A Lever Function to a New Codomain with Adequate Indeterminacy *

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Abstract: The key transform of the REESSE1+ cryptosystem is $C_i \equiv (A_i W^{\ell(i)})^{\delta}$ (% M) with $\ell(i) \in \Omega = \{5, 7, …, 2n + 3\}$ for $i = 1, …, n$, where $\ell(i)$ is called a lever function. In this paper, the authors give a simplified transform $C_i \equiv A_i W^{\ell(i)}$ (% M) and a new codomain $\Omega_\pm = \{+/–5, +/–6, …, +/–(n + 4)\}$, where "+/–" means the selection of the "+" or "–" sign. Discuss the necessity of $\ell(.)$ to $\Omega_\pm$ that a simplified private key is insecure if $\ell(.)$ is only a fixed integer, and the sufficiency that a simplified private key is secure (namely $C_i \equiv A_i W^{\ell(i)}$ (% M) is not faced with determinate polynomial time attack) if $\ell(.)$ is a one-to-one function. The sufficiency is expounded from five aspects: indeterminacy of $\ell(.)$ to $\Omega_\pm$, insufficiency of each of the four judgment conditions for counteraction of powers of W and W^{-1} even if $\Omega_\pm = \{5, 6, …, n + 4\}$, verifying by examples, running times of continued fraction attack and indeterministic intersection attack most efficient now, and a relation between a lever function and a random oracle.

Keywords: Public key cryptosystem; Coprime sequence; Lever function; Continued fraction attack; Random oracle

1 Introduction

Theories of computational complexity such as the class P, the class NP, one-way functions, and trapdoor functions provide public key cryptosystems with foundation stones [1][2][3]. For instance, the RSA cryptosystem is founded on the integer factorization problem (IFP) [4], and the ElGamal cryptosystem is founded on the discrete logarithm problem (DLP) [5]. It appeals to people whether polynomial time algorithms for solving IFP and DLP on electronic computers exist or not since IFP and DLP are not proved NP-complete.

To $N = pq$ with p and q prime, if N is given, the values of p and q are determined. To $y \equiv g^x$ (% p) with g a generator of $(\mathbb{Z}_p^*, \cdot)$, if y is given, the value of x is also determined. Nevertheless there exists such a class of computational problems, which looks very ordinary, but leads indeterminacy into a public key cryptosystem — a permutation problem for example.

In the REESSE1+ public key cryptosystem [6], the key transform is $C_i \equiv (A_i W^{\ell(i)})^{\delta}$ (% M) with $\ell(i) \in \Omega = \{5, 7, …, 2n + 3\}$. The analysis in [6] shows that a REESSE1+ private key ($\{A_i\}$, $\{\ell(i)\}$, W, δ) is secure without doubt due to the existence of $\delta \in [1, M - 1]$.

If $\delta = 1$ and $C_i \equiv A_i W^{\ell(i)}$ (% M) with $\ell(i) \in \Omega_\pm = \{+/–5, +/–6, …, +/–(n + 4)\}$ new, what is the thing?

In this paper, starting on the security of the simplified transform $C_i \equiv A_i W^{\ell(i)}$ (% M), we will investigate the effect of the lever function $\ell(.)$ from $\{1, 2, …, n\}$ to $\Omega_\pm$ with indeterminacy.

Throughout the paper, unless otherwise specified, $n \ge 80$ is the bit-length of a plaintext block or the item-length of a sequence, the sign % means "modulo", $\bar{M}$ does "$M - 1$" with M prime, lg$x$ denotes a logarithm of x to the base 2, $\neg x$ does the opposite of a bit x, Ᵽ does the maximal prime allowed in coprime sequences, $|x|$ does the absolute value of an integer x, ¦S¦ does the size of a set S, and gcd(a, b) represents the greatest common divisor of two integers a and b. Without ambiguity, "% M" is usually omitted in expressions.

2 Simplified REESSE1+ Encryption Scheme

To probe the indeterminacy of the lever function $\ell(.)$ to $\Omega_\pm$, let $1 = \delta$ in the key transform of the REESSE1+ cryptosystem.

We first observe the simplified REESSE1+ encryption scheme with $\delta = 1$.

2.1 Two Definitions

Definition 1: If $A_1, …, A_n$ are n pairwise distinct positive integers such that $\forall A_i, A_j$ ($i \ne j$), either gcd(A_i, A_j) = 1, or gcd(A_i, A_j) = $F \ne 1$ with $(A_i / F) \nmid A_k$ and $(A_j / F) \nmid A_k$ $\forall k$ ($\ne i, j$) $\in [1, n]$, these

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integers are called a coprime sequence, denoted by $\{A_1, \ldots, A_n\}$, and shortly $\{A_i\}$.

Notice that the elements of a coprime sequence are not necessarily pairwise coprime, but a sequence whose elements are pairwise coprime must be a coprime sequence.

Property 1: Let $\{A_1, \ldots, A_n\}$ be a coprime sequence. If randomly select $m \in [1, n]$ elements $A_{x_1}, \ldots, A_{x_m}$ from the sequence, then the mapping from a subset $\{A_{x_1}, \ldots, A_{x_m}\}$ to a subset product $G = \prod_{i=1}^{m} A_{x_i}$ is one-to-one, namely the mapping from $b_1 \ldots b_n$ to $G = \prod_{i=1}^{n} A_i^{b_i}$ is one-to-one, where $b_1 \ldots b_n$ is a bit string.

