Parallel Gauss Sieve Algorithm: Solving the SVP Challenge over a 128-Dimensional Ideal Lattice

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Abstract. In this paper, we report that we have solved the SVP Challenge over a 128-dimensional lattice in Ideal Lattice Challenge from TU Darmstadt, which is currently the highest dimension in the challenge that has ever been solved. The security of lattice-based cryptography is based on the hardness of solving the shortest vector problem (SVP) in lattices. In 2010, Micciancio and Voulgaris proposed a Gauss Sieve algorithm for heuristically solving the SVP using a list L of Gauss-reduced vectors. Milde and Schneider proposed a parallel implementation method for the Gauss Sieve algorithm. However, the efficiency of the more than 10 threads in their implementation decreased due to the large number of non-Gauss-reduced vectors appearing in the distributed list of each thread. In this paper, we propose a more practical parallelized Gauss Sieve algorithm. Our algorithm deploys an additional Gauss-reduced list V of sample vectors assigned to each thread, and all vectors in list L remain Gauss-reduced by mutually reducing them using all sample vectors in V. Therefore, our algorithm allows the Gauss Sieve algorithm to run for large dimensions with a small communication overhead. Finally, we succeeded in solving the SVP Challenge over a 128-dimensional ideal lattice generated by the cyclotomic polynomial $x^{128}+1$ using about 30,000 CPU hours.

Keywords: shortest vector problem, lattice-based cryptography, ideal lattice, Gauss Sieve algorithm, parallel algorithm

1 Introduction

Lattice-based cryptography has been considered a powerful primitive for constructing useful cryptographic protocols. The security of lattice-based cryptography is based on the hardness of solving the shortest vector problem (SVP), which involves searching for the shortest nonzero vectors in lattices. Ajtai proved that the worst case complexity of solving the SVP is NP-hard under randomized reductions [1]. The α-SVP [18] is an approximation problem of the SVP, which searches for elements with the size of the shortest vector multiplied by a small approximation factor α. Many cryptographic primitives have been built on lattices due to their security against quantum computers and their novel functionalities: Ajtai-Dwork scheme [2], NTRU [12], fully-homomorphic cryptosystems [10], and multi-linear maps [9].

There are several approaches for solving the SVP and the α-SVP. The fastest deterministic algorithm is the Voronoi cell algorithm [19], which runs in exponential time $2^{O(n)}$ and space $2^{O(n)}$ for n-dimensional lattices. The sieving algorithms, which are explained in the next subsection, are probabilistic algorithms that require exponential time $2^{O(n)}$ and space $2^{O(n)}$ [3, 22, 6, 5]. The enumeration algorithms are exhaustive search algorithms that need time $2^{O(n^2)}$ or $2^{O(n\log n)}$, but only the polynomial size of space [31, 32, 8], and they are suitable for parallelization using multicore CPUs and GPUs. Moreover, the lattice basis reduction such as LLL or BKZ is a polynomial-time approximation algorithm [17, 30]. Generally, enumeration algorithms are also used in lattice basis reduction algorithms as a subroutine for solving the α-SVP. On the other hand, sieving algorithms are used only for solving the SVP.

1.1 Sieving Algorithms and Ideal Lattices

In 2001 Ajtai *et al.* proposed the first sieve algorithm for solving the SVP [3]. There are many variants of the sieving algorithm [22, 6, 5] that try to improve the computational costs of the algorithm. In 2009 Micciancio and Voulgaris proposed a practical sieving algorithm, called the Gauss Sieve algorithm [20]. The Gauss Sieve algorithm consists of a list L of vectors in the lattice and a reduction algorithm that outputs a shorter vector from two input vectors. List L manages the vectors reduced by the reduction algorithm. The number of vectors in L increases but the norm of several vectors L is shrunk by the reduction algorithm, and eventually the shortest nonzero vector can be found in list L.

The theoretical upper boundary of the computation time of the Gauss Sieve algorithm is not yet proved; however, the Gauss Sieve algorithm is faster than any other sieve algorithm in practice, because it deploys a list L of pair-wise Gauss-reduced vectors that can gradually reduce the norm of vectors in the list. The time complexity of the Gauss sieve is estimated to be asymptotically $2^{0.52n}$ for n-dimensional lattices [20]. In 2011 Milde and Schneider considered a parallelization variant of the Gauss Sieve algorithm. The distributed list $L_i(i = 1, 2, ..., t)$ of $L = \cup_i L_i$ with a queue $Q_i(i = 1, 2, ..., t)$ is assigned to each thread, where L_i is connected to adjacent list L_{i+1} and the Gauss-reduced vectors are transferred from list L_i to L_{i+1} using queue Q_i for $t \in \mathbb{N}$, where we set $L_{t+1} = L_1$. However, several vectors in the whole list $L = \cup_i L_i$ are no longer Gauss-reduced without exact synchronization of queues Q_i for all threads, and thus the efficiency decreases as the number of threads increases. From the experiment by Milde and Schneider, once the number of threads increases to more than ten, the speed-up factor does not exceed around five. Therefore, it is difficult to apply to large-scale parallel computation.

In order to realize efficient construction of lattice-based cryptography, ideal lattices are often used. Using ideal lattices, many cryptographic primitives work faster and require less storage [12, 9]. One of the open problems is whether the computational problems related to the ideal lattices are easier to solve compared with those of random lattices [24]. First, Micciancio and Voulgaris mentioned the possibility of speeding up the sieving algorithm for ideal lattices [20]. In ideal lattices, several vectors of similar norms have a rotation structure, and thus it is possible to compute the set of vectors in the reduction algorithm derived from the sieve algorithm without a large overhead. Schneider proposed the Ideal Gauss Sieve algorithm, which uses the rotation structure of the *Anti-cyclic lattice* generated by the polynomial $x^n + 1$ where n is a power of two [28]. Then, their proposed algorithm enables the Gauss Sieve algorithm to run about 25 times faster on 60-dimensional ideal lattices.

1.2 Our Contribution

We propose a parallelized Gauss Sieve algorithm using an additional list V generated by the multisampling technique of vectors in the lattice. Our algorithm mutually reduces the vectors in both L and V, so that all vectors in both lists V and L remain pair-wisely Gauss-reduced. Using this technique, the reduction algorithm can be easily parallelized. Additionally, even if the number of threads increases, our algorithm keeps the vector set pairwise-reduced and efficiency is maintained. Therefore, our algorithm enables the Gauss Sieve algorithm to run without excessive overhead even in a large-scale parallel computation.

With the result of our proposed algorithm, we succeeded in solving the SVP Challenge over a 128-dimensional ideal lattice generated by the cyclotomic polynomial $x^{128} + 1$ using about 30,000 CPU hours. In our experiment, we used 84 instances and each instance runs 32 threads, namely the number of threads is 2,688 in total. The communication overhead among threads was less than ten percents of the total running time.

2 Definitions and Problems

In this section, we provide a short overview of the definition of the SVP on the lattice. We then explain the definitions of Gauss-reduced and pairwise-reduced for a set of vectors on the lattice used for the Gauss Sieve algorithm.

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a set of n linearly independent vectors in $\mathbb{R}^m$. The lattice generated by B is the set $\mathcal{L}(B) = \mathcal{L}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = \{\sum_{1 \le i \le n} x_i \mathbf{b}_i,\ x_i \in \mathbb{Z}\}$ of all integer linear combinations of the vectors in B. The set B is called *basis* of the lattice $\mathcal{L}(B)$. In the following, we denote by $\mathcal{L}(\mathbf{B})$ the lattice of basis B as the matrix representation $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathbb{R}^{m \times n}$. If $n = m$, the lattice $\mathcal{L}(\mathbf{B})$ is called full-rank. In this paper, for the sake of simplicity, we will consider only full-rank lattices and assume that all the basis vectors $\mathbf{b}_i(i = 1, 2, ..., n)$ have only integer entries.

The Euclidean norm of vector $\mathbf{v} = (v_0, \ldots, v_{n-1}) \in \mathcal{L}(\mathbf{B})$ is denoted by $||\mathbf{v}|| = \sum_{0 \le i < n} v_i^2$. The norm of the shortest nonzero vectors in $\mathcal{L}(\mathbf{B})$ is denoted by $\lambda_1(\mathcal{L}(\mathbf{B}))$. The inner product of two vectors $\mathbf{a} = (a_0, \ldots, a_{n-1}), \mathbf{b} = (b_0, \ldots, b_{n-1}) \in \mathcal{L}(\mathbf{B})$ is defined by $\langle \mathbf{a} \cdot \mathbf{b} \rangle = \sum_{0 \le i < n} a_i b_i$. For $x \in \mathbb{R}$, $\lfloor x \rceil$ denotes the nearest integer to x, namely $\lfloor x + 1/2 \rfloor$.

We define the shortest vector problem (SVP) on a lattice as follow.

Definition 1 (Shortest vector problem on a lattice) *Given a lattice $\mathcal{L}(\mathbf{B})$, find a shortest nonzero vector of the length $\lambda_1(\mathcal{L}(\mathbf{B}))$ in $\mathcal{L}(\mathbf{B})$.*

From the Gaussian heuristic, the length of a shortest vector in lattice $\mathcal{L}(\mathbf{B})$ is estimated to be $\lambda_1(\mathcal{L}(\mathbf{B})) = (1/\sqrt{\pi})\Gamma(\frac{n}{2}+1)^{\frac{1}{n}} \cdot \det(\mathcal{L}(\mathbf{B}))^{\frac{1}{n}}$, where $\Gamma(x)$ is the gamma-function and $\det(\mathbf{B})$ is the determinant of matrix $\mathbf{B}$.

Let $g(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree n, and let I be an ideal of ring $\mathbb{Z}[x]/(g(x))$. All elements of ideal I are represented by polynomials $\mathbf{v}(x) = \sum_{0 \le i < n} v_i x^i$ in $\mathbb{Z}[x]/(g(x))$. We identify $\mathbf{v}(x)$ with vectors $\mathbf{v} = (v_0, \ldots, v_{n-1}) \in \mathbb{Z}^n$. The ideal I is an additive subgroup of $\mathbb{Z}[x]/(g(x))$, and the set $\{\mathbf{v} = (v_0, \ldots, v_{n-1}) \in \mathbb{Z}^n | \mathbf{v}(x) = \sum_{0 \le i < n} v_i x^i \in I\}$ becomes a lattice. This is called the ideal lattice generated by $\mathbf{v}(x)$, and its basis B consists of the rotation vectors $x^i\mathbf{v}(x) \in \mathbb{Z}[x]/(g(x))$ for $i = 0, 1, ..., n-1$. The cyclotomic polynomials, such as $g(x) = x^n + 1$ for $n = 2^h$ with some positive integer h, are often used for generating the ideal lattice in cryptography.

2.1 Gauss-reduced and Pairwise-reduced

We define Gauss-reduced and pairwise-reduced for a set of vectors on lattice $\mathcal{L}(\mathbf{B})$. We then explain an algorithm for determining and reducing two given vectors of lattice $\mathcal{L}(\mathbf{B})$.

First, the definition of Gauss-reduced is as follows.

