mathcal{R}(P_{t+1}, A, B, Q, R, \mathbb{E}_N(N)), \tag{22}$$

$$Y_t = \mathcal{R}(Y_{t+1}, A^{22}, B^{22}, Q^{22}, R^{22}, \mathbb{E}_{N^2}(N^2)), \tag{23}$$

$$r_t = r_{t+1} + \mathrm{trace}\left(\begin{bmatrix} P_{t+1}^{11} & P_{t+1}^{12} \\ P_{t+1}^{21} & Y_{t+1}\end{bmatrix}\Sigma_v\right), \tag{24}$$

with $P_T = Q_T$, $Y_T = P_T^{22}$, and $r_T = 0$. Define J_t and K_t to be

$$K_t = \left(\mathbb{E}_N(N(R + B^T P_{t+1} B)N)\right)^{-1}\mathbb{E}_N(N)B^T P_{t+1} A, \tag{25}$$

$$J_t = \left(\mathbb{E}_{N^2}(N^2)(N^2(R^{22} + B^{22,T} Y_{t+1} B^{22})N^2)\right)^{-1} \times \mathbb{E}_{N^2}(N^2){B^{22}}^T Y_{t+1} A^{22}. \tag{26}$$

The optimal controller is given by (7) with $\xi_0 = 0$,

$$A_t^K(N_t^1) = A^{22} - B^{21}N_t^1 K_t^{12} - B^{22}\mathbb{E}_{N^2}(N^2)K_t^{22}, \tag{27}$$

$$B_t^K(N_t^1) = A^{21} - B^{21}N_t^1 K_t^{11} - B^{22}\mathbb{E}_{N^2}(N^2)K_t^{21}, \tag{28}$$

$$C^K = \begin{bmatrix} -K_t^{12} \\ J_t - K_t^{22}\end{bmatrix}, \tag{29}$$

$$D^K = \begin{bmatrix} -K_t^{11} & 0 \\ -K_t^{21} & -J_t\end{bmatrix}. \tag{30}$$

The optimal cost $\min_\mu \mathcal{J}_T(\mu)$ is

$$\sum_{t=1}^{T}\mathrm{trace}\left(\begin{bmatrix} P_t^{11} & P_t^{12} \\ P_t^{21} & Y_t\end{bmatrix}\Sigma_v\right) + \mathrm{trace}\left(\begin{bmatrix} P_0^{11} & P_0^{12} \\ P_0^{21} & Y_0\end{bmatrix}\Sigma_s\right).$$

According to (7), the control policy μ^1 is given by

$$u_t^1 = -K_t^{11}x_t^1 - \sum_{i=0}^{t-1}(K_t^{12}A_{t-1}^K \cdots A_{i+1}^K B_i^K)x_i^1, \tag{31}$$

and similarly for μ^2 and u_t^2. Note that the set of policies μ depends on N_t^1 with $t \in \mathbb{Z}_+$ through A_t^K and B_t^K. These N_t^1 are known to the controllers through observations..

The sequence of optimal finite horizon costs divided by their horizon, $\min_\mu \mathcal{J}_T(\mu)/(T+1)$, does not necessarily converge as $T \to \infty$. This sequence corresponds to $f_k(w_k)$ in Corollary 10. We consider when this sequence converges as $T \to \infty$ in the next section.

6 Convergence of Modified Riccati Recursions and Finite Horizon Costs

We first consider the convergence of the modified Riccati recursions of P_t and Y_t. We then show that when P_t and Y_t converge, the set of policies converges to an infinite horizon policy, and the sequence of optimal average costs converges. .

Definition 12. *Define the operator $g_N(P)$ for $P \geq 0$ by*

$$g_N(P) = \mathcal{R}(P, A, B, Q, R, \mathbb{E}(N)),$$

where A, B, Q, R are given with $Q \geq 0$ and $R > 0$.

