

# Chapter 1

## The Foundation: Logic and Proof

### 1.1 Propositional Logic

# 1. Introduction

- Logic is the basis for mathematical reasoning.
  - Application:
    1. verification the correctness of programs;
    2. ....

# 1. Introduction

## □ Example 1

### ■ Algebraic Laws of Programming Languages

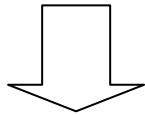
Language:

Skip

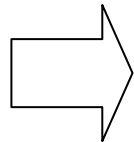
$x := e$

$P \triangleleft b \triangleright Q$

while b do P



Semantics using  
Logic Formulae



Laws:

$$(1) x := e; x := f(x) = x := f(x)$$

$$(2) (P; Q); R = P; (Q; R)$$

$$(3) P \triangleleft b \triangleright P = P$$

$$(4) (P \triangleleft b \triangleright Q) \triangleleft c \triangleright R \\ = P \triangleleft b \wedge c \triangleright (Q \triangleleft c \triangleright R)$$

$$(5) \text{while } b \text{ do } P = \text{if } b \text{ then} \\ (P ; \text{while } b \text{ do } P) \text{ else Skip}$$

# 2. Propositions (命题)

- Definition of proposition
  - A proposition is a declarative statement that is true or false, but not both. (即:表示判断的语句称为命题)
- Example 1 (see page 2)
  - All the following declarative sentences are propositions:
    1. Washington, D.C. is the capital of the United States of America.
    2. Toronto is the capital of Canada.
    3.  $1 + 1 = 2$ .
    4.  $2 + 2 = 3$ .
  - Propositions 1 and 3 are true, whereas 2 and 4 are false

## 2. Propositions (命题)

- Example 2 (not propositions)
  - Consider the following sentences.
    1. What time is it now?
    2. Read it carefully?
    3.  $x+1 = 2$
    4.  $x+y = z$
    - Sentences 1 and 2 are not declarative sentences.
    - Sentences 3 and 4 are neither true or false.
- The truth value of proposition is true, denoted by T.
- The false value of proposition is false, denoted by F.

# 2. Propositions (命题)

## □ Negation of a Proposition (“非”)

### ■ Definition 1

□ Let  $p$  be a proposition. The statement “It is not the case that  $p$ ” is another proposition, called the negation of  $p$ .

1. The negation of  $p$  is denoted by  $\neg p$ .

2. The proposition “ $\neg p$ ” is read “not  $p$ ”.

### □ Truth Table

$p$	$\neg p$
T	F
F	T

## 2. Propositions (命题)

- Negation of a Proposition (“非”)
  - Example 3
    - Find the negation of “Today is Friday.”
    - Solution:
      1. “It is not the case that today is Friday,”
      2. or “Today is not Friday,”
      3. or “It is not Friday today.”

# 2. Propositions (命题)

## □ Conjunction (“并且”又称“合取”)

### ■ Definition

- Let  $p$  and  $q$  be propositions. The proposition “ $p$  and  $q$ ”, denoted as “ $p \wedge q$ ” is the proposition that is true when both of them are true and is false otherwise. The proposition “ $p \wedge q$ ” is called the conjunction of  $p$  and  $q$ .

### □ Truth Table

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F



# 2. Propositions (命题)

- Conjunction

- Example 5 (page 4)

- Find the conjunction of the propositions  $p$  and  $q$  where  $p$  is the proposition “Today is Friday” and  $q$  is the proposition “It is raining today.”

- Solution:

1.  $p \wedge q$  is the proposition “Today is Friday and it is raining today.”

2. When is  $p \wedge q$  true?

## 2. Propositions (命题)

### □ Disjunction (“或者”又称“析取”)

#### ■ Definition

- Let  $p$  and  $q$  be propositions. The proposition “ $p$  or  $q$ ”, denoted as  $p \vee q$ , is the proposition that is false when  $p$  and  $q$  are both false and true otherwise. The proposition  $p \vee q$  is called the disjunction of  $p$  and  $q$ .