Refer to [6] for its proof.

Definition 2: The secret parameter $\ell(i)$ in the key transform of a public key cryptosystem is called a lever function, if it has the following features:

- $\ell(.)$ is an injection from the domain $\{1, \ldots, n\}$ to the codomain $\Omega \subset \{5, \ldots, \bar{M}\}$, where $\bar{M}$ is large;
- the mapping between i and $\ell(i)$ is established randomly without an analytical expression;
- an attacker has to be faced with all the arrangements of n elements in Ω when extracting a related private key from a public key;
- the owner of a private key only needs to consider the accumulative sum of n elements in Ω when recovering a related plaintext from a ciphertext.

The latter two points manifest that if n is large enough, it is infeasible for the attacker to search all the permutations of elements in Ω exhaustively while the decryption of a normal ciphertext is feasible in some time being polynomial in n. Thus, there are the large amount of calculation on $\ell(.)$ at "a public terminal", and the small amount of calculation on $\ell(.)$ at "a private terminal".

Notice that ① in modular $\bar{M}$ arithmetic, $-x$ represents $\bar{M} - x$; ② the number of elements of Ω is not less than n; ③ considering the speed of decryption, the absolute values of all the elements should be comparatively small; ④ the lower limit 5 will make seeking the root W from $W^{\ell(i)} \equiv A_i^{-1} C_i$ (% M) face an unsolvable Galois group when $A_i \le 1201$ is guessed [7].

2.2 Key Generation Algorithm

In the simplified REESSE1+ encryption scheme, we substitute $\Omega = \{5, 7, \ldots, 2n + 3\}$ with $\Omega_\pm$.

Let $|\Omega_\pm|$ be the set of absolute values of all the elements in $\Omega_\pm$.

Let $\Lambda = \{2, \ldots, \mathcal{P}\}$, where $\mathcal{P} = 863, 937, 991$, or 1201 when $n = 80, 96, 112$, or 128.

This algorithm is employed by a certificate authority or the owner of a key pair.

INPUT: the integer n; the set Λ.

S1: Randomly generate $\Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n + 4)\}$.

S2: Randomly produce pairwise coprime $A_1, \ldots, A_n \in \Lambda$.

S3: Find a prime $M > \prod_{i=1}^{n} A_i$ making $q^2 \mid \bar{M}\ \forall\ q$ (prime) $\in |\Omega_\pm|$.

S4: Stochastically pick the integer $W \in (1, \bar{M})$.

S5: Randomly produce pairwise distinct $\ell(1), \ldots, \ell(n) \in \Omega_\pm$.

S6: Compute $C_i \leftarrow A_i W^{\ell(i)}$ % M for $i = 1, \ldots, n$.

OUTPUT: a public key $(\{C_i\}, M)$; a private key $(\{A_i\}, W, M)(\{\ell(i)\}$ may be discarded).

Notice that at S1, $\Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n + 4)\}$ indicates that $\Omega_\pm$ is one of 2^n potential sets, and indeterminate, where "+/–" means the selection of the "+" sign or the "–" sign.

2.3 Encryption Algorithm

This algorithm is employed by a person who wants to encrypt plaintexts.

INPUT: a public key $(\{C_i\}, M)$; an n-bit plaintext block $b_1 \ldots b_n$.

S1: Set $\bar{G} \leftarrow 1$, $i \leftarrow 1$.

S2: If $b_i = 1$ then let $\bar{G} \leftarrow \bar{G} C_i$ % M.

S3: Let $i \leftarrow i + 1$.

S4: If $i \le n$ then goto S2; else end.

OUTPUT: the ciphertext $\bar{G} \equiv \prod_{i=1}^{n} C_i^{b_i}$ (% M).

Definition 3: Given $\bar{G}$ and $(\{C_i\}, M)$, seeking $b_1 \ldots b_n$ from $\bar{G} \equiv \prod_{i=1}^{n} C_i^{b_i}$ (% M) is called a subset product problem, shortly SPP [6][8].

Notice that when $\lceil \lg M \rceil < 1024$, a discrete logarithm can be found in tolerable time.

Let g be a generator of $(\mathbb{Z}_M^*, \cdot)$, $\bar{G} \equiv g^u$ (% M), $C_1 \equiv g^{v_1}$ (% M), …, $C_n \equiv g^{v_n}$ (% M), and then the subset product problem $\bar{G} \equiv \prod_{i=1}^{n} C_i^{b_i}$ (% M) is degenerated to a subset sum problem $u \equiv b_1 v_1 + \ldots + b_n v_n$ (% $\bar{M}$) of density less than 1, which indicates $\bar{G}$ is not robust [9].

Therefore, only if $\lceil \lg M \rceil \ge 1024$, can simplified REESSE1+ have practical sense.

2.4 Decryption Algorithm

This algorithm is employed by a person who wants to decrypt ciphertexts.

INPUT: a private key ($\{A_i\}$, W, M); a ciphertext $\bar{G}$.

S1: Set $X_0 \leftarrow \bar{G}$, $X_1 \leftarrow X_0$, $h \leftarrow 0$.

S2: Set $b_1 \ldots b_n \leftarrow 0$, $G \leftarrow X_h$, $i \leftarrow 1$.

S3: If $A_i \mid G$ then let $b_i \leftarrow 1$, $G \leftarrow G / A_i$.

S4: Let $i \leftarrow i + 1$.

If $i \le n$ and $G \ne 1$ then goto S3.

