Chapter 2
Basic Structures: Sets, Functions, Sequences and Sum

2.4 Cardinality

1. Introduction

- Definition 4 (page 158)
  - The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.
- Definition 5 (page 158)
  - A set that is either finite or has the same cardinality as the set of positive integers is countable (可数的或可列的).
  - A set that is not countable is called uncountable.
  - When an infinite set S is countable, we denote the cardinality of S by 特殊符号(见书)

1. Introduction

- Example 158 (page 234)
  - Show that the set of odd positive integer is a countable set.
    - Proof: Consider the function

      $f(n)=2n-1$

      – from Z+ to the set of odd positive integers.
    - Now we need to show that f is a one-to-one correspondence .
    - For the details, please see page 158.

1. Introduction

- Further explanation of countable sets.
  - An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
  - The reason for this is:a_1, a_2,….., a_n, ...
  - where： $a_1=f(1)$, $a_2=f(2)$, ..., $a_n=f(n)$ (see page 159)
- Example 19 (page 159)
  - Show that the set of positive rational numbers is countable.
  - Proof: See book.

1. Introduction

- 定理：开区间(0,1)上的所有实数组成一个不可列集 (Similar to page 160)
  - 证明： 倘若开区间(0,1)可列，那么(0,1)上的所有实数可以排列成：

$s_1, s_2, s_3, \ldots\ldots, s_n, \ldots\ldots$

现将s_n $(n \in Z+)$写成十进制无限小数的形式：

$s_1 = 0.a_{11}a_{12}a_{13}a_{14}\ldots\ldots$,

$s_2 = 0.a_{21}a_{22}a_{23}a_{24}\ldots\ldots$,

$s_3 = 0.a_{31}a_{32}a_{33}a_{34}\ldots\ldots$,

……………………………………

$s_n = 0.a_{n1}a_{n2}a_{n3}a_{n4}\ldots\ldots$,

……………………………………

1. Introduction

- 定理: 开区间(0,1)上的所有实数组成一个不可列集 (Similar to page 160)
  - 证明(续):

    那么，对任何$n \in Z+$)，令

    若$a_{nn} \neq 1$, 那么$b_n = 1$,

    若$a_{nn} = 1$，那么$b_n = 2$

    由此得到无限十进制小数 $r_1 = 0.b_1 b_2 b_3 b_4 \ldots\ldots$,

    $r \in (0,1)$但对任何n，$r \neq s_n$，这与(0,1)中元素排列成：

    s_1, s_2, s_3, ……, s_n, …….矛盾。

    由此可得开区间(0,1)不是可列集。

1. Introduction

- 推论: 实数集R不是可数集.
  - 证明：倘若R 是可列集，那么R的无限子集(0,1)也是可列集，但(0,1)不是可列集, 所以R一定不是可列集。
- 定理: (0,1)~R, 即等势.
  - 证明：Please see blackboard.

Homework

- Page 160~163
  - 35, 36, 37, 40, 41, 42