Theorem 13. *Let the operator $\Phi(K, P) = \mathbb{E}_N(F^T PF + V)$ where $F = A - BNK$ and $V = Q + K^T NRNK$. Suppose there exists matrices $\bar{K}$ and $\bar{P} > 0$ such that $\bar{P} > \Phi(\bar{K}, \bar{P})$, then the sequence $P_{t+1} = g_N(P_t)$ converges to a unique positive semidefinite fixed point $\tilde{P}$ of the modified Riccati equation $P = g_N(P)$ for any initial condition $P_0 \geq 0$.*

Proof. See [3] for proof of a similar result. ∎

In our decentralized problem, we need to find sufficient conditions for both modified Riccati recursions (12) and (13) to converge.

Proof. (Theorem 6) Given that there exists a pair (K, P) such that $P > 0$ and $P > \mathbb{E}_N((A-BNK)^T P(A-BNK) + Q + K^T NRNK)$, then modified Riccati recursion (12) converges to a unique fixed point P_∞ for any $P_0 \geq 0$. Similarly, if there exists a pair (V, Y) such that $Y > 0$ and $Y > \mathbb{E}_{N^2}((A^{22} - B^{22}N^2V)^T Y(A^{22} - B^{22}N^2V) + Q^{22} + V^T N^2 R^{22} N^2 V)$, the modified Riccati recursion (13) converges to a unique fixed point Y_∞ for any $Y_0 \geq 0$. ∎

Given that P_t and Y_t converge to P_∞ and Y_∞ as $t \to \infty$, the limit of optimal finite horizon costs divided by the time horizon by taking $T \to \infty$ is

$$\lim_{T\to\infty} \frac{1}{T+1} \mathcal{J}_T(\mu) = \operatorname{trace}\left(\begin{bmatrix} P_\infty^{11} & P_\infty^{12} \\ P_\infty^{21} & Y_\infty \end{bmatrix} \Sigma_v \right).$$

Furthermore, (25) (26) (27) (28) (29) (30) converge to (15) (16) (17) (18) (19) (20) by continuity. Therefore, the coefficients of (31) converge as follows:

$$u_t^1 = -K_\infty^{11} x_t^1 - \sum_{i=0}^{t-1} (K_\infty^{12} (A_\infty^K)^{t-i-1} B_\infty^K) x_i^1, \qquad (32)$$

where $t-i-1$ here is the power of A_∞^K. Similarly for the coefficients of u_t^2.

In other words, we form a set of decentralized infinite horizon policies as stated in Theorem 8. This set will be our choice for $\hat{w}$ in Corollary 10. Up to this point, we do not know if this controller is feasible and optimal.

7 Decentralized Controller Synthesis

We first examine the closed-loop system with infinite horizon policy as stated in Theorem 8. The closed-loop system is given by (8) and (9). Note that, since $\mathbb{E}(x_0) = 0$ and $\xi_0 = 0$, we have $\mathbb{E}x_t = 0$ and $\mathbb{E}\xi_t = 0$. We also have

$$\operatorname{cov}\left(\begin{bmatrix} x_{t+1} \\ \xi_{t+1} \end{bmatrix} \right) = \mathbb{E}_N \left\{ A_{cl} \operatorname{cov}\left(\begin{bmatrix} x_t \\ \xi_t \end{bmatrix} \right) A_{cl}^T \right\} + \begin{bmatrix} \Sigma_v & 0 \\ 0 & 0 \end{bmatrix}.$$

The following Lemma gives a sufficient condition when the sequence of covariance matrices is bounded.

Lemma 14. *Define the linear map $L(S) = \mathbb{E}_N(F^T SF)$ where $F = A - BNK$. Let the sequence $\{S_0, S_1, \ldots\}$ follow the recursion $S_{t+1} = L(S_t) + Z$ with $S_0 \geq 0$ and $Z \geq 0$. Suppose there exists $\bar{S} > 0$ such that $\bar{S} > L(\bar{S})$, then the sequence S_t is bounded above.*

Proof. See [3] for proof of a similar result. ∎

Proof. (Theorem 7) The result is evident by applying Lemma 14 to the covariance matrix recursion above, with proper modifications on A, B, N, K, and Z in Lemma 14 according to (9). ∎

We now show that the third assumption in Corollary 10 is true for our particular problem.