Definition 2 (Gauss-reduced) *If two different vectors $\mathbf{a}, \mathbf{b} \in \mathcal{L}(\mathbf{B})$ satisfy $||\mathbf{a} \pm \mathbf{b}|| \ge \max(||\mathbf{a}||, ||\mathbf{b}||)$, then $\mathbf{a}, \mathbf{b}$ are called Gauss-reduced.*

Micciancio and Voulgaris explained about the way to convert two vectors $\mathbf{a}, \mathbf{b}$ in $\mathcal{L}(\mathbf{B})$ to be Gauss-reduced. The conversion algorithm uses the Reduce algorithm (Alg.2 at Appendix A), which outputs vectors $\mathbf{a}'$ for two vectors $\mathbf{a}, \mathbf{b}$ in $\mathcal{L}(\mathbf{B})$. The reduced vector $\mathbf{a}'$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, which has a shorter norm than $\max(\mathbf{a}, \mathbf{b})$, or otherwise $\mathbf{a}' = \mathbf{a}$. From this, we can determine whether two vectors $\mathbf{a}, \mathbf{b}$ in $\mathcal{L}(\mathbf{B})$ are Gauss-reduced. Indeed, we can easily prove the following lemma.

Lemma 1. *Let $\mathbf{a}, \mathbf{b}$ be two vectors in $\mathcal{L}(\mathbf{B})$. We set $\mathbf{a}'$ =Reduce$(\mathbf{a}, \mathbf{b})$ and $\mathbf{b}'$ =Reduce$(\mathbf{b}, \mathbf{a})$. If both $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$ hold, then $\mathbf{a}, \mathbf{b}$ are Gauss-reduced.*

If two vectors $\mathbf{a}, \mathbf{b}$ are not Gauss-reduced, then $\mathbf{a} \ne \mathbf{a}'$ or $\mathbf{b} \ne \mathbf{b}'$ holds by Lemma 1. Recall that the reduced vector $\mathbf{a}' \leftarrow$Reduce$(\mathbf{a}, \mathbf{b})$ has the property $||\mathbf{a}'|| \le ||\mathbf{a}||$. After performing both Reduce$(\mathbf{a}, \mathbf{b})$ and Reduce$(\mathbf{b}, \mathbf{a})$, we know that the resulting vectors $(\mathbf{a}', \mathbf{b}')$ are either Gauss-reduced or $\mathbf{a}'$ (or $\mathbf{b}'$) is strictly shorter than $\mathbf{a}$ (or $\mathbf{b}$), respectively. If we repeatedly run the Reduce algorithm for $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$, then we expect the resulting vectors $(\mathbf{a}', \mathbf{b}')$ to become Gauss-reduced. From our experiments in the 100-dimensional lattices, we can obtain the Gauss-reduced vectors after at most 10 iterations in most cases.

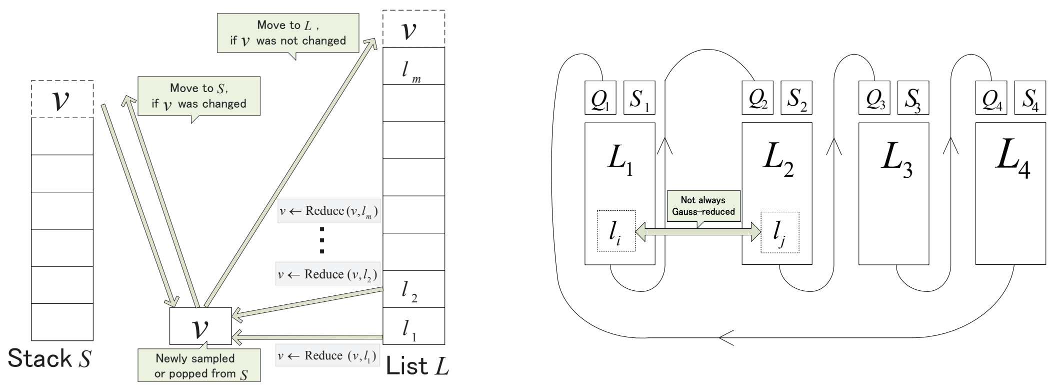
If $\mathbf{a}, \mathbf{b}$ are linearly dependent, the output of Reduce$(\mathbf{a}, \mathbf{b})$ is always the zero vector, *i.e.*, $||\mathbf{a}'|| = 0$, which is called a “collision”. The collision is used as the condition for terminating the Gauss Sieve algorithm.

Definition 3 (Pairwise-reduced) *Let A be a set of d vectors in $\mathcal{L}(\mathbf{B})$. If every pair of two vectors $(\mathbf{a}_i, \mathbf{a}_j)$ in A for $i, j = 1, \ldots, d, i \ne j$ is Gauss-reduced, then the A is called pairwise-reduced.*

In general, if we append a vector $\mathbf{b} \in \mathcal{L}(\mathbf{B})$ to a pairwise-reduced set A, then $A \cup \{\mathbf{b}\}$ is not always pairwise-reduced. If every pair of two vectors $(\mathbf{a}_i, \mathbf{b})$ for $\mathbf{a}_1, ..., \mathbf{a}_d \in A$ is Gauss-reduced, then the union $A \cup \{\mathbf{b}\}$ becomes pairwise-reduced from the definition. Obviously we can prove the following lemma that shows that the union of two pairwise-reduced sets of vectors becomes pairwise-reduced by checking whether the all pairs of two vectors from A and B are Gauss-reduced.

Lemma 2 (Combining Lemma). *Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ and $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ be sets of vectors in $\mathcal{L}(\mathbf{B})$. Assume that both A and B are pairwise-reduced. If every pair of two vectors $(\mathbf{a}_i, \mathbf{b}_j)$ in A, B for $1 \le i \le r, 1 \le j \le m$ is Gauss-reduced, then the union $A \cup B$ is pairwise-reduced.*

This lemma is used for constructing our proposed parallel algorithm for the Gauss Sieve algorithm.


Move to L, if v was not changed
Move to S, if v was changed
v
l_m
$v \leftarrow$ Reduce (v, l_m)
$v \leftarrow$ Reduce (v, l_2)
$v \leftarrow$ Reduce (v, l_1)
l_2
l_1
Newly sampled or popped from S
Stack S
List L
Q_1 S_1
Q_2 S_2
Q_3 S_3
Q_4 S_4
L_1
L_2
L_3
L_4
Not always Gauss-reduced
l_i
l_j

(a) Gauss Sieve [20] (b) Parallel implementation [21]

Fig. 1. Flows of the Gauss Sieve algorithm (left) and its parallel implementation (right) of the Gauss Sieve algorithm

3 Gauss Sieve Algorithm

In this section, we briefly explain the Gauss Sieve algorithm [20], its parallel implementation [21], and the Ideal Gauss Sieve algorithm [28].

3.1 Gauss Sieve [20]

The Gauss Sieve (GS) algorithm was proposed by Micciancio and Voulgaris in 2009 [20] and it was implemented as **gsieve** library by Voulgaris [34]. This algorithm is shown in Alg.1 and Alg.3 at Appendix A. We prepare two auxiliary lists L and S, where L and S are defined by a set of vectors and a stack of vectors, respectively. L and S are initially assigned as empty. In the beginning of the GS algorithm, a new vector $\mathbf{v}$ is randomly sampled using Klein's randomized rounding algorithm [13].

The GS algorithm runs a subroutine, Gauss_Reduce, which updates $\mathbf{v}, L, S$ by the steps in the following two parts. The first part (steps 4 to 8) runs the Reduce algorithm using a list L for updating $\mathbf{v}' = \text{Reduce}(\mathbf{v}, \boldsymbol{\ell}_i)$ for all vectors $\boldsymbol{\ell}_i \in L$. Once the $\mathbf{v}'$ is not equal to $\mathbf{v}$, this vector $\mathbf{v}'$ is moved to stack S. The reason is that if $\mathbf{v}$ is reduced using $\boldsymbol{\ell}_i \in L$, then $\mathbf{v}'$ and $\boldsymbol{\ell}_j, (i > j)$ are not always Gauss-reduced. If the $\mathbf{v}$ is not changed by $\text{Reduce}(\mathbf{v}, \boldsymbol{\ell}_i)$ for all $\boldsymbol{\ell}_i \in L$, the steps in the second part (steps 9 to 13) are performed. The second part runs the Reduce algorithm using a list L that makes the list pairwise-reduced. If $\boldsymbol{\ell}'_i \neq \boldsymbol{\ell}_i$ holds for $\boldsymbol{\ell}'_i = \text{Reduce}(\boldsymbol{\ell}_i, \mathbf{v})$, then the vector $\boldsymbol{\ell}'_i$ is moved to stack S and deleted from L. By the above steps, all pairs $(\mathbf{v}, \boldsymbol{\ell}_i)$ are always Gauss-reduced, where $\boldsymbol{\ell}_i \in L$. Therefore, $L \cup \boldsymbol{v}$ becomes pairwise-reduced by Lemma 2. Then L is updated by $L \cup \mathbf{v}$ and the iteration is continued (step 2 in Alg.1). If the stack is not empty, $\mathbf{v}$ is popped from the stack S, otherwise, $\mathbf{v}$ is newly sampled at step 4 in Alg.1.

The termination condition of the GS algorithm is determined by the number of collisions of the zero vector ($||\mathbf{a}'|| = 0$) that appears in L. The variable K in Alg.1 is the total number of collisions. When the value of K exceeds the threshold condition $\alpha|L| + \beta$, then the GS algorithm is terminated. In the **gsieve** library [34], $\alpha = 1/10$, and $\beta = 200$ are chosen as the threshold values. The theoretical upper bound of the complexity of the GS algorithm is not yet proved; however, in practice, the GS algorithm is faster than any other sieving algorithms. According to Micciancio and Voulgaris [20], the complexity of the GS algorithm is asymptotically estimated as time $2^{0.52n}$ and space $2^{0.2n}$. Moreover, Micciancio and Voulgaris showed some experiments that the GS algorithm outputs a shortest vector in some lattices of up to 60 dimensions, but it is not theoretically proved that the GS algorithm always outputs a shortest vector [20].

The GS algorithm cannot be easily parallelized for the following reason. If the list is $L = \{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, ...\}$ in the first part (steps 4 to 8) in the Gauss_Reduce algorithm (Alg.3), then the algorithm runs $\mathbf{v}' \leftarrow \text{Reduce}(\mathbf{v}, \boldsymbol{\ell}_1)$, $\mathbf{v}'' \leftarrow \text{Reduce}(\mathbf{v}', \boldsymbol{\ell}_2)$,... in this iteration. Hence, the first part must be executed sequentially step by step. In contrast, it is easy to parallelize the second part (steps 9 to 13) of reducing the vectors $\boldsymbol{\ell}_1, ..., \boldsymbol{\ell}_t$ by $\text{Reduce}(\boldsymbol{\ell}_i, \mathbf{v})$ for fixed $\mathbf{v}$. Therefore, at most, only half of the Gauss_Reduce algorithm appearing in the GS algorithm can be parallelized. We show the flow of the GS algorithm in Figure 1(a).

