Chapter 1
The Foundation: Logic and Proof

1.4 Nested Quantifiers

1. Introduction

- Nested quantifiers
  - occur within the scope of other quantifiers
  - $\forall x\ \exists y(x+y=0)$
- Example 1 (page 51)
  - domain for x and y -----all real numbers
    - $\forall x \forall y\ (x+y=y+x)$---true
    - $\forall x \exists y\ (x+y=0)$------true
    - $x \forall y \forall z\ (x+(y+z)=(x+y)+z)$------true

1. Introduction

- Example 2 (page 51)
  - $\forall x \forall y ((x>0) \land (y<0) \rightarrow (xy<0))$
    - domain: all real numbers
    - English meaning
    - Value:
      - true

2. The order of Quantifiers

- Example 3 (page 52)
  - P(x,y)-----"x+y=y+x"
    - domain for all variables: all real numbers
    - How about

1. $\forall x\ \forall y\ P(x,y)$----------true
2. $\forall y\ \forall x\ P(x,y)$----------true

- We have: $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$

2. The Order of Quantifiers

- Example 4 (page 52)
  - $Q(x,y)$------"x+y=0"
    - universe of discourse: all real numbers
    - How about
      - $\exists y \forall x\, Q(x,y)$ and $\forall x \exists y\, Q(x,y)$?
    - Solution:
      - $\exists y \forall x\, Q(x,y)$----There is a real number y such that for every real number x, Q(x,y). ----false
      - $\forall x \exists y\, Q(x,y)$----For every real number x there is a real number y such that Q(x,y). -----true
      - $\exists y \forall x\, Q(x,y)$-----$\forall x \exists y\, Q(x,y)$(not equivalent)

2. The order of Quantifiers

- Summary (see Table 1 on page 34)

Statement	When True?	When False?
$\forall x\ \forall y\ P(x,y)$ $\forall y\ \forall x\ P(x,y)$		
$\forall x\ \exists y\ P(x,y)$		
$\exists x \forall y\ P(x,y)$		
$\exists\ x\ \exists\ y\ P(x,y)$ $\exists\ y\ \exists\ x\ P(x,y)$		

2. The order of Quantifiers

- Further,
  - If $\exists y \forall x\, P(x,y)$ is true, then $\forall x \exists y\, P(x,y)$ is true.
  - If $\forall x \exists y P(x,y)$ is true, then it is not necessary for $\exists y \forall x\, P(x,y)$ to be true.
  - Please see Exercise 22and 24 at the end of this chapter (page 107).

2. The order of Quantifiers

- Example 5 (page 53)
  - Q(x,y,z)-----"x+y=z"
    - domain:all real numbers
    - How about
      - $\forall x \forall y \exists z\, Q(x,y,z)$
      - $\exists z \forall x \forall y\, Q(x,y,z)$
    - Solution:
      - $\forall x \forall y \exists z\, Q(x,y,z)$ is true.
      - $\exists z \forall x \forall y\, Q(x,y,z)$ is false.

3. Translating Mathematical Statements into Statements Involving Nested Quantifiers

- Example 6
  - Translate the statement "The sum of two positive integers is always positive" into a logical expression.
  - Solution:
    - Way1: domain for x and y----all integers
      - $\forall x \forall y\ ((x>0) \wedge (y>0) \rightarrow (x+y>0))$
    - Way 2: domain for x and y----all positive integers
      - $\forall x \forall y\ (x+y>0)$

3. Translating Mathematical Statements into Statements Involving Nested Quantifiers

- Example 7
  - Translate the statement “Every real number except zero has a multiplicative inverse”
  - Solution:
    - Domain for x and y-----all real numbers
      - $\forall x\,((x \neq 0) \rightarrow \exists y\,(xy=1))$

4. Translating from Nested Quantifiers into English

- Example 9 (page 55)
  - $\forall x\ (C(x) \vee \exists y\ (C(y) \wedge F(x,y)))$
    - C(x)----”x has a computer”
    - F(x,y)----”x,y are friends”
    - universe of discourse for both x and y
      - all students in the school??? What does the formula mean?

4. Translating from Nested Quantifiers into English

- Example 10 (page 55)
  - $\exists x\ \forall y\ \forall z\ ((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$
    - F(a,b)----a and b are friends
    - domain for x, y and z: all students in your school
    - What does this formula mean?

5. Translating English Sentences Into Logical Expression

- Example 11 (page 56)
  - "If a person is female and is parent, then this person is someone's mother."
    - domain------all people
    - Solution:
      - also can be expressed as "For every person, if person x is a female and person x is a parent, then there exists a person y such that person x is the mother of person x."
      - F(x)-----x is female; P(x)-----x is a parent
      - M(x,y)-----x is the mother of y, Then, the formula is:

1. $\forall x\ ((F(x) \land P(x)) \rightarrow \exists y\ M(x,y))$ or
2. $\forall x\ \exists y\ ((F(x) \land P(x)) \rightarrow M(x,y))$

5. Translating English Sentences Into Logical Expression

- Example 12 (page 46)
  - "Everyone has exactly one best friend"
  - domain: all people
  - Solution:
    - "For every person x, person x has exactly one best friend"
      - B(x,y) ----- y is the best friend of x
      - $\exists y\ (\ B(x,y) \land \forall z\ (\ (z \neq y) \rightarrow \neg B(x,z)\ )\ )$
      - $\forall x$(………the above formula…………….)

6. Negating Nested Quantifiers

- Example 14 (page 57)
  - Express the negation of $\forall x \exists y\ (xy=1)$ so that no negation precedes a quantifier.
  - Solution:

$$\neg \forall x \exists y\ (xy=1)$$
$$\equiv \exists x \neg \exists y\ (xy=1)$$
$$\equiv \exists x \forall y \neg (xy=1)$$
$$\equiv \exists x \forall y\ (xy\neq 1)$$

Homework

- page 58~62
  - 26, 28, 30, 40