Lemma 15. *Consider any feasible infinite horizon policy μ, we have*

$$\lim_{T\to\infty} \frac{1}{T+1} \mathcal{J}_T(\mathcal{Q}_T \mu) = \mathcal{J}_\infty(\mu),$$

where $\mathcal{Q}_T \mu = (\mu_0, \ldots, \mu_{T-1})$.

Proof. We first consider the centralized case.

$$\begin{aligned}
&\mathcal{J}_\infty(\mu) = \left(\lim_{T\to\infty} \frac{1}{T+1} \mathcal{J}_T \right)(\mu) \\
&= \lim_{T\to\infty} \frac{1}{T+1} \mathbb{E} \left\{ \sum_{t=0}^{T-1} \left(x_t^T Q x_t + u_t^T N_t R N_t u_t \right) \right. \\
&\qquad \left. + x_T^T Q_T x_T \middle| (1), u_t = \mu_t(I_t), \forall t \in \mathbb{Z}+ \right\} \\
&= \lim_{T\to\infty} \frac{1}{T+1} \mathbb{E} \left\{ \sum_{t=0}^{T-1} \left(x_t^T Q x_t + u_t^T N_t R N_t u_t \right) \right. \\
&\qquad \left. + x_T^T Q_T x_T \middle| (1), u_t = \mu_t(I_t), \forall t \in \{0, \ldots, T-1\} \right\} \\
&= \lim_{T\to\infty} \frac{1}{T+1} \mathcal{J}_T(\mathcal{Q}_T \mu).
\end{aligned}$$

Similarly for the decentralized case when $u_t^1 = \mu_t^1(I_t^1)$ and $u_t^2 = \mu_t^2(I_t^2)$. ■

Lemma 16. *Let μ be the limit of the sequence of the set of optimal finite horizon policies defined in Section 6. Consider the finite horizon decentralized problems as in Problem 2. Instead of applying the optimal actions defined in Theorem 11, we apply the actions defined in Theorem 8, by using only the first T policies of μ when the time horizon is $T+1$. Then the corresponding cost* $\mathcal{J}_T(\mathcal{Q}_T\mu) =$

$$T \operatorname{trace}\left(\begin{bmatrix} P_\infty^{11} & P_\infty^{12} \\ P_\infty^{21} & Y_\infty \end{bmatrix}\Sigma_v\right) + \operatorname{trace}\left(\begin{bmatrix} P_\infty^{11} & P_\infty^{12} \\ P_\infty^{21} & Y_\infty \end{bmatrix}\Sigma_s\right) + \mathbb{E}\left(\begin{bmatrix} x_T \\ x_T^2 - \xi_T \end{bmatrix}^T \begin{bmatrix} Q_T - P_\infty & 0 \\ 0 & P_T^{22} - Y_\infty \end{bmatrix}\begin{bmatrix} x_T \\ x_T^2 - \xi_T \end{bmatrix}\right).$$

Proof. See [1] for a similar proof. Instead of letting $P_T = Q_T$, $Y_T = P_T^{22}$, and $r_T = 0$, we let $P_T = P_\infty$, $Y_T = Y_\infty$, and $r_T =$

$$\mathbb{E}\left(\begin{bmatrix} x_T \\ x_T^2 - \xi_T \end{bmatrix}^T \begin{bmatrix} Q_T - P_\infty & 0 \\ 0 & P_T^{22} - Y_\infty \end{bmatrix}\begin{bmatrix} x_T \\ x_T^2 - \xi_T \end{bmatrix}\right).$$