S5: If $G \ne 1$ then do $h \leftarrow \neg h$, $X_h \leftarrow X_h W^{(-1)^h} \% M$, goto S2;

else end.

OUTPUT: the original plaintext block $b_1 \ldots b_n$.

Notice that as long as $\bar{G}$ is a true ciphertext, this algorithm can always terminates normally.

3 Necessity of the Lever Function $\ell(.)$

We will discuss the necessity of the lever function $\ell(.)$ from $[1, \ldots, n]$ to $\Omega_\pm$ for resisting continued fraction attack and intersection attack.

The necessity of the lever function $\ell(.)$ to $\Omega_\pm$ means that if a simplified REESSE1+ private key is secure, $\ell(.)$ as a one-to-one function must exist in the key transform. The equivalent contrapositive assertion is that if $\ell(.)$ as a one-to-one function does not exist (namely every $\ell(i)$ is mapped to the same integer $\bar{e}$), a simplified REESSE1+ private key will be insecure.

3.1 Continued Fraction Attack on a Simplified Private Key

Theorem 1: If α is an irrational number, $r, s > 0$ are two integers, and r / s is a rational in the lowest terms such that $|\alpha - r / s| < 1 / (2s^2)$, then r / s is a convergent of the simple continued fraction expansion of α.

Refer to [10] for the proof.

Notice that theorem 1 also holds when α is a rational number [10].

For a public key cryptosystem, if a private key is insecure, a plaintext must be insecure. Hence, the security of a private key is most foundational [11].

***Definition 4**:* Attack on $C_i \equiv A_i W^{\ell(i)}$ (% M) with $\ell(i) \in \Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n + 4)\}$ for $i = 1, \ldots, n$ by a convergent of the continued fraction of G_z / M, where $G_z \equiv (C_{x_1} \ldots C_{x_m})(C_{y_1} \ldots C_{y_h})^{-1}$ with $m \in [1, n - 1]$, $h \in [1, n - m]$, and $x_j \ne y_k \; \forall \, j \in [1, m]$ and $k \in [1, h]$, is called continued fraction attack.

***Property 2**:* Let $\bar{e} \in [1, \bar{M}]$ be any integer. If the key transform of the simplified REESSE1+ cryptosystem is $C_i \equiv A_i W^{\bar{e}}$ (% M), namely $\ell(i) = \bar{e}$ for $i = 1, \ldots, n$, a simplified REESSE1+ private key ($\{A_1, \ldots, A_n\}$, $W^{\bar{e}}$) is insecure.

Proof.

Assume that $\ell(1) = \ldots = \ell(n) = \bar{e}$, where $\bar{e}$ is a fixed integer.

Then, the key transform becomes as

$$C_i \equiv A_i W^{\bar{e}} \ (\% \ M),$$

and especially when $\bar{e} = 1$, $C_i \equiv A_i W$ (% M) for $i = 1, \ldots, n$.

Since $(\mathbb{Z}^*_M, \cdot)$ is an Abelian group [7], of course, there is

$$C_i^{-1} \equiv (A_i W^{\bar{e}})^{-1} \ (\% \ M).$$

$\forall x \in [1, n - 1]$, let

$$G_z \equiv C_x C_n^{-1} \ (\% \ M).$$

Substituting $A_x W^{\bar{e}}$ and $A_n W^{\bar{e}}$ respectively for C_x and C_n in the above congruence yields

$$G_z \equiv A_x W^{\bar{e}} (A_n W^{\bar{e}})^{-1} \ (\% \ M)$$

$$A_n G_z \equiv A_x \ (\% \ M)$$

$$A_n G_z - LM = A_x,$$

where L is a positive integer.

The either side of the equation is divided by $A_n M$ gives

$$G_z / M - L / A_n = A_x / (A_n M). \tag{1}$$

Due to $M > \prod_{i=1}^{n} A_i$ and $A_i \ge 2$, there is

$$G_z / M - L / A_n < A_x / (A_n \prod_{i=1}^{n} A_i)$$
$$= A_x / (A_n^2 \prod_{i=1}^{n-1} A_i) \le 1 / (2^{n-2} A_n^2),$$

that is,

$$G_z / M - L / A_n < 1 / (2^{n-2} A_n^2). \quad (2)$$

Evidently, as $n > 2$, there is

$$G_z / M - L / A_n < 1 / (2 A_n^2). \quad (2')$$

In terms of theorem 1, L / A_n is a convergent of the continued fraction of G_z / M.

Thus, L / A_n, namely A_n may be determined by (2′) in polynomial time since the length of the continued fraction will not exceed $\lceil \lg M \rceil$, and further $W^{\bar{e}} \equiv C_n A_n^{-1}$ (% M) may be computed, which indicates the original coprime sequence $\{A_1, \ldots, A_n\}$ with $A_i \le Ɀ$ can almost be recovered. □

Because W in every C_i has the same exponent, and the powers of W and W^{-1} in any $C_x C_n^{-1}$ % M always counteract each other, when $\ell(i)$ is a fixed integer $\underline{k}$, there does not exist the indeterministic reasoning problem.

It should be noted that when a convergent of the continued fraction of G_z / M satisfies (2′), the some subsequent convergents also possibly satisfies (2′), and if so, it will bring about the nonuniqueness of value of A_n. Therefore, we say that $\{A_1, \ldots, A_n\}$ with $A_i \le Ɀ$ can almost be recovered.