3.2 Parallel Implementation of the Gauss Sieve Algorithm [21]

In 2011, Milde and Schneider proposed parallel implementation of the Gauss Sieve algorithm [21]. This method tries to extend the single Gauss Sieve algorithm into a parallel variant.

Let t be the number of threads in this method. Each thread has an instance that consists of list L_i, stack S_i, and queue Q_i, where $L = \cup_i L_i$, $S = \cup_i S_i$, $i = 1, 2, ..., t$, and Q_i is used as a buffer for the next thread. The individual instances are connected together in a ring fashion. Each instance deals with a sample vector $\mathbf{v}$ in the distributed list L_i independently just as in the original Gauss Sieve algorithm. First, an instance runs Reduce($\mathbf{v}, \boldsymbol{\ell}_i$), where $\boldsymbol{\ell}_i \in L_i$. Second, the instance runs Reduce($\boldsymbol{\ell}_i, \mathbf{v}$) inversely. After that, if it is not changed by the above reduction steps, the vector $\mathbf{v}$ is sent to the buffer Q_{i+1} of the next instance. Otherwise, $\mathbf{v}$ is moved to the distributed stack S_i in its own instance. If a vector $\mathbf{v}$ passed through all instances, the vector $\mathbf{v}$ is added to the distributed list L_i. If Q_i is empty, the instance generates a new sample vector. We show the flow of this parallel implementation of the Gauss Sieve algorithm in Figure 1(b).

In this method, each instance runs the Reduce algorithm in parallel. However, this method cannot ensure that the whole list $L = L_1 \cup \cdots \cup L_t$ remains pairwise-reduced. This is because when a vector is added to a distributed list, another instance may add other vectors. Therefore, the number of non-Gauss-reduced pairs increases as the number of threads increases, and thus the overall performance of this parallel algorithm can not be accelerated for a large number of threads. In the experiment by Milde and Schneider [21], once the number of threads increases to more than ten, the speed-up factor does not exceed around five.

3.3 Ideal Gauss Sieve Algorithm [28]

Schneider proposed an Ideal Gauss Sieve algorithm [28] that uses the structure of an ideal lattice to improve the processing speed of the Gauss Sieve algorithm. The following ideal lattices support the rotation operation without additional cost and are suitable for speed-up[1].

- *Prime cyclotomic lattice*
  If $n+1$ is prime, an ideal lattice generated by the cyclotomic polynomial $\boldsymbol{g}(x) = x^n + x^{n-1} + \cdots + x + 1$ is called a *Prime cyclotomic lattice*. In this type, the rotation of vector $\mathbf{v}$ is $\mathbf{rot}(\mathbf{v}) = (-v_{n-1}, v_0 - v_{n-1}, \ldots, v_{n-2} - v_{n-1})$.
- *Anti-cyclic lattice*
  If n is a power of two, an ideal lattice generated by the cyclotomic polynomial $\boldsymbol{g}(x) = x^n + 1$ is called an *Anti-cyclic lattice*. In this type, the rotation of vector $\mathbf{v}$ is $\mathbf{rot}(\mathbf{v}) = (-v_{n-1}, v_0, \ldots, v_{n-2})$.

The rotation maps of the above ideal lattices can generate new vectors that have a similar norm virtually for free. Therefore, we can implement the Gauss Sieve algorithm using the list L with the rotated vectors $\mathbf{rot}^i(\mathbf{v})$ for $i = 1, 2, ..., n-1$ in addition to $\mathbf{v}$ with a small overhead. The algorithm enables the Gauss Sieve algorithm to run about 25 times faster on 60-dimensional ideal lattices [28].

Unfortunately, upper bound of a running time has not yet been proven theoretically. However, Micciancio and Vougalris shows experimentally the running time is about $2^{0.52n}$ asymptotically [19]. Furthermore, they show the space complexity and the number of iteration is about $2^{0.2n}$ and $2^{0.29n}$ respectively.

4 Proposed Parallel Gauss Sieve Algorithm

In this section, we propose the parallelized algorithm derived from the Gauss Sieve algorithm. We design our algorithm so that the list L remains pairwise-reduced as with the Gauss Sieve algorithm, even though this algorithm works in parallel.

[1] Schneider also discussed another type of ideal lattice, called the *Cyclic Lattice*. However, this type is generated by a non-cyclotomic polynomial. Because we focus here on the ideal lattice generated by a cyclotomic polynomial for the Ideal Lattice Challenge, we do not deal with the *Cyclic Lattice* in this paper.

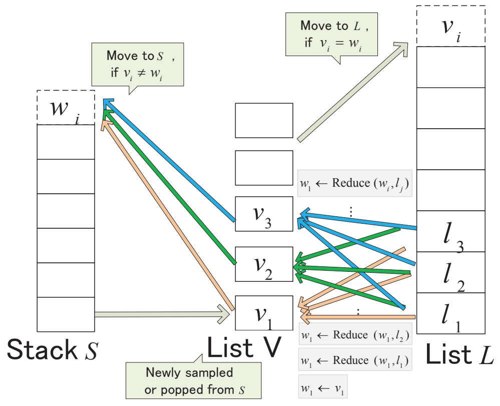

Move to S, if $v_i \neq w_i$
Move to L, if $v_i = w_i$
w_i
v_i
v_3
v_2
v_1
l_3
l_2
l_1
$w_1 \leftarrow$ Reduce (w_i, l_j)
$w_1 \leftarrow$ Reduce (w_1, l_2)
$w_1 \leftarrow$ Reduce (w_1, l_1)
$w_1 \leftarrow v_1$
Stack S
List V
List L
Newly sampled or popped from S

Fig. 2. Flow of the **Reduction sample vectors using list vectors** in the proposed parallel Gauss Sieve algorithm. Each task in different threads is indicated by color (*e.g.* blue, green, orange).

4.1 Overview

Let t be the number of threads used in our algorithm. Our algorithm prepares the auxiliary list V of r vectors in $\mathcal{L}(\mathbf{B})$, where each thread treats at most $s = \lfloor r/t \rfloor$ sample vectors for the list V. We also use the same list L and stack S in the Gauss Sieve algorithm, and the vectors in list L remain pairwise-reduced during our algorithm by control with list V. Each thread has list V, list L, and stack S, where we write $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ and $L = \{\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_m\}$. After each iteration of the loop in our algorithm, we pop vectors from the stack S to list V. If the size of V is smaller than r, we generate new sample vectors by the multisampling techniques. We explain how to construct the proposed threads in the following. There are three different reduction steps in our algorithm, namely **Reduction sample vectors using list vectors**, **Reduction sample vectors using sample vectors**, and **Reduction list vectors using sample vectors**. Our algorithm requires to use Alg.2 at most $\max(rm, r^2)$ times in each step, in other words, at most $\max(\lfloor rm/t \rfloor, \lfloor r^2/t \rfloor)$ times in each thread.

In the **Reduction sample vectors using list vectors**, let $s = \lfloor r/t \rfloor$ be the number of sample vectors treated by a thread, where r is the size of list V. Each thread has the distributed list $V_i = \{\mathbf{v}_{(i-1)s+1}, \ldots, \mathbf{v}_{is}\}$ and list L, where $V = \cup_i V_i$ and $i = 1, 2, ..., t$. Each thread i independently deals with list L and the sample vectors V_i, and runs $\mathbf{v}'_k = \text{Reduce}(\mathbf{v}_k, \boldsymbol{\ell}_j)$, where $\mathbf{v}_k \in V_i, \boldsymbol{\ell}_j \in L$, identical to a Gauss Sieve algorithm. If $\mathbf{v}'_k \neq \mathbf{v}_k$ holds, then the thread i moves the reduced vector $\mathbf{v}'_k$ into the stack S, otherwise, the thread i moves this vector $\mathbf{v}'_k$ into new list V'. At the end of this part, any vector $\mathbf{v}$ in list V' satisfies $\mathbf{v} = \text{Reduce}(\mathbf{v}, \boldsymbol{\ell})$ for all vectors $\boldsymbol{\ell}$ in list L. We show the flow of this part in Figure 2.

In the **Reduction sample vectors using sample vectors**, each thread has list V', which consists of r' vectors on a lattice. Let $s' = \lfloor r'/t \rfloor$ be the number of sample vectors treated by a thread. Each thread i deals with only a sample list V' and runs $\mathbf{v}'_k = \text{Reduce}(\mathbf{v}_k, \mathbf{v}_j)$, where $\mathbf{v}_k \in \{\mathbf{v}_{(i-1)s'+1}, \ldots, \mathbf{v}_{is'}\}, \mathbf{v}_j \in V'$ with $k \neq j$. If $\mathbf{v}'_k \neq \mathbf{v}_k$ holds, then the thread i moves the reduced vectors $\mathbf{v}'_k$ into the stack S, otherwise, the thread i moves the vectors $\mathbf{v}'_k$ into new list V''. At the end of this part, list V'' becomes pairwise-reduced and we have the relationship $V'' \subset V' \subset V$.

In the **Reduction list vectors using sample vectors**, let $\bar{s} = \lfloor m/t \rfloor$ be the number of list vectors treated by a thread, where m is the size of list L. Each thread has list $L_i = \{\boldsymbol{\ell}_{(i-1)\bar{s}+1}, \ldots, \boldsymbol{\ell}_{i\bar{s}}\}$ and V'', where $L = \cup_i L_i$, and $i = 1, 2, ..., t$. From our assumption, L is pairwise-reduced before processing this part. Each thread i deals with a distributed list L_i and a list V'' and runs $\boldsymbol{\ell}'_k = \text{Reduce}(\boldsymbol{\ell}_k, \mathbf{v}_j)$, where $\boldsymbol{\ell}_k \in L_i, \mathbf{v}_j \in V''$. If $\boldsymbol{\ell}'_k \neq \boldsymbol{\ell}_k$ holds, then the thread i moves the reduced vector $\boldsymbol{\ell}'_k$ into the stack S, otherwise, the thread i moves the vectors $\boldsymbol{\ell}_k$ into new list L'. At the end of this part, any vector $\boldsymbol{\ell}_k$ in the new list L' satisfies $\boldsymbol{\ell}_k = \text{Reduce}(\boldsymbol{\ell}_k, \mathbf{v}_j)$ for all vectors $\mathbf{v}_j$ in list V''. Here both L' and V'' are pairwise-reduced due to relationship $L' \subset L$ and $V'' \subset V'$, respectively.

After the above three reduction steps, our algorithm merges list L' and list V'' to create the new list $L = L' \cup V''$. Note that $\boldsymbol{\ell} = \text{Reduce}(\boldsymbol{\ell}, \mathbf{v})$ and $\mathbf{v} = \text{Reduce}(\mathbf{v}, \boldsymbol{\ell})$ hold for any vector $\boldsymbol{\ell} \in L'$ and $\mathbf{v} \in V''$. Therefore, any pair of two vectors $(\boldsymbol{\ell}, \mathbf{v})$ in L', V'' is Gauss-reduced by Lemma 1, and thus the union $L = L' \cup V''$ becomes pairwise-reduced by Lemma 2.