■

We are now ready to show that the second assumption in Corollary 10 is true. According to Lemma 14, if there exists $\Sigma > 0$ such that

$$\Sigma > \mathbb{E}_{N_t}\left\{A_{cl}\Sigma A_{cl}^T\right\} + \begin{bmatrix} \Sigma_v & 0 \\ 0 & 0 \end{bmatrix},$$

we must have $\operatorname{cov}\left(\begin{bmatrix} x_t \\ \xi_t \end{bmatrix}\right)$ upper bounded, so are $\operatorname{cov}(x_t)$ and $\operatorname{cov}(x_t^2 - \xi_t)$. Consequently,

$$\begin{aligned} r_T &= \operatorname{trace}((Q_T - P_\infty)\operatorname{cov}(x_T)) \\ &\quad + \operatorname{trace}((P_T^{22} - Y_\infty)\operatorname{cov}(x_T^2 - \xi_T)) \end{aligned}$$

is bounded, and we have

$$\lim_{T\to\infty} \frac{1}{T+1}\mathcal{J}_T(\mathcal{Q}_T\mu) = \operatorname{trace}\left(\begin{bmatrix} P_\infty^{11} & P_\infty^{12} \\ P_\infty^{21} & Y_\infty \end{bmatrix}\Sigma_v\right).$$

By the third assumption in Corollary 10, we must have

$$\mathcal{J}_\infty(\mu) = \operatorname{trace}\left(\begin{bmatrix} P_\infty^{11} & P_\infty^{12} \\ P_\infty^{21} & Y_\infty \end{bmatrix}\Sigma_v\right),$$

which coincides with the limit of optimal finite horizon costs divided by the time horizon by taking $T \to \infty$, thereby meeting the second assumption of Corollary 10.

We are now ready to prove the main Theorem 8.

Proof. (Theorem 8) Given the conditions in the statement of Theorem 8, we must have P_t and Y_t converge to P_∞ and Y_∞, respectively. Then K_t and J_t, which depend on t through P_{t+1} and Y_{t+1}, become time-invariant as in (15) and (16). Replacing K_t and J_t in (27), (28), (29), and (30), we have (17), (18), (19), and (20), respectively. The optimal cost converges to (21) evidently. Since all assumptions of Corollary 10 are true, this policy does minimize the infinite horizon cost (6). ■

8 Conclusion

In this paper, we have extended the boundaries of two-player decentralized LQG control over TCP erasure network to infinite horizon cases. Unlike the classical approaches which use spectral factorization, we derived an infinite horizon synthesis by showing that the sequence of finite horizon optimal controllers converges to an optimal infinite horizon synthesis under three assumptions set out in Corollary 10. To meet these assumptions, we derived a sufficient condition with two inequalities for two MAREs to converge in Theorem 6. Under this sufficient condition on convergence, along with a sufficient condition on stability as in Theorem 7, an optimal two-player synthesis was provided in Theorem 8.

References

[1] Chung-Ching Chang and Sanjay Lall. An explicit solution for optimal two-player decentralized control over a tcp erasure channel with state feedback. In *Proceedings of the American Control Conference*, 2011.

[2] E. Garone, B. Sinopoli, A. Goldsmith, and A. Casavola. Lqg control for distributed systems over tcp-like erasure channels. In *Proceedings of IEEE Conference on Decision and Control*, 2007.

[3] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry. Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control, Special Issue on Networked Control Systems*, 2004.

[4] John Swigart and Sanjay Lall. An explicit dynamic programming solution for a decentralized two-player optimal linear-quadratic regulator. In *Proceedings of the International Symposium on Mathematical Theory of Networks and Systems*, 2010.

[5] John Swigart and Sanjay Lall. An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator. In *Proceedings of the American Control Conference*, 2010.

[6] John Swigart and Sanjay Lall. Optimal synthesis and explicit state-space solution for a decentralized two-player linear-quadratic regulator. In *Proceedings of IEEE Conference on Decision and Control*, 2010.