3.2 Intersection Attack on a Simplified Private Key

Assume that $\ell(1) = \ldots = \ell(n) = \bar{e}$, where $\bar{e}$ is a fixed integer. Then the key transform turns to $C_i \equiv A_i W^{\bar{e}}$ (% M) for $i = 1, \ldots, n$. Hence, there exists the following attack.

Algorithm 3.2:

INPUT: a public key ($\{C_1, \ldots, C_n\}$, M)

S1: Let A_i traverse Λ for every i:

S1.1: Compute $W^{\bar{e}}$ such that $W^{\bar{e}} \equiv C_i A_i^{-1}$ (% M) for every possible value of A_i.

S1.2: Place the pair ($W^{\bar{e}}$, A_i) into the set Θ_i for every possible value of A_i.

S2: Seek the intersection $\Theta = \Theta_1 \cap \ldots \cap \Theta_n$ on $W^{\bar{e}}$. (Note that $¦\Theta¦$ is pretty limited, and at least 1)

S3: Extract $W^{\bar{e}}$ from Θ and corresponding A_i from Θ_i.

OUTPUT: a private key ($\{A_1, \ldots, A_n\}$, $W^{\bar{e}}$).

It is not difficult to understand that the time complexity of the above attack is dominantly involved in S1 and S2. Concretely speaking, the time complexity is $O(¦\Lambda¦n + ¦\Lambda¦n) = O(¦\Lambda¦n)$, and polynomial in n.

Section 3.1 and 3.2 manifest that when every $\ell(i)$ is a fixed integer $\bar{e}$, a related private key can be deduced from a public key, and further a related plaintext can be inferred from a ciphertext. Thus, the one-to-one lever function $\ell(.)$ is necessary to the security of a simplified REESSE1+ private key.

4 Sufficiency of the Lever Function $\ell(.)$

The sufficiency of the lever function $\ell(.)$ to $\Omega_\pm$ for resisting continued fraction attack and indeterministic intersection attack which are most efficient currently means that if $\ell(1), \ldots, \ell(n) \in \Omega_\pm$ are pairwise distinct, a simplified REESSE1+ private key will be secure.

The analysis in this section will show that the continued fraction attack is utterly ineffectual if $\Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n+4)\}$ is indeterminate, and do not always threaten $C_i \equiv A_i W^{\ell(i)}$ (% M) even if $\Omega_\pm = \{5, \ldots, n+4\}$ is adventitiously selected and known to adversaries.

4.1 Indeterminacy of the Lever Function $\ell(.)$

According to Section 2.2, if the lever function $\ell(.)$ exists, we have

$$C_i \equiv A_i W^{\ell(i)} \text{ (\% } M),$$

where $A_i \in \Lambda = \{2, \ldots, Ɀ\}$, and $\ell(i) \in \Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n+4)\}$ for $i = 1, \ldots, n$.

The lever function $\ell(.)$ brings adversaries at least two difficulties:

- No method in terms of which one can directly judge whether the power of W in $C_{x_1} \ldots C_{x_m}$ is counteracted by the power of W^{-1} in $(C_{y_1} \ldots C_{y_h})^{-1}$ or not;
- No criterion in terms of which one can verify the presupposition of an indeterministic reasoning in polynomial time.

The indeterministic reasoning based on continued fractions means that ones first presuppose that the powers of the parameter W and the inverse W^{-1} counteract each other in a product, and then judge whether the presupposition holds or not by the consequence.

According to Section 3, first select $m \in [1, n-1]$ elements and $h \in [1, n-m]$ other elements from $\{C_1, \ldots, C_n\}$. Let

$$G_x \equiv C_{x_1} \ldots C_{x_m} \ (\% \ M),$$
$$G_y \equiv C_{y_1} \ldots C_{y_h} \ (\% \ M),$$

where $x_j \neq y_k \ \forall j \in [1, m]$ and $k \in [1, h]$.

Let

$$G_z \equiv G_x G_y^{-1} \ (\% \ M).$$

Since $\{\ell(1), \ldots, \ell(n)\}$ is any arrangement of n elements in $\Omega_\pm$, it is impossible to predicate that G_z does not contain the factor W or W^{-1}. For a further deduction, we have to *presuppose* that the power of W in G_x is exactly counteracted by the power of W^{-1} in G_y^{-1}, and then,

$$G_z \equiv (A_{x_1} \ldots A_{x_m})(A_{y_1} \ldots A_{y_h})^{-1} \ (\% \ M)$$
$$G_z (A_{y_1} \ldots A_{y_h}) \equiv A_{x_1} \ldots A_{x_m} \ (\% \ M)$$
$$G_z (A_{y_1} \ldots A_{y_h}) - L M = A_{x_1} \ldots A_{x_m}$$
$$G_z / M - L / (A_{y_1} \ldots A_{y_h}) = (A_{x_1} \ldots A_{x_m}) / (M A_{y_1} \ldots A_{y_h}),$$

where L is a positive integer.

Denoting the product $A_{y_1} \ldots A_{y_h}$ by $\bar{A}_y$ yields

$$G_z / M - L / \bar{A}_y = (A_{x_1} \ldots A_{x_m}) / (M \bar{A}_y). \quad (3)$$

Due to $M > \prod_{i=1}^{n} A_i$ and $A_i \geq 2$, we have

$$G_z / M - L / \bar{A}_y < 1 / (2^{n-m-h} \bar{A}_y^{\,2}). \quad (4)$$

Obviously, when $n > m + h$, (4) may have a variant, namely

$$G_z / M - L / \bar{A}_y < 1 / (2 \bar{A}_y^{\,2}). \quad (4')$$

Notice that when $n = m + h$, if $M > 2(\prod_{i=1}^{n} A_i)$, (4′) still holds.