We show the algorithm derived from the proposed parallelized Gauss Sieve Algorithm in Alg.4 at Appendix B. The inputs of this algorithm are a lattice on basis $\mathbf{B}$, the number of samplings $r \in \mathbb{N}$, and termination conditions α, β. Here r is determined by the experimental scale, for example, the number of CPU cores or the available memory (we discuss the most suitable value based on an experiment described in section 5). In the following, we explain the details of the proposed algorithm.

4.2 Multisampling of Vectors (Steps from 3 to 9 in Alg.4)

We sample r vectors in lattice $\mathcal{L}(\mathbf{B})$ and construct a list $V = (\mathbf{v}_1, \ldots, \mathbf{v}_r)$ at the beginning of the iteration from step 3 to step 9 in Alg.4. Sample vector $\mathbf{v}_i$ is samples in two ways, (*i.e.,* popping from stack S or newly generating just as in the case the Gauss Sieve algorithm). If $|S| \geq r$, all vectors $\mathbf{v}_i$ are popped from the stack S, where $1 \leq i \leq r$. If $0 < |S| < r$, we pop $|S|$ vectors from the stack S and generate $(r - |S|)$ vectors using Klein's sampling algorithm. If S is empty, all vectors $\mathbf{v}_i$ are newly generated using Klein's sampling algorithm.

4.3 Reduction of Sample Vectors using List Vectors (Steps from 12 to 22 in Alg.4)

In this part, by reducing the sample vectors in V using all vectors in list L we will construct the list V', which consists of vectors $\mathbf{v}_i \in V$ that satisfy Reduce$(\mathbf{v}_i, \boldsymbol{\ell}_j) = \mathbf{v}_i$ for all $\boldsymbol{\ell}_j \in L$. Here denote $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ and $L = \{\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_m\}$. At the beginning of this part, we assign $\mathbf{w}_i \leftarrow \mathbf{v}_i$ at step 13 in Alg.4. For $i = 1, 2, ..., r$, this part runs Reduce$(\mathbf{w}_i, \boldsymbol{\ell}_j)$ from $j = 1$ to m for the fixed first input $\mathbf{w}_i$ and updates $\mathbf{w}_i$ using its output repeatedly. After running Reduce$(\mathbf{w}_i, \boldsymbol{\ell}_j)$ for $\boldsymbol{\ell}_j \in L$, if $\mathbf{w}_i$ is changed (*i.e.,* $\mathbf{w}_i \neq$ Reduce$(\mathbf{w}_i, \boldsymbol{\ell}_j)$ for some $\boldsymbol{\ell}_j$), this vector $\mathbf{w}_i$ is moved to stack S, otherwise, $\mathbf{w}_i (= \mathbf{v}_i)$ is moved to the distributed list V'. This part runs the Reduce algorithm in the following order.

$$\boxed{\begin{array}{ccc}
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \boldsymbol{\ell}_1) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \boldsymbol{\ell}_1) & \vdots \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \boldsymbol{\ell}_2) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \boldsymbol{\ell}_2) & \mathbf{w}_r \leftarrow \mathit{Reduce}(\mathbf{w}_r, \boldsymbol{\ell}_1) \\
\vdots & \vdots & \mathbf{w}_r \leftarrow \mathit{Reduce}(\mathbf{w}_r, \boldsymbol{\ell}_2) \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \boldsymbol{\ell}_m) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \boldsymbol{\ell}_m) & \vdots \\
\vdots & \vdots & \mathbf{w}_r \leftarrow \mathit{Reduce}(\mathbf{w}_r, \boldsymbol{\ell}_m)
\end{array}}$$

At the end of this part, we re-index the vectors in V' from 1 to r' in no particular order, and rename the vectors in list V' from $\{\mathbf{w}_1, ..., \mathbf{w}_{r'}\}$ to $\{\mathbf{v}_1, ..., \mathbf{v}_{r'}\}$ at step 22 in Alg.4. Recall that any vector $\mathbf{v}_i$ in list V' satisfies $\mathbf{v}_i$ =Reduce$(\mathbf{v}_i, \boldsymbol{\ell}_j)$ for all vectors $\boldsymbol{\ell}_j$ in list L. We have the relationship $V' \subseteq V$ and $|V'| = r' \leq r$.

This part can simply be parallelized without heavy overhead. Let t be the number of threads and s be the number of sample vectors treated by a thread, where $s = \lfloor r/t \rfloor$. While a thread $i (1 \leq i \leq t)$ computes Reduce$(\mathbf{w}_i, \boldsymbol{\ell}_1)$ to Reduce$(\mathbf{w}_i, \boldsymbol{\ell}_m)$, another thread $j (j \neq i)$ can compute Reduce$(\mathbf{w}_j, \boldsymbol{\ell}_1)$ to Reduce$(\mathbf{w}_j, \boldsymbol{\ell}_m)$, because the vectors $\boldsymbol{\ell}_k$ in list L are not changed in this part. Therefore, the inner loop (from step 14 to step 21) can be fully parallelized and the degree of parallelization is at most r, if we set $s = 1$. If $s > 1$, the thread i has $V_i = \{\mathbf{v}_{(i-1)s+1}, \ldots, \mathbf{v}_{is}\}$ and list L, where $V = \cup_i V_i$. And then the thread i runs Reduce$(\mathbf{w}_{(i-1)s+1}, \boldsymbol{\ell}_1)$ to Reduce$(\mathbf{w}_{is}, \boldsymbol{\ell}_m)$ sequentially in the following order.

$$\boxed{\begin{array}{c}
\boxed{\text{Thread } 1} \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \boldsymbol{\ell}_1) \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \boldsymbol{\ell}_2) \\
\vdots \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \boldsymbol{\ell}_m) \\
\vdots \\
\mathbf{w}'_s \leftarrow \mathit{Reduce}(\mathbf{w}_s, \boldsymbol{\ell}_1) \\
\mathbf{w}'_s \leftarrow \mathit{Reduce}(\mathbf{w}_s, \boldsymbol{\ell}_2) \\
\vdots \\
\mathbf{w}'_s \leftarrow \mathit{Reduce}(\mathbf{w}_s, \boldsymbol{\ell}_m)
\end{array}}
\quad
\boxed{\begin{array}{c}
\boxed{\text{Thread } 2} \\
\mathbf{w}_{s+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s+1}, \boldsymbol{\ell}_1) \\
\mathbf{w}_{s+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s+1}, \boldsymbol{\ell}_2) \\
\vdots \\
\mathbf{w}_{s+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s+1}, \boldsymbol{\ell}_m) \\
\vdots \\
\mathbf{w}_{2s} \leftarrow \mathit{Reduce}(\mathbf{w}_{2s}, \boldsymbol{\ell}_1) \\
\mathbf{w}_{2s} \leftarrow \mathit{Reduce}(\mathbf{w}_{2s}, \boldsymbol{\ell}_2) \\
\vdots \\
\mathbf{w}_{2s} \leftarrow \mathit{Reduce}(\mathbf{w}_{2s}, \boldsymbol{\ell}_m)
\end{array}}
\quad \cdots \quad
\boxed{\begin{array}{c}
\boxed{\text{Thread } t} \\
\mathbf{w}_{s(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s(t-1)+1}, \boldsymbol{\ell}_1) \\
\mathbf{w}_{s(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s(t-1)+1}, \boldsymbol{\ell}_2) \\
\vdots \\
\mathbf{w}_{s(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s(t-1)+1}, \boldsymbol{\ell}_m) \\
\vdots \\
\mathbf{w}_{st} \leftarrow \mathit{Reduce}(\mathbf{w}_{st}, \boldsymbol{\ell}_1) \\
\mathbf{w}_{st} \leftarrow \mathit{Reduce}(\mathbf{w}_{st}, \boldsymbol{\ell}_2) \\
\vdots \\
\mathbf{w}_{st} \leftarrow \mathit{Reduce}(\mathbf{w}_{st}, \boldsymbol{\ell}_m)
\end{array}}$$

4.4 Reduction of Sample Vectors using Sample Vectors (Steps from 23 to 34 in Alg.4)

In this part we try to convert the list $V' = \{\mathbf{v}_1, \ldots, \mathbf{v}_{r'}\}$ to be a pairwise-reduced list V''. We reduce sample vectors $\mathbf{v}_i \in V'$ using all vectors in $V' \setminus \{\mathbf{v}_i\}$ and construct list V'', which consists of vectors $\mathbf{v}_i$ that satisfy Reduce$(\mathbf{v}_i, \mathbf{v}_j) = \mathbf{v}_i$ for all $\mathbf{v}_j \in V''$ with $i \neq j$. At the beginning of this part, we assign $\mathbf{w}_i \leftarrow \mathbf{v}_i$ at step 24 in Alg.4. For $i = 1, 2, ..., r'$, this part runs Reduce$(\mathbf{w}_i, \mathbf{v}_j)$ from $j = 1$ to m without $j = i$ for the fixed first input $\mathbf{w}_i$ and updates $\mathbf{w}_i$ using its output repeatedly. During all reductions, just after $\mathbf{w}_i$ is reduced even once, this vector $\mathbf{w}_i$ is moved to stack S as in the first reduction part. If $\mathbf{w}_i$ is not reduced ($\mathbf{w}_i =$ Reduce$(\mathbf{w}_i, \mathbf{v}_j)$), this vector $\mathbf{w}_i (= \mathbf{v}_i)$ is moved to list V''. This part runs the Reduce algorithm in the following order.

$$\begin{array}{lll}
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_2) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \mathbf{v}_1) & \vdots \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_3) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \mathbf{v}_2) & \mathbf{w}_{r'} \leftarrow \mathit{Reduce}(\mathbf{w}_{r'}, \mathbf{v}_1) \\
\vdots & \vdots & \mathbf{w}_{r'} \leftarrow \mathit{Reduce}(\mathbf{w}_{r'}, \mathbf{v}_2) \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_{r'}) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \mathbf{v}_{r'}) & \vdots \\
\vdots & \vdots & \mathbf{w}_{r'} \leftarrow \mathit{Reduce}(\mathbf{w}_{r'}, \mathbf{v}_{r'-1})
\end{array}$$

At the end of this part, we re-index the vectors in V'' from 1 to r'' in no particular order, and rename the vectors in list V'' from $\{\mathbf{w}_1, ..., \mathbf{w}_{r''}\}$ to $\{\mathbf{v}_1, ..., \mathbf{v}_{r''}\}$ at step 34 in Alg.4. Recall that list V'' becomes pairwise-reduced because Reduce$(\mathbf{v}_i, \mathbf{v}_j) = \mathbf{v}_i$ holds for all vectors $\mathbf{v}_i, \mathbf{v}_j \in V''$ with $i \neq j$. We then have relationship $V'' \subseteq V' \subseteq V$ and $|V''| = r'' \leq r' \leq r$.