#### □ Truth Table

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

## 2. Propositions (命题)

### □ Example 6

- Find the disjunction of the propositions  $p$  or  $q$  where  $p$  is the proposition “Today is Friday” and  $q$  is the proposition “It is raining today.”
- Solution:
  - $p \vee q$  is the proposition “Today is Friday or it is raining today.”
  - When is  $p \vee q$  true?

## 2. Propositions (命题)

### □ Exclusive (“异或”)

#### ■ Definition

- Let  $p$  and  $q$  be propositions. The exclusive or of  $p$  and  $q$ , denoted by  $p \oplus q$ , is the proposition that is true when exactly one of  $p$  and  $q$  is true and is false otherwise

#### □ Truth Table

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

# 2. Propositions (命题)

## □ Implication (“蕴含”)/Conditional Statement

### ■ Definition

- Let  $p$  and  $q$  be propositions. The implication  $p \rightarrow q$  is the proposition that is false when  $p$  is true and  $q$  is false, and is true otherwise.
- In this implication  $p$  is called the hypothesis (or antecedent or premise) and  $q$  is called the conclusion (or consequence)

### □ Truth Table

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

# 2. Propositions (命题)

## □ Remark

- a variety of terminology to express  $p \rightarrow q$  (page 6).
- When is  $p \rightarrow q$  false?
  - How about the case that  $p$  is false?
- $q \rightarrow p$  is called the converse of  $p \rightarrow q$  (逆命题).
- $\neg q \rightarrow \neg p$  is called contrapositive of  $p \rightarrow q$  (逆否命题).
- $\neg p \rightarrow \neg q$  is called inverse of  $p \rightarrow q$  (否命题).

## □ Example 9 (see page 8)

- What are the converse, contrapositive, inverse of the implication "The home team wins whenever it is raining."?
- Solution:
  - The implication can be rewritten as: "If it is raining, then the home team wins" Then .....

## 2. Propositions (命题)

### □ Biconditional (“当且仅当”又称“等价”)

#### ■ Definition

- Let  $p$  and  $q$  be propositions. The biconditional  $p \leftrightarrow q$  is the proposition that is true when  $p$  and  $q$  have the same truth values, and is false otherwise.

#### □ Truth Table

$p$	$q$	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

## 2. Propositions (命题)

- $p \leftrightarrow q$  ----- "p if and only if q"
- $p \leftrightarrow q$  has the same truth table of  $(p \rightarrow q) \wedge (q \rightarrow p)$
- Example 10 (see page 9)
  - Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket."
  - Then  $p \leftrightarrow q$  is the statement "You can take the flight if and only if you buy a ticket."



## 2. Propositions (命题)

### □ Truth Tables of compound Propositions

#### ■ Example 11

□ Construct the truth table of the compound proposition  $(p \vee \neg q) \rightarrow (p \wedge q)$

□ Solution: The Truth table is:

$p$	$q$	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

# 3. Precedence of Logical Operations

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

□ Example:

1.  $p \wedge q \vee r$  means  $(p \wedge q) \vee r$  rather than  $p \wedge (q \vee r)$
2.  $p \vee q \rightarrow r$  is the same as  $(p \vee q) \rightarrow r$ .
  - Remark: We will use parentheses when the order of the conditional operator and biconditional operator is at issue.

# 4. Translating English Sentences

## □ Example 12

- How can this English sentence be translated into a logical expression.
  - “You can access the internet from the campus only if (page 6) you are a computer science major or you are not a freshman (新生).”
- Solution:
  - a-----“You can access the internet from the campus.”
  - b-----“You are a computer science major.”
  - c-----“You are a freshman.”
  - Then this sentence can be expressed as  
 $a \rightarrow (c \vee \neg b)$

# 4. Translating English Sentences

## □ Example 13

- How can this English sentence be translated into logical expression?

- "You cannot ride the roller coaster (过山车) if you are under 4 feet tall unless you are older than 16 years older."

- Solution:

- $q$ -----"You can ride the roller coaster."

- $r$ -----"You are under 4 feet tall."

- $s$ -----"You are older than 16 years older."

- Then the sentence can be translated to

$$(r \wedge \neg s) \rightarrow \neg q.$$

# Homework

□ Page 16~19

■ 2, 4, 6, 8, 12, 26, 28, 30, 32,