Especially, when $n > 3$, $h = 1$, and $m = 2$, there exists

$$G_z / M - L / A_{y_1} < 1 / (2^{n-3} A_{y_1}^{\,2}) < 1 / (2 A_{y_1}^{\,2}). \quad (4'')$$

Obviously, as a discriminant, (4) is stricter than (4′) and (4″). (4″) is consistent with theorem 1.

***Property 3**:* Let $h + m \leq n$. If $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$, the subset product $\bar{A}_y = A_{y_1} \ldots A_{y_h}$ in (4′) will be found in polynomial time.

Proof.

$\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$ means that the exponent on W in $C_{x_1} \ldots C_{x_m}$ is counteracted by the exponent on W^{-1} in $(C_{y_1} \ldots C_{y_h})^{-1}$, and thus (4′) holds.

In terms of theorem 1, $L / \bar{A}_y$ is inevitably a convergent of the continued fraction of G_z / M, and thus $\bar{A}_y = A_{y_1} \ldots A_{y_h}$ can be found in polynomial time. □

Notice that (4′) is insufficient for $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$ (see Property 7), and $\bar{A}_y$ is faced with nonuniqueness because there may possibly exist several convergents of the continued fraction of G_z / M which all satisfy (4′).

Property 4* (Indeterminacy of $\ell(.)$)*: Let $h + m \leq n$. $\forall x_1, \ldots, x_m, y_1, \ldots, y_h \in [1, n]$, and $\|W\| \neq \bar{M}$.

① When $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$, and $m \neq h$, there is

$$\ell(x_1) + \|W\| + \ldots + \ell(x_m) + \|W\| \neq \ell(y_1) + \|W\| + \ldots + \ell(y_h) + \|W\| \ (\% \ \bar{M});$$

② when $\ell(x_1) + \ldots + \ell(x_m) \neq \ell(y_1) + \ldots + \ell(y_h)$, there always exist

$$C_{x_1} \equiv A'_{x_1} W'^{\,\ell(x_1)}, \ \ldots, \ C_{x_m} \equiv A'_{x_m} W'^{\,\ell(x_m)},$$
$$C_{y_1} \equiv A'_{y_1} W'^{\,\ell(y_1)}, \ \ldots, \ C_{y_h} \equiv A'_{y_h} W'^{\,\ell(y_h)} \ (\% \ M),$$

such that $\ell'(x_1) + \ldots + \ell'(x_m) \equiv \ell'(y_1) + \ldots + \ell'(y_h) \ (\% \ \bar{M})$ with $A'_{y_1} \ldots A'_{y_h} \leq \mathcal{P}^h$;

③ when $\ell(x_1) + \ldots + \ell(x_m) \neq \ell(y_1) + \ldots + \ell(y_h)$, probability that $C_{x_1}, \ldots, C_{x_m}, C_{y_1}, \ldots, C_{y_h}$ make (4) with $A'_{y_1} \ldots A'_{y_h} \leq \mathcal{P}^h$ hold is roughly $1 / 2^{n-m-h-1}$.

Proof.

① It is easy to understand that

$$W^{\ell(x_1)} \equiv W^{\ell(x_1) + \|W\|}, \ \ldots, \ W^{\ell(x_m)} \equiv W^{\ell(x_m) + \|W\|} \ (\% \ M),$$
$$W^{\ell(y_1)} \equiv W^{\ell(y_1) + \|W\|}, \ \ldots, \ W^{\ell(y_h)} \equiv W^{\ell(y_h) + \|W\|} \ (\% \ M),$$

Due to $\|W\| \neq \bar{M}$, $m\|W\| \neq h\|W\|$, and $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$, it follows that

$$\ell(x_1) + \ldots + \ell(x_m) + m\|W\| \neq \ell(y_1) + \ldots + \ell(y_h) + h\|W\| \ (\% \ \bar{M}).$$

② Because $A'y_1\ldots A'y_h$ need be observed, the constraint $A'y_1\ldots A'y_h \le \mathcal{P}^h$ is demanded while because $A'x_1, \ldots, A'x_m$ need not be observed, the constraints $A'x_1 \le \mathcal{P}, \ldots, A'x_m \le \mathcal{P}$ are not demanded.

Let $\bar{O}_d$ be an oracle on a discrete logarithm.

Suppose that $W' \in [1, \bar{M}]$ is a generator of $(\mathbb{Z}^*_M, \cdot)$.

Let $\mu = \ell'(y_1) + \ldots + \ell'(y_h)$. In terms of group theories, $\forall A'y_1, \ldots, A'y_h \in [2, \mathcal{P}]$ which need not be pairwise coprime, the equation

$$Cy_1\ldots Cy_h \equiv A'y_1\ldots A'y_h W'^{\mu}\ (\%\ M)$$

in μ has a solution. μ may be obtained through $\bar{O}_d$.

$\forall\ \ell'(y_1), \ldots, \ell'(y_{h-1}) \in [1, \bar{M}]$, let $\ell'(y_h) \equiv \mu - (\ell'(y_1) + \ldots + \ell'(y_{h-1}))\ (\%\ \bar{M})$.