This part also can be parallelized in a similar way as the first part. Let t be the number of threads and s' be the number of sample vectors treated by a thread, where $s' = \lfloor r'/t \rfloor$. Each thread i deals with only a sample list V' and runs $\mathbf{w}_k \leftarrow$ Reduce$(\mathbf{w}_k, \mathbf{v}_j)$, where $(i-1)s' + 1 \leq k \leq is', \mathbf{v}_j \in V'$ with $k \neq j$. When thread i computes $\mathbf{w}_i \leftarrow$ Reduce$(\mathbf{w}_i, \mathbf{v}_j)$, another thread h can compute $\mathbf{w}_h \leftarrow$Reduce$(\mathbf{w}_h, \mathbf{v}_j)$ for all $\mathbf{v}_j \in V'$. More specifically, in this part, each thread runs the Reduce algorithm in the following order in parallel.

$$\begin{array}{c|c|c|c}
\boxed{\text{Thread 1}} & \boxed{\text{Thread 2}} & & \boxed{\text{Thread } t} \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_2) & \mathbf{w}_{s'+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s'+1}, \mathbf{v}_1) & & \mathbf{w}_{s'(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s'(t-1)+1}, \mathbf{v}_1) \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_3) & \mathbf{w}_{s'+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s'+1}, \mathbf{v}_2) & & \mathbf{w}_{s'(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s'(t-1)+1}, \mathbf{v}_2) \\
\vdots & \vdots & & \vdots \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_{r'}) & \mathbf{w}_{s'+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s'+1}, \mathbf{v}_{r'}) & \cdots & \mathbf{w}_{s'(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{s'(t-1)+1}, \mathbf{v}_{r'-1}) \\
\vdots & \vdots & & \vdots \\
\mathbf{w}_s \leftarrow \mathit{Reduce}(\mathbf{w}_s, \mathbf{v}_2) & \mathbf{w}_{2s'} \leftarrow \mathit{Reduce}(\mathbf{w}_{2s'}, \mathbf{v}_1) & & \mathbf{w}_{s't} \leftarrow \mathit{Reduce}(\mathbf{w}_{s't}, \mathbf{v}_1) \\
\mathbf{w}_s \leftarrow \mathit{Reduce}(\mathbf{w}_s, \mathbf{v}_3) & \mathbf{w}_{2s'} \leftarrow \mathit{Reduce}(\mathbf{w}_{2s'}, \mathbf{v}_2) & & \mathbf{w}_{s't} \leftarrow \mathit{Reduce}(\mathbf{w}_{s't}, \mathbf{v}_2) \\
\vdots & \vdots & & \vdots \\
\mathbf{w}_s \leftarrow \mathit{Reduce}(\mathbf{w}_s, \mathbf{v}_{r'}) & \mathbf{w}_{2s'} \leftarrow \mathit{Reduce}(\mathbf{w}_{2s'}, \mathbf{v}_{r'}) & & \mathbf{w}_{s't} \leftarrow \mathit{Reduce}(\mathbf{w}_{s't}, \mathbf{v}_{r'})
\end{array}$$

4.5 Reduction of List Vectors using Sample Vectors (Steps from 35 to 45 in Alg.4)

In this part, by reducing the vectors $\boldsymbol{\ell}_i$ in L using all sample vectors in $V'' = \{\mathbf{v}_1, \ldots, \mathbf{v}_{r''}\}$, we will construct the list L', which consists of vectors $\boldsymbol{\ell}_i \in L$ that satisfy Reduce$(\boldsymbol{\ell}_i, \mathbf{v}_j) = \boldsymbol{\ell}_i$ for all $\mathbf{v}_j \in V''$. At the beginning of this part, we assign $\mathbf{w}_i \leftarrow \boldsymbol{\ell}_i$ at step 36 in Alg.4. For $i = 1, 2, ..., m$, this part runs Reduce$(\mathbf{w}_i, \mathbf{v}_j)$ from $j = 1$ to r'' for the fixed first input $\mathbf{w}_i$ and updates $\mathbf{w}_i$ using its output repeatedly. During all reduction steps, if $\mathbf{w}_i$ is changed (*i.e.,* $\mathbf{w}_i \neq$ Reduce$(\mathbf{w}_i, \mathbf{v}_i)$ for some $\mathbf{v}_i$), this vector $\mathbf{w}_i$ is moved to stack S, otherwise, this vector $\mathbf{w}_i (= \boldsymbol{\ell}_i)$ is moved to the distributed list L'. This part runs the Reduce algorithm in the following order.

$$\begin{array}{lll}
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_1) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \mathbf{v}_1) & \vdots \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_2) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \mathbf{v}_2) & \mathbf{w}_m \leftarrow \mathit{Reduce}(\mathbf{w}_m, \mathbf{v}_1) \\
\vdots & \vdots & \mathbf{w}_m \leftarrow \mathit{Reduce}(\mathbf{w}_m, \mathbf{v}_2) \\
\mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_{r''}) & \mathbf{w}_i \leftarrow \mathit{Reduce}(\mathbf{w}_i, \mathbf{v}_{r''}) & \vdots \\
\vdots & \vdots & \mathbf{w}_m \leftarrow \mathit{Reduce}(\mathbf{w}_m, \mathbf{v}_{r''})
\end{array}$$

Table 1. Experiment environment used in our experiment

Instance type	cc1.8xlarge
CPU	Intel Xeon E5-2670 2.6GHz ×2
Core	8 core ×2
Memory	64GB
OS	Ubuntu12.10
Compiler	g++ 4.1.2
Library	OpenMP, OpenMPI, NTL5.5.2

At the end of this part, we re-index the vectors in L' from 1 to m' in no particular order, and rename the vectors in list L' from $\{\mathbf{w}_1, ..., \mathbf{w}_{m'}\}$ to $\{\boldsymbol{\ell}_1, ..., \boldsymbol{\ell}_{m'}\}$ at Step 45 in Alg.4. Recall that any vector $\boldsymbol{\ell}_i$ in list L' satisfies Reduce$(\boldsymbol{\ell}_i, \mathbf{v}_j) = \boldsymbol{\ell}_i$ for all vectors $\mathbf{v}_j$ in list V''. We then have relationships $L' \subseteq L$ and $|L'| = m' \leq m$. After this part, our algorithm merges list L' and list V'' to become the new list $L = L' \cup V''$ at Step 46 in Alg.4.

This step can be simply parallelized without heavy overhead in a similar way as the first part, and the degree of parallelization is at most r''. Each thread of index i updates $\bar{s}$ vectors in list L_i (*i.e.,* $L_i = \{\boldsymbol{\ell}_{(i-1)\bar{s}+1}, \ldots, \boldsymbol{\ell}_{i\bar{s}}\}$, where $\bar{s} = \lfloor m/r'' \rfloor$). This part runs the Reduce algorithm in the following order in parallel.

$$\begin{array}{c} \boxed{\text{Thread 1}} \\ \mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_1) \\ \mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_2) \\ \vdots \\ \mathbf{w}_1 \leftarrow \mathit{Reduce}(\mathbf{w}_1, \mathbf{v}_{r''}) \\ \vdots \\ \mathbf{w}_{\bar{s}} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}}, \mathbf{v}_1) \\ \mathbf{w}_{\bar{s}} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}}, \mathbf{v}_2) \\ \vdots \\ \mathbf{w}_{\bar{s}} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}}, \mathbf{v}_{r''}) \end{array} \quad \begin{array}{c} \boxed{\text{Thread 2}} \\ \mathbf{w}_{\bar{s}+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}+1}, \mathbf{v}_1) \\ \mathbf{w}_{\bar{s}+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}+1}, \mathbf{v}_2) \\ \vdots \\ \mathbf{w}_{\bar{s}+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}+1}, \mathbf{v}_{r''}) \\ \vdots \\ \mathbf{w}_{2\bar{s}} \leftarrow \mathit{Reduce}(\mathbf{w}_{2\bar{s}}, \mathbf{v}_1) \\ \mathbf{w}_{2\bar{s}} \leftarrow \mathit{Reduce}(\mathbf{w}_{2\bar{s}}, \mathbf{v}_2) \\ \vdots \\ \mathbf{w}_{2\bar{s}} \leftarrow \mathit{Reduce}(\mathbf{w}_{2\bar{s}}, \mathbf{v}_{r''}) \end{array} \quad \cdots \quad \begin{array}{c} \boxed{\text{Thread } t} \\ \mathbf{w}_{\bar{s}(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}(t-1)+1}, \mathbf{v}_1) \\ \mathbf{w}_{\bar{s}(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}(t-1)+1}, \mathbf{v}_2) \\ \vdots \\ \mathbf{w}_{\bar{s}(t-1)+1} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}(t-1)+1}, \mathbf{v}_{r''}) \\ \vdots \\ \mathbf{w}_{\bar{s}t} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}t}, \mathbf{v}_1) \\ \mathbf{w}_{\bar{s}t} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}t}, \mathbf{v}_2) \\ \vdots \\ \mathbf{w}_{\bar{s}t} \leftarrow \mathit{Reduce}(\mathbf{w}_{\bar{s}t}, \mathbf{v}_{r''}) \end{array}$$

4.6 Properties of the Proposed Algorithm

In our algorithm, list L remains pairwise-reduced at any iteration for the following reasons. After the three reduction steps, our algorithm merges list L' and list V'' to become the new list $L = L' \cup V''$. Note that $\boldsymbol{\ell} = \text{Reduce}(\boldsymbol{\ell}, \mathbf{v})$ and $\mathbf{v} = \text{Reduce}(\mathbf{v}, \boldsymbol{\ell})$ hold for any vector $\boldsymbol{\ell}$ in L' and $\mathbf{v} \in V''$ by the first and third reduction parts. And then, V'' is pair-wise reduced by the second part. Therefore, any pair of two vectors $(\boldsymbol{\ell}, \mathbf{v})$ in L', V'' is Gauss-reduced by Lemma 1, and thus the union $L = L' \cup V''$ becomes pairwise-reduced by Lemma 2.

Our algorithm is a natural extension of the Gauss Sieve algorithm. If only one vector is sampled (*i.e.,* $r = 1$), all the pairs of $(\boldsymbol{\ell}_j, \mathbf{v}_1)$ and $(\mathbf{v}_1, \boldsymbol{\ell}_j)$ are Gauss-reduced by the first and third reduction part, where $\boldsymbol{\ell}_j \in L$. There is nothing to do in the second reduction part. Therefore, this algorithm is equal to the Gauss Sieve algorithm when $r = 1$.

5 Implementation and Experimental Results

In this section, we explain the parallel implementation of the proposed parallel Gauss Sieve algorithm on a multicore CPU, and we also present some algorithmic improvement in our experiment.

5.1 Implementation using Amazon EC2

We use the instance cc1.8xlarge in AmazonEC2 [4]. Our experimental environment is shown in Table 1. Our implementation is based on the **gsieve** library, published by Voulgaris [34] and written in C++. We assume the following properties from our preliminary experiment:

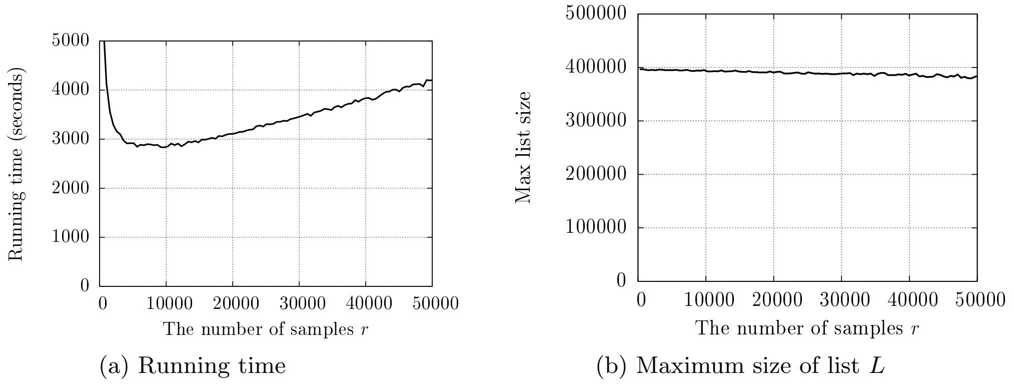


Fig. 3. Results for solving the SVP Challenge of a 80-dimensional lattice. Fig (a) shows the running time using one instance (32 threads). Fig (b) shows the maximum size of list L. The horizontal axis indicates the number of sample vector r.

– *all absolute values of entries of vectors are less than* 2^{16}
– *the computational cost of the inner product is dominant (step 1 in Alg.2)*

We optimize the code for the inner product (step 1 in Alg.2) using the SIMD operation. Intel Xeon E5-2670 and g++4.1.2 support SSE4.2, and we can use a 128-bit SSE register. Using the SSE, we can treat eight elements in one SSE operation in parallel. This technique enables our program to run about four times faster.

In this experiment, we fixed the number of threads at 32 per instance, *i.e.,* at double the number of CPU cores, because the instance supports hyper-threading technology. For examples, we can use 2,688 threads in total for 84 instances.

5.2 Space Complexity

In this section, we discuss the space complexity with a large number of sample vectors r and a fixed number of threads t. The space complexity of our algorithm is dominated by the size of lists L, V, and stack S. We evaluate the size of a list by the number of vectors in the list. In our experiment of solving the SVP Challenge of 80 dimensions [29], the sizes of list L between Gauss Sieve algorithm ($r = 1$) and our algorithm ($r > 1$) are similar within several percent. Indeed, Figure 3(b) shows the maximum size of list L for $r = 1, 2, \ldots, 5000$ and fixed $t = 32$ using one instance, and there is no increase of the maximum size of list L from 400,000 even if r increases.

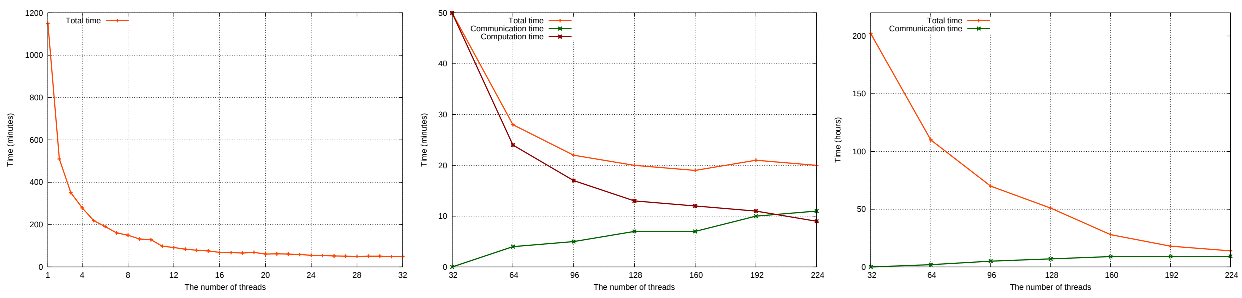
Next, in our algorithm, the maximum size of list V is at most r because V is selected by r random vectors on a lattice at the beginning of iteration (from step 12) and then the size of V shrinks by each iteration from step 12 to step 46. If we choose a suitable value of r which minimizes the total running time of our proposed algorithm, then r is much smaller than the maximum size of list L. Indeed, Figure 3(a) shows that the running time for solving the SVP Challenge of 80 dimensions becomes relatively fast when the number of sample vectors r is in the range of about 4,000 to 10,000.

Finally, in our experiment, the size of stack S in our proposed algorithm does not increase that of the original Gauss Sieve algorithm. As a result, the space complexity of our algorithm with a large r is not greater than that of the Gauss Sieve algorithm of $2^{0.2n}$.

5.3 Communication Complexity

In this section, we discuss the communication complexity between threads in our proposed parallel algorithm. We evaluate the communication comlexity in terms of the size of the lists communicated among the threads.

At first, we describe the details of data communication among threads in our algorithm. In our algorithm, the lists L, V, and stack S are required for the synchronization with each thread. In the beginning of the iteration (from step 12 to step 46 in Alg.4), a main thread (for example, thread 0) distributes



(a) 80 dimensions (b) 80 dimensions (c) 96 dimensions

Fig. 4. Results for solving the SVP Challenge on lattices of 80 and 90 dimensions. Fig (a) shows the running time of solving the SVP Challenge of 80 dimensions for $t = 1, 2, \ldots, 32$. Fig (b) shows the running and communication time of solving the SVP Challenge of 80 dimensions for $t = 32, 64, \ldots, 224$. Fig (c) shows the running and communication time of solving the SVP Challenge of 96 dimensions for $t = 32, 64, \ldots, 224$.

the whole of the list L and the part of the list $V_i = \{\mathbf{v}_{(i-1)/2+1}, \ldots, \mathbf{v}_{is}\}$ to i-th thread, where the s is the total number of list V. Each thread also has stack S, which is initialized to an empty set. After the **Reduction sample vectors using list vectors** part, a main thread gathers the distributed list V_i and the stack S into the new list V' and new stack S, respectively. Before the **Reduction sample vectors using sample vectors** part, the whole of the list V' are broadcast to all threads. In this part, each i-th thread updates the list $V_i = \{\mathbf{v}_{s'(i-1)+1}, .., \mathbf{v}_{s'i}\}$ in the list V'. Therefore, after this part, the main thread gathers the part of the list V' into new list V'' and distributes it to all threads. In the **Reduction sample vectors using sample vectors** part, i-thread uses the whole of the list V'' and the list L_i in the list L. After this part, the main thread gathers all L_i into the new list L'. Finally, the main thread gathers the stack S in all threads and continues the iteration (from step 12).

Next, we estimate the communication complexity of our algorithm. The dominant part of the communication complexity of our algorithm is the timing of broadcasting the whole list L in the beginning of iterations (from step 12) because the size of the list V is much smaller than that of the list L for large dimensions n. In the previous section, we estimated that the space complexity of our algorithm was $2^{0.2n}$, which was the maximum size of list L. In the following, we estimate the number of broadcasting the list L among threads in our algorithm. A main thread broadcasts the whole list L to t threads in each iteration (from step 12 to step 46 in Alg.4), and thus the communication complexity of our algorithm becomes $t2^{0.2n}$ per one iteration. Therefore, the total communication complexity of broadcasting the list L is $t\gamma 2^{0.2n}$, where γ is the number of iterations (from step 12 to step 46 in Alg.4). Here, the number of iterations γ can be estimated as $2^{0.29n}$ in the case of $r = 1$ and $t = 1$ [19]. On the other hand, in our experiment of the proposed algorithm in from 60 to 80 dimensions, γ was estimated as $2^{0.25n}$ for $r = 8192$ and $t = 32$. Note that the number of iterations γ is independent of t and, γ remains the same for a fixed number of sampling r. If r is bigger than 8192 with fixed t, then we have more samples r in the beginning of the iteration (from step 12 to step 46 in Alg.4) and γ is not greater than $2^{0.25n}$. Therefore, the communication complexity of our algorithm is at most $2^{0.45n}$ which is smaller than the computation time of each thread, *i.e.,* $2^{0.52n}$.

Finally, we describe some experiments on both the running and communication time for solving the SVP Challenge [29] of 80 and 96 dimensions for changing the number of threads t. Figure 4 shows the running and communication time of our algorithm for solving the SVP Challenge of 80 and 96 dimensions by changing the number of threads for $t = 1, \ldots, 224$. Figure 4(a) shows the total time for solving the SVP Challenge of 80 dimensions for $t = 1, \ldots, 32$ using one instance that has 32 threads. Note that there is no communication cost among 32 threads in one instance because they share one common memory in the instance. The total time becomes $1/t$ by using t threads for $t \leq 16$. The number of cores is 16 in one instance, and the improvement becomes smaller than $1/t$ for $r > 16$ due to the overhead of hyper-threading technology. Next, Figures 4(b) and 4(c) show the running and communication time for solving the SVP Challenge of 80 and 96 dimensions by changing the number of instance from 1 to 7, namely $t = 32$ to 224. In this experiment, the communication time becomes greater if the number of threads t increases. The communication time of our algorithm is about ten percent of the total running time for 64 threads and

Table 2. Maximum, average and minimum value of the norm of all the sample vectors (step 9 in Alg.4) and all the vectors in the final list L (step 47 in Alg.4) for solving the SVP Challenge over a lattice of 80 dimensions. GH stands for the Gaussian heuristic bound that is equal to 2179 for the lattice.

	sample vectors (step 9 in Alg.4) using $d = \log n$		sample vectors (step 9 in Alg.4) using $d = \log n/70$		vectors in L (step 47 in Alg.4) using $d = \log n/70$	
	norm	norm/GH	norm	norm/GH	norm	norm/GH
maximum	23060	10.58	4521	2.07	2991	1.37
average	13607	6.24	3639	1.66	2715	1.24
minimum	8692	3.98	2752	1.26	2272	1.04

128 threads in 80 dimensions (Figure 4(b)) and 96 dimensions (Figure 4(c)), respectively. Therefore, we expect that the rate of communication time relatively decreases for larger dimensions n.

5.4 Sampling Short Vectors and Shrinking Ratio

If we are able to sample shorter vectors at step 9 in Alg.4, then the running time of the proposed Gauss Sieve algorithm can be improved. However, it takes longer time to sample such shorter vectors on a lattice in general. Therefore, we try to adjust the parameter which determines the tradeoff between the length of the norm of sample vectors and the running time of our algorithm.

In the **gsieve** library [34], Klein's randomized rounding algorithm [13] is implemented. The details of the algorithm are explained by Gentry *et al.* [11]. In the following we adjust the parameter of the core subroutine in the **gsieve** library, namely the *SampleD* algorithm described in [11]. For the two inputs (u, c), *SampleD* chooses an integer x from the range $[c - u \cdot d, c + u \cdot d]$, where $d = \log n$ in the **gsieve** library. We determine a more suitable value of d instead of $d = \log n$ used in the **gsieve** library. The *SampleD* outputs x with probability $\rho_{u,c}(x - c)$, otherwise repeats choosing x, where $\rho_{u,c}(x)$ denoted a Gaussian function on $\mathbb{R}$ that is defined by $\rho_{u,c}(x) = \exp(-\pi|x - c|^2/u^2)$ for any $x \in \mathbb{R}$. Klein's sampling algorithm generates a new vector on a lattice that is a linear combination of the basis vector and the coefficient vector $\mathbf{x}$ using the *SampleD* algorithm n times. Therefore, if the *SampleD* algorithm outputs a smaller integer, Klein's sampling algorithm outputs a shorter vector. However, the computational time of the *SampleD* algorithm increases as the length of the output vector decreases.