Similarly, $\forall\ \ell'(x_1), \ldots, \ell'(x_{m-1}) \in [1, \bar{M}]$, let $\ell'(x_m) \equiv \mu - (\ell'(x_1) + \ldots + \ell'(x_{m-1}))\ (\%\ \bar{M})$.

Further, from $Cx_1 \equiv A'x_1 W'^{\ell(x_1)}, \ldots, Cx_m \equiv A'x_m W'^{\ell(x_m)}\ (\%\ M)$, we can obtain a tuple $(A'x_1, \ldots, A'x_m)$, where $A'x_1, \ldots, A'x_m \in (1, M)$, and $\ell'(x_1) + \ldots + \ell'(x_m) \equiv \ell'(y_1) + \ldots + \ell'(y_h)\ (\%\ \bar{M})$.

Thus, Property 4.1 is proven.

③ Let $G_z \equiv Cx_1\ldots Cx_m(Cy_1\ldots Cy_h)^{-1}\ (\%\ M)$. Then in terms of Property 4.1, there is

$$Cx_1\ldots Cx_m(Cy_1\ldots Cy_h)^{-1} \equiv A'x_1\ldots A'x_m W'^{\ell(x_1)+\ldots+\ell(x_m)}(A'y_1\ldots A'y_h W'^{\ell(y_1)+\ldots+\ell(y_h)})^{-1}$$

with $\ell'(x_1) + \ldots + \ell'(x_m) \equiv \ell'(y_1) + \ldots + \ell'(y_h)\ (\%\ \bar{M})$.

Further, there is

$$A'x_1\ldots A'x_m \equiv Cx_1\ldots Cx_m(Cy_1\ldots Cy_h)^{-1}A'y_1\ldots A'y_h\ (\%\ M).$$

The above equation manifests that the values of W' and $(\ell'(y_1) + \ldots + \ell'(y_h)$ or $\ell'(x_1) + \ldots + \ell'(x_m))$ do not influence the value of the product $A'x_1\ldots A'x_m$.

If $A'y_1\ldots A'y_h \in [2^h, \mathcal{P}^h]$ changes, the product $A'x_1\ldots A'x_m$ also changes, where $A'y_1\ldots A'y_h$ is a composite integer. Therefore, $\forall\ x_1, \ldots, x_m, y_1, \ldots, y_h \in [1, n]$, the number of potential values of $A'x_1\ldots A'x_m$ is roughly $(\mathcal{P}^h - 2^h + 1)$.

Let $M = q\mathcal{P}^m(A'y_1\ldots A'y_h)2^{n-m-h}$, where q is a rational number.

According to (3),

$$\begin{aligned} G_z / M - L / (A'y_1\ldots A'y_h) &= (A'x_1 \ldots A'x_m) / (M A'y_1 \ldots A'y_h) \\ &= (A'x_1\ldots A'x_m) / (q\mathcal{P}^m 2^{n-m-h}(A'y_1\ldots A'y_h)^2). \end{aligned}$$

When $A'x_1\ldots A'x_m \le q\mathcal{P}^m$, there is

$$\begin{aligned} G_z / M - L / (A'y_1\ldots A'y_h) &\le q\mathcal{P}^m / (q\mathcal{P}^m 2^{n-m-h}(A'y_1\ldots A'y_h)^2) \\ &= 1 / (2^{n-m-h}(A'y_1\ldots A'y_h)^2), \end{aligned}$$

which satisfies (4).

Assume that the value of $A'x_1\ldots A'x_m$ distributes uniformly on the interval $(1, M)$. If $A'y_1\ldots A'y_h$ is a certain concrete value, the probability that $A'x_1\ldots A'x_m$ makes (4) hold at a specific value of $A'y_1\ldots A'y_h$ is

$$\begin{aligned} q\mathcal{P}^m / M &= q\mathcal{P}^m / (q\mathcal{P}^m(A'y_1\ldots A'y_h)2^{n-m-h}) \\ &= 1 / ((A'y_1\ldots A'y_h)2^{n-m-h}). \end{aligned}$$

In fact, it is possible that $A'y_1\ldots A'y_h$ take every value in the interval $[2^h, \mathcal{P}^h]$ when $Cx_1, \ldots, Cx_m, Cy_1, \ldots, Cy_h$ are fixed. Thus, the probability that $A'x_1\ldots A'x_m$ makes (4) hold is

$$\begin{aligned} P_{\forall x_1, \ldots, x_m, y_1, \ldots, y_h \in [1, n]} &= (1 / (2^{n-m-h}))(1/2^h + 1/(2^h+1) + \ldots + 1/\mathcal{P}^h) \\ &> (1/2^{n-m-h})(2(\mathcal{P}^h - 2^h + 1)/(\mathcal{P}^h + 2^h)) \\ &= (\mathcal{P}^h - 2^h + 1)/(2^{n-m-h-1}(\mathcal{P}^h + 2^h)) \\ &\approx 1/2^{n-m-h-1}. \end{aligned}$$

Obviously, the larger $m + h$ is, the larger the probability is, and the smaller n is, the larger the probability is also. □

Property 5*:* Let $h + m \le n$. $\forall\ x_1, \ldots, x_m, y_1, \ldots, y_h \in [1, n]$, when $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$, the probability that another $\bar{A}_y$ makes (4) with $\bar{A}_y \le \mathcal{P}^h$ hold is roughly $1/2^{n-m-h-1}$.

Proof.