We implemented the Klein algorithm in **gsieve** for the vectors reduced by BKZ with a block size of 30 using NTL library. Table 2 shows the maximum, average and minimum values of the norm of all the sample vectors (at step 9 in Alg.4) for solving the SVP Challenge of 80 dimensions [29]. In our experiment, we found the parameter $d = \log n/70$ which is most suitable for speeding up our proposed parallel Gauss sieve algorithm. In this case, the average value of the norms of all the sample vectors using the parameter $d = \log n/70$ becomes 3.7 times shorter than that using the parameter $d = \log n$ in the **gsieve** library. This technique enables our proposed algorithm to run about two times faster.

Next we estimate how the norm of sample vectors becomes smaller in the final list L in our proposed algorithm. Our proposed algorithm terminates and outputs a shorter vector from the final list L at step 47 in Alg.4. The right hand side of Table 2 presents the maximum, average and minimum values of the norm of all the vectors in the final list L (at step 47 in Alg.4) for solving the same SVP Challenge above. Here denote by GH the Gaussian heuristic bound $(1/\sqrt{\pi})\Gamma(\frac{n}{2}+1)^{\frac{1}{n}} \cdot \det(\mathcal{L}(\mathbf{B}))^{\frac{1}{n}}$ for a lattice $\mathcal{L}(\mathbf{B})$ of dimensions n, which is heuristically estimated as the length $\lambda_1(\mathcal{L}(\mathbf{B}))$ of a shortest vector in $\mathcal{L}(\mathbf{B})$. In our experiment, we used a lattice $\mathcal{L}(\mathbf{B})$ of 80 dimensions whose GH is equal to 2179. Table 2 also shows that the multiple value of the norm of all the sample vectors (at step 9 in Alg.4) and vectors in the final list L (at step 47 in Alg.4) by the GH bound. The average value of the norm of all the sample vectors is 1.66 GH and that of vectors in the final list L is 1.24 GH. The norm of the shortest vector in the final list L at the termination of our proposed algorithm achieves 1.04 GH.

5.5 Improvement of the Ideal Gauss Sieve

In this section, we show some techniques for accelerating the Ideal Gauss Sieve algorithm [28] by using the bi-directional rotation structure of some cyclotomic polynomials.

Selecting Cyclotomic Polynomials: In [28], there are three types of ideal lattices generated by specific polynomials (including two cyclotomic polynomials), which are suitable for the rotate operation $\mathbf{rot}(\mathbf{v})$ of a vector $\mathbf{v}$. We define new types of an ideal lattice, which is called a *Trinomial lattice*. A *Trinomial lattice* is generated by the trinomials in the cyclotomic polynomials. Note that the *Trinomial lattice* is not used in cryptography, but we use this type for the speeding up for solving the SVP Challenge in Ideal lattice Challenge [23].

There are two conditions for a *Trinomial lattice*, as follows:

– *Condition 1*
  If $n/2$ is a power of three, where n is an even dimension of a lattice, an ideal lattice generated by a cyclotomic polynomial $\boldsymbol{g}(x) = x^n + x^{n/2} + 1$ and one basis B is called a *Trinomial lattice*. In this type, the rotation of vector $\mathbf{v}$ is $\mathbf{rot}(\mathbf{v}) = (-v_{n-1}, v_0, \dots, v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1} - v_{n-1}, v_{\frac{n}{2}}, \dots, v_{n-2})$.
– *Condition 2*
  If the dimension n is the product of both a power of two and a power of three, an ideal lattice generated by the cyclotomic polynomial $\boldsymbol{g}(x) = x^n - x^{n/2} + 1$ and one basis B is called a *Trinomial lattice*. In this type, the rotation of vector $\mathbf{v}$ is $\mathbf{rot}(\mathbf{v}) = (-v_{n-1}, v_0, \dots, v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1} + v_{n-1}, v_{\frac{n}{2}}, \dots, v_{n-2})$.

Gallot proved that the polynomials that satisfy the above conditions are cyclotomic polynomials (Theorem 3.1 in [7]). The rotate operation $\mathbf{rot}(\mathbf{v})$ using the *Trinomial lattice* requires no greater computational cost than that using the *Anti-cyclic lattice*.

Inverse Rotation: In [28], the rotation $\mathbf{rot}(\mathbf{v})$ of vector $\mathbf{v}$ is used for accelerating the Gauss Sieve algorithm. In the same manner, we define the *inverse rotation* $\mathbf{rot}^{-1}(\mathbf{v})$ of vector $\mathbf{v}$. The inverse rotation is defined by $\mathbf{rot}^{-1}(\mathbf{v}) = x^{-1}\mathbf{v} \bmod \boldsymbol{g}(x)$, where $\mathbf{v}(x)$ is a polynomial representation of vector $\mathbf{v}$. In *Trinomial lattice*, x^{-1} is $-x^{n-1} \pm x^{\frac{n}{2}-1}$. The computational costs of $\mathbf{rot}(\mathbf{v})$ and $\mathbf{rot}^{-1}(\mathbf{v})$ using the *Prime cyclotomic lattice*, *Anti-cyclic lattice* and the *Trinomial lattice* are quite similar.

Updating Vectors: In a *Trinomial lattice*, we can find shorter vectors using rotation and inverse rotation. If $\mathbf{v} \in \mathcal{L}(\mathbf{B})$, the difference of the Euclidean norm between vector $\mathbf{v}$ and rotated vector $\mathbf{rot}(\mathbf{v})$ (or inversely rotated vector $\mathbf{rot}^{-1}(\mathbf{v})$) is represented as follows:

$$||\mathbf{rot}(\mathbf{v})|| - ||\mathbf{v}|| = (v_{n-1})^2 + 2v_{\frac{n}{2}-1}v_{n-1},$$
$$||\mathbf{rot}^{-1}(\mathbf{v})|| - ||\mathbf{v}|| = (v_0)^2 + 2v_{\frac{n}{2}}v_0.$$

Therefore, the Euclidean norm of a(n) (inversely) rotated vector becomes shorter than that of the original vector with a non-negligible probability.

In a *Trinomial lattice*, repeating the rotate operation increases the norm gradually. Therefore, the total running time of our algorithm increases with too large a number of rotate operations. Then, we derived the most suitable number of rotate operations from the experiment to solve the SVP Challenge of 72 dimensions with each number of rotations. In our experiment, it was found that the most suitable number was 6, and this technique enables our parallel Gauss Sieve algorithm to run about 5.5 times faster.

5.6 Solving the SVP Challenge

We have solved several problems in the SVP Challenge over random lattices [29] and the Ideal Lattice Challenge [23]. The problem setting in these challenges has been published in [24]. We pre-computed the BKZ-reduced basis with a block size of 30 using NTL library [33]. Because this precomputation requires much less time than the Gauss Sieve algorithm, we do not include the timing in the following. In our experiment, we used the instance cc1.8xlarge described in Table 1. We fix the number of threads at 32 per an instance. We show the results of our experiments in Table 3.

In the SVP Challenge over random lattices [29], we solved the SVP challenges of 80 and 96 dimensions given as filename “svpchallengedim80seed0.txt” and “svpchallengedim90seed0.txt”. As we explained in section 5.1, our parallel algorithm solved the SVP Challenges of 80 dimensions in about one CPU hour using one instance which deploys 32 threads and 8,192 sample vectors. According to the results of Schneider [26], their program for the Gauss Sieve requires about 10^6 seconds $\approx$ 278 hours using one thread for the same problem. Hence, our parallel algorithm enables the Gauss Sieve algorithm to run about like 200 times faster. We also solved the SVP Challenge of 96 dimensions using four instances of 128 threads and 32,768 sample vectors. As a result, our parallel algorithm required about 200 CPU hours. This SVP Challenge

Table 3. Results of the SVP Challenge [29] and Ideal Lattice Challenge [23]. SVP Challenges of 96 and 128 dimensions in the Ideal Lattice Challenge are generated by cyclotomic polynomials $x^{96} - x^{48} + 1$ and $x^{128} + 1$, respectively. The other SVP Challenges are non-special type.

	dimension	instance hours	#instance	#thread t	#sample vectors r	type
SVP Challenge	80	0.9	1	32	8,192	*Random lattice*
	96	200	4	128	32,768	*Random lattice*
Ideal Lattice Challenge	80	0.9	1	32	8,192	*Ideal lattice*
	96	8	1	32	8,192	*Trinomial lattice*
	128	29,994	84	2,688	688,128	*Anti-cyclic lattice*

of 96 dimensions is the largest problem that has been solved to date by using a sieving algorithm on a random lattice.

In the Ideal Lattice Challenge [23], we solved the SVP Challenges of 80, 96 and 128 dimensions. In this challenge, a basis of n-dimensional ideal lattice is generated from one of cyclotomic polynomials of degree n. In our experiment we chose the 80-dimensional lattice generated by cyclotomic polynomial $g(x) = x^{80}+x^{78}-x^{70}-x^{68}+x^{60}-x^{56}-x^{50}+x^{46}+x^{40}+x^{34}-x^{30}-x^{24}+x^{20}-x^{12}-x^{10}+x^{2}+1$ given as a filename "ideallatticedim80index220seed0.txt". The basis of 96-dimensional lattice was selected to be a *Trinomial lattice* generated by $g(x) = x^{96} - x^{48} + 1$ given as filename "ideallatticedim96index288seed0.txt", and that of 128-dimensional SVP Challenge was selected to be an *Anti-cyclic lattice* generated by cyclotomic polynomial $g(x) = x^{128} + 1$ given as filename "ideallatticedim128index256seed0.txt". In our experiment of the 80-dimensional ideal lattice, our parallel algorithm required about one CPU hour using 32 threads and 8,192 sample vectors, which are the same time cost compared with our above experiment for a random lattice in the SVP Challenge. Additionally, in our experiment of the 96-dimensional ideal lattice, our parallel algorithm required about 8 CPU hours using 32 threads and 8,192 sample vectors. The proposed two techniques (*inverse rotation* and *updating vectors*) enable us to speedup about 25 times faster than the random lattice of the same dimension.

In our experiment of the 128-dimensional ideal lattice, our parallel algorithm require $29,994$ CPU hours using 84 instances, where we can set that the number of total threads and sample vectors are $t = 2,688$ and $r = 688,128$, respectively. As a result, our parallel algorithm outputs a short vector:

```
(-613, -20, -146, -249, 237, 161, 290, 518, -204, -207, -39, 333, -97, 30, 53, 579,
93, -634, 297, 223, -201, 75, -98, -85, -68, 100, 21, -87, -442, -63, -211, 358, -143,
239, -39, 240, -9, -382, -38, -285, -10, 275, 108, 116, -288, -165, 509, 589, 445,
-137, -230, -131, -84, -26, -37, 442, -115, 267, 642, 168, -226, 361, 212, -193, 379,
59, 45, 215, -48, -12, 53, 48, 83, -156, 184, -103, 102, -427, -400, 363, -69, -142,
562, -145, -118, -51, -31, -96, 604, 260, -371, -361, -553, -292, -222, 74, -51, 179,
-162, -431, -24, 159, -180, 8, -85, 57, 264, 157, 4, -232, 272, -638, -58, 68, 3,
314, -11, -395, -88, -129, -29, 219, -223, -186, 42, 73, 399, -146).
```

The Euclidean norm of this vector is 2,959 which is 1.03 times larger than the Gaussian heuristic bound of this ideal lattice, namely this vector is a solution of SVP Challenge.