Let

$$G_x \equiv Cx_1\ldots Cx_m \equiv (Ax_1\ldots Ax_m)\, W^{\ell(x_1)+\ldots+\ell(x_m)}\ (\%\ M),$$
$$G_y \equiv Cy_1\ldots Cy_h \equiv (Ay_1\ldots Ay_h)\, W^{\ell(y_1)+\ldots+\ell(y_h)}\ (\%\ M).$$

Due to $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$, there is

$$G_z \equiv G_x G_y^{-1} \equiv (Ax_1\ldots Ax_m)(Ay_1\ldots Ay_h)^{-1} \equiv (Ax_1\ldots Ax_m)\bar{A}_y^{-1}\ (\%\ M).$$

According to the derivation of (4″), $\bar{A}_y$ will occur in a convergent of the continued fraction of G_z / M.

Let $p_1 / q_1, \ldots, p_x / q_x = L / \bar{A}_y, p_{x+1} / q_{x+1}, \ldots, p_t / q_t$ be the convergent sequence of the continued fraction of G_z / M, where $t \le \lceil \lg M \rceil$.

Because of $G_z / M - L / \bar{A}_y < 1 / (2^{n-m-h} \bar{A}_y^2)$, it will lead

$$|G_z / M - p_{x+1} / q_{x+1}| < 1 / (2^{n-m-h} q_{x+1}^2) \text{ with } q_{x+1} \le \mathcal{P}^h,$$

……, or

$$|G_z / M - p_t / q_t| < 1 / (2^{n-m-h} q_t^2) \text{ with } q_t \le \mathcal{P}^h$$

to probably hold, and in terms of Property 4.2, the probability is roughly $1 / 2^{n-m-h-1}$.

Notice that in this case, there is $\ell'(x_1) + \ldots + \ell'(x_m) \equiv \ell'(y_1) + \ldots + \ell'(y_h)$ (% $\bar{M}$) with $A'_{y_1} \ldots A'_{y_h} \le \mathcal{P}^h$, where $\ell'(x_1), \ldots, \ell'(x_m), \ell'(y_1), \ldots, \ell'(y_h)$ satisfy

$$C_{x_1} \equiv A'_{x_1} W'^{\ell(x_1)}, \ldots, C_{x_m} \equiv A'_{x_m} W'^{\ell(x_m)}, C_{y_1} \equiv A'_{y_1} W'^{\ell(y_1)}, \ldots, C_{y_h} \equiv A'_{y_h} W'^{\ell(y_h)} \text{ (\% } M).$$

End. □

Property 5 illuminates that the nonuniqueness of $\bar{A}_y$, namely there may exist the disturbance of $\bar{A}_y$. The smaller $m + h$ is, the less the disturbance is.

4.2 Some Conditions Are Only Necessary But Insufficient

Property 6: (4) is necessary but insufficient for $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$ with $x_1, \ldots, x_m, y_1, \ldots, y_h \in [1, n]$, namely for the powers of W and W^{-1} in G_z to counteract each other.

Proof. Necessity:

Suppose that $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$.

Let $\{C_1, \ldots, C_n\}$ be a public key sequence, and M be a modulus, where $C_i \equiv A_i W^{\ell(i)}$ (% M).

Let $G_x \equiv C_{x_1} \ldots C_{x_m}$ (% M), $G_y \equiv C_{y_1} \ldots C_{y_h}$ (% M), and $G_z \equiv G_x G_y^{-1}$ (% M).

Further, $G_z \equiv (A_{x_1} \ldots A_{x_m})(A_{y_1} \ldots A_{y_h})^{-1}$ (% M).

Denote the product $A_{y_1} \ldots A_{y_h}$ by $\bar{A}_y$. Similar to Section 4.1, we have

$$G_z / M - L / \bar{A}_y < 1 / (2^{n-m-h} \bar{A}_y^2),$$

Namely (4) holds.

Insufficiency:

Suppose that (4) holds.

The contrapositive of the proposition that if (4) holds, $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$ holds is that if $\ell(x_1) + \ldots + \ell(x_m) \ne \ell(y_1) + \ldots + \ell(y_h)$, (4) does not hold.

Hence, we need to prove that when $\ell(x_1) + \ldots + \ell(x_m) \ne \ell(y_1) + \ldots + \ell(y_h)$, (4) still holds.

In terms of Property 4.2, when $\ell(x_1) + \ldots + \ell(x_m) \ne \ell(y_1) + \ldots + \ell(y_h)$, the (4) holds with the probability $1 / 2^{n-m-h-1}$, which reminds us that when $\{C_1, \ldots, C_n\}$ is generated, some subsequences in the forms $\{C_{x_1}, \ldots, C_{x_m}\}$ and $\{C_{y_1}, \ldots, C_{y_h}\}$ which are verified to satisfy (4) with $\ell(x_1) + \ldots + \ell(x_m) \ne \ell(y_1) + \ldots + \ell(y_h)$ can always be found beforehand through adjusting the values of W and some elements in $\{A_1, A_2, \ldots, A_i\}$ or $\{\ell(1), \ell(2), \ldots, \ell(n)\}$.

Hence, the (4) is not sufficient for $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$. □

Property 7: (4′) is necessary but not sufficient for $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$ with $x_1, \ldots, x_m, y_1, \ldots, y_h \in [1, n]$, for the powers of W and W^{-1} in G_z to counteract each other.

Proof.