In the experiment, the communication overhead among threads for solving the SVP Challenge of 128 dimensions was less than ten percents for the total running time of our proposed parallel Gauss Sieve algorithm.

According to the estimated complexity $2^{0.52n}$ [20] and our implementation result of solving the SVP Challenge of 96 dimensions, if the time complexity of our algorithm is the same as the Gauss Sieve algorithm, solving the SVP Challenge of 128 dimensions over a random lattice requires $200 \times 2^{0.52\Delta n} \approx 2.0 \times 10^{7}$ CPU hours using the parallel Gauss Sieve algorithm. Therefore, we heuristically estimate that solving the SVP Challenge of 128 dimensions over an *Anti-cyclic lattice* is about 600 times faster than that over random lattices.

Finally, we compare our result with other records of the SVP Challenge over random lattices [29]. Chen and Nguyen solved the SVP Challenge of 126 dimensions by BKZ and enumeration algorithms and spent 800 CPU days (about 19,200 CPU hours) which is close to our experiment. Kashiwabara and Fukase solved the SVP Challenge of 128 dimensions using several hundreds of CPU hours ("a few weeks using two personal computers"). The details of their experiments are not published yet in the contest page [29].

6 Conclusion

In this paper, we proposed a parallel Gauss Sieve algorithm, which is an extension of Gauss Sieve algorithm suitable for parallel computation of a large number of threads. We implemented the proposed parallel Gauss Sieve algorithm by the SIMD operation in AmazonEC2 which supports hyper-threading technology. Our experiment deploys 32 threads per instance cc1.8xlarge of 16 CPU cores. We tried to solve the SVP Challenge in the Ideal Lattice Challenge from TU Darmstadt (`http://www.latticechallenge.org/`).

Then we successfully solved the SVP Challenge of 128 dimensions on the ideal lattice generated by the cyclotomic polynomial $x^{128} + 1$, where this type of ideal lattice is often used for efficient implementation of lattice-based cryptography. Our experiment required 29,994 CPU hours by executing 2,688 threads over 84 instances in total. In the experiment, the communication overhead among threads is less than ten percents of the total running time. To the best of our knowledge, this is currently the highest dimensions of solving the SVP Challenge over ideal lattices.

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A Gauss Sieve Algorithm [20]

In this appendix, we present the Gauss Sieve algorithm [20]. Alg.1 is a main algorithm of the Gauss Sieve algorithm, and Alg.2 and Alg.3 are its subroutines.

Algorithm 1 Gauss Sieve (GS) [20]

```
Require: Lattice basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}, \alpha, \beta > 0 \in \mathbb{R}$
Ensure: A shortest vector in $\mathcal{L}(\mathbf{B})$ /* It is not theoretically proved that the GS algorithm always outputs a
    shortest vector, but the norm of the vector obtained by the GS algorithm is shortest in the lattices of the
    experiment by Micciancio and Voulgaris for up to 60 dimensions [20] */
 1: $L \leftarrow \{\}, S \leftarrow \{\}, K \leftarrow 0$
 2: while $K < \alpha|L| + \beta$ do
 3:   if $S \neq \{\}$ then
 4:     Pop from Stack $S$ to $\mathbf{v}$
 5:   else
 6:     Generate a new vector $\mathbf{v}$ using Klein's randomized rounding algorithm [13] (Our implementation employed
        gsieve and BKZ with a block size of 30 using NTL library. See Section 5.4.)
 7:   $(\mathbf{v}', L, S) \leftarrow$ Gauss_Reduce$(\mathbf{v}, L, S)$   /* Alg.3 */
 8:   if $||\mathbf{v}'|| = 0$ then
 9:     /* $\mathbf{v}'$ is a multiple of one of vectors in the list $L$ */
10:     $K \leftarrow K + 1$
11:   else
12:     $L \leftarrow L \cup \{\mathbf{v}'\}$
13: return a shortest vector in $L$
```

Algorithm 2 Reduce [20]

```
Require: Vectors $\boldsymbol{p}_1, \boldsymbol{p}_2$ in lattice $\mathcal{L}(\mathbf{B})$
Ensure: Vector $\boldsymbol{p}_1$ in lattice $\mathcal{L}(\mathbf{B})$ s.t. $|\frac{\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle}{\langle \boldsymbol{p}_2, \boldsymbol{p}_2 \rangle}| \leq \frac{1}{2}$
 1: if $|2 \cdot \langle \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 \rangle| > \langle \boldsymbol{p}_2 \cdot \boldsymbol{p}_2 \rangle$ then
 2:   $\boldsymbol{p}_1 \leftarrow \boldsymbol{p}_1 - \left\lfloor \frac{\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle}{\langle \boldsymbol{p}_2, \boldsymbol{p}_2 \rangle} \right\rceil \cdot \boldsymbol{p}_2$   /* Make $\boldsymbol{p}_1$ closest to $\boldsymbol{p}_2$ in $\boldsymbol{p}_1 + \boldsymbol{p}_2\mathbb{Z}$ */
 3: return $\boldsymbol{p}_1$
```

Algorithm 3 Gauss_Reduce [20]

```
Require: Vector $\mathbf{v}$ on lattice $\mathcal{L}(\mathbf{B})$, list $L$, stack $S$
Ensure: Vector $\mathbf{v}$, list $L$, stack $S$, s.t. $\{v\} \cup L$ is pairwise-reduced
 1: $reduce\_flag \leftarrow true$
 2: while $reduce\_flag = true$ do
 3:   $reduce\_flag \leftarrow false$
 4:   for $\boldsymbol{\ell} \in L$ do
 5:     $\mathbf{v}' \leftarrow$ Reduce$(\mathbf{v}, \boldsymbol{\ell})$   /* Alg.2 */
 6:     if $\mathbf{v}' \neq \mathbf{v}$ then
 7:       $reduce\_flag \leftarrow true$
 8:       $\mathbf{v} \leftarrow \mathbf{v}'$
 9: while $\boldsymbol{\ell} \in L$ do
10:   $\boldsymbol{\ell}' \leftarrow$ Reduce$(\boldsymbol{\ell}, \mathbf{v})$   /* Alg.2 */
11:   if $\boldsymbol{\ell}' \neq \boldsymbol{\ell}$ then
12:     $S \leftarrow S \cup \{\boldsymbol{\ell}'\}, L \leftarrow L \backslash \{\boldsymbol{\ell}\}$
13: return $(\mathbf{v}, L, S)$
```

B Proposed Parallel Gauss Sieve

Alg.4 is our proposed parallelized Gauss Sieve algorithm, named Parallel Gauss Sieve.

Algorithm 4 Proposed Parallel Gauss Sieve

```
Require: Lattice basis $\mathbf{B}$, the number of sample vectors $r \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$
Ensure: A shortest vector $\mathbf{v}$ in $\mathcal{L}(\mathbf{B})$ /* The outputs from our proposed algorithm over the lattices in our
    experiment for up to 70 dimensions were exactly same with those from the Gauss Sieve algorithm which is
    expected to solve the SVP */
 1: $L \leftarrow \{\}, V \leftarrow \{\}, S \leftarrow \{\}, K \leftarrow 0$
    /* Steps from 2 to 9 are described in 4.2 Multisampling of vectors */
 2: while $K < \alpha|L| + \beta$ do
 3:   if $|S| \neq 0$ then
 4:     $t \leftarrow \min(r, |S|)$
 5:     for $j = 1, \dots, t$ do
 6:       Pop from Stack $S$ to $\mathbf{v}_j$
 7:   if $|S| < r$ then
 8:     for $j = |S| + 1, \dots, r$ do
 9:       Generate a new vector $\mathbf{v}_j$ using Klein's randomized rounding algorithm [13] (We use gsieve and BKZ
          with a block size of 30 using NTL library. See Section 5.4.)
10:   $V \leftarrow \{\mathbf{v}_1, ..., \mathbf{v}_r\}, V' \leftarrow \{\}, V'' \leftarrow \{\}, L' \leftarrow \{\}$
11:   $L = \{\boldsymbol{\ell}_1, ..., \boldsymbol{\ell}_m\}$
      /* Steps from 12 to 22 are described in 4.3 Reduction sample vectors using*/
12:   for $i = 1, \dots, r$ do
13:     $\mathbf{w}_i \leftarrow \mathbf{v}_i$
14:     for $j = 1, \dots, m$ do
15:       $\mathbf{w}_i \leftarrow \mathrm{Reduce}(\mathbf{w}_i, \boldsymbol{\ell}_j)$  /* This step can be ran in parallel */
16:     if $||\mathbf{w}_i|| = 0$ then
17:       $K \leftarrow K + 1$
18:     else if $\mathbf{w}_i \neq \mathbf{v}_i$ then
19:       $S \leftarrow S \cup \{\mathbf{w}_i\}$
20:     else
21:       $V' \leftarrow V' \cup \{\mathbf{w}_i\}$
22:   $V' = \{\mathbf{v}_1, ..., \mathbf{v}_{r'}\}$
      /* Steps from 23 to 34 are described in 4.4 Reduction sample vectors using sample vectors */
23:   for $i = 1, \dots, r'$ do
24:     $\mathbf{w}_i \leftarrow \mathbf{v}_i$
25:     for $j = 1, \dots, r'$ do
26:       if $i \neq j$ then
27:         $\mathbf{w}_i \leftarrow \mathrm{Reduce}(\mathbf{w}_i, \mathbf{v}_j)$  /* This step can be ran in parallel */
28:     if $||\mathbf{w}_i|| = 0$ then
29:       $K \leftarrow K + 1$
30:     else if $\mathbf{w}_i \neq \mathbf{v}_i$ then
31:       $S \leftarrow S \cup \{\mathbf{w}_i\}$
32:     else
33:       $V'' \leftarrow V'' \cup \{\mathbf{w}_i\}$
34:   $V'' = \{\mathbf{v}_1, ..., \mathbf{v}_{r''}\}$
      /* Steps from 35 to 45 are described in 4.5 Reduction list vectors using sample vectors*/
35:   for $i = 1, \dots, m$ do
36:     $\mathbf{w}_i \leftarrow \boldsymbol{\ell}_i$
37:     for $j = 1, \dots, r''$ do
38:       $\mathbf{w}_i \leftarrow \mathrm{Reduce}(\mathbf{w}_i, \mathbf{v}_j)$  /* This step can be ran in parallel */
39:     if $||\mathbf{w}_i|| = 0$ then
40:       $K \leftarrow K + 1$
41:     else if $\mathbf{w}_i \neq \boldsymbol{\ell}_i$ then
42:       $S \leftarrow S \cup \{\mathbf{w}_i\}$
43:     else
44:       $L' \leftarrow L' \cup \{\mathbf{w}_i\}$
45:   $L' = \{\boldsymbol{\ell}_1, ..., \boldsymbol{\ell}_{m'}\}$
46:   $L \leftarrow L' \cup V''$
47: return a shortest vector in $L$
```