Because (4′) is derived from (4), and Property 6 holds, naturally Property 7 holds. □

Property 8: Let $m = 2$ and $h = 1$. $\forall\, x_1, x_2, y_1 \in [1, n]$, when $\ell(x_1) + \ell(x_2) \ne \ell(y_1)$,

① there always exist

$$C_{x_1} \equiv A'_{x_1} W'^{\ell'(x_1)}, C_{x_1} \equiv A'_{x_2} W'^{\ell'(x_2)}, C_{y_1} \equiv A'_{y_1} W'^{\ell'(y_1)} \text{ (\% } M),$$

such that $\ell'(x_1) + \ell'(x_2) \equiv \ell'(y_1)$ (% $\bar{M}$) with $A'_{y_1} \le \mathcal{P}$;

② $C_{x_1}, C_{x_2}, C_{y_1}$ make (4″) with $A'_{y_1} \le \mathcal{P}$ hold in all probability.

Proof.

① It is similar to the proving process of Property 4.1.

② Let

$$G_z \equiv C_{x_1} C_{x_2} C_{y_1}^{-1} \equiv A'_{x_1} A'_{x_2} W'^{\ell'(x_1)+\ell'(x_2)} (A'_{y_1} W'^{\ell'(y_1)})^{-1} \text{ (\% } M)$$

with $\ell'(x_1) + \ell'(x_2) \equiv \ell'(y_1)$ (% $\bar{M}$).

Further, there is $A'_{x_1} A'_{x_2} \equiv C_{x_1} C_{x_2} C_{y_1}^{-1} A'_{y_1}$ (% M).

It is easily seen from the above equations that the values of W' and $\ell'(y_1)$ do not influence the value of $(A'_{x_1} A'_{x_2})$.

If $A'_{y_1} \in [2, \mathcal{P}]$ changes, $A'_{x_1} A'_{x_2}$ also changes. Thus, $\forall\, x_1, x_2, y_1 \in [1, n]$, the number of potential values of $A'x_1 A'x_2$ is $Ƥ - 1$.

Let $M = 2qƤ^2 A'y_1$, where q is a rational number.

According to (3),

$$G_z / M - L / A'y_1 = A'x_1 A'x_2 / (M A'y_1)$$
$$= A'x_1 A'x_2 / (2qƤ^2 A'y_1{}^2).$$

When $A'x_1 A'x_2 \le qƤ^2$, there is

$$G_z / M - L / A'y_1 \le qƤ^2 / (2qƤ^2 A'y_1{}^2)$$
$$= 1 / (2A'y_1{}^2),$$

which satisfies (4″).

Assume that the value of $A'x_1 A'x_2$ distributes uniformly on $(1, M)$. Then, the probability that $A'x_1 A'x_2$ makes (4″) hold is

$$P_{\forall x_1, x_2, y_1 \in [1,\, n]} = (qƤ^2 / (2qƤ^2))(1/2 + \ldots + 1/Ƥ)$$
$$\ge (1/2)(2(Ƥ - 1)/(Ƥ + 2))$$
$$= 1 - 3/(Ƥ + 2).$$

Apparently, $P_{\forall x_1, x_2, y_1 \in [1,\, n]}$ is very large, and especially when $Ƥ$ is pretty large, it is close to 1. □

According to Property 8.2, for a certain Cy_1 and $\forall\ Cx_1, Cx_2 \in \{C_1, \ldots, C_n\}$, attack by (4″) will produce roughly $n^2/2$ possible values of Ay_1, including the repeated, while attack by (4) may filter out most of the disturbing data of Ay_1. Because every $Ay_1 \le Ƥ < n^2/2$ in REESSE1, the number of potential values of Ay_1 is at most $Ƥ$ in terms of the pigeonhole principle, which indicates the running time of discriminating the original coprime sequence from the values of A_1, …, the values of A_n is $O(Ƥ^n)$.

Property 9: (4″) is necessary but not sufficient for $\ell(x_1) + \ell(x_2) = \ell(y_1)$ with $x_1, x_2, y_1 \in [1, n]$, namely for the powers of W and W^{-1} in G_z to counteract each other.

Proof.

Because (4″) is derived from (4), and Property 6 holds, naturally Property 9 holds. □

It should be noted that Property 2, 3, …, 9 do not depend on the selection of codomain of the lever function $\ell(.)$, namely regardless of selecting the old Ω or the new $\Omega_\pm$, Property 2, 3, …, 9 still hold.

4.3 Two Discrepant Cases

The cases of $h = 1$ and $h \ne 1$ need to be treated distinguishingly.

4.3.1 Case of *h* = 1: Verifying by Examples

The $h = 1$ means that $\bar{A}_y = Ay_1$. If $\bar{A}_y$ is determined, a certain Ay_1 might be exposed directly. A single Ay_1 may be either prime or composite, and thus "whether Ay_1 is prime" may not be regarded as the criterion of the powers of W and W^{-1} counteracting each other.

If take $m = 2$ and $h = 1$, in terms of Property 4.2, the probability $P_{\forall x_1, x_2, y_1 \in [1,\, n]}$ that $A'x_1 A'x_2$ makes (4) hold is roughly $1/2^{n-4}$, and the number of rationals formed as G_z/M which lead (4) to hold is roughly $n^3/2^{n-4}$ when the interval $[1, n]$ is traversed by x_1, x_2, y_1 separately. Notice that $P_{\forall x_1, x_2, y_1 \in [1,\, n]}$ is with respect to (4), but not with respect to (4′) or (4″).

Notice that due to $\Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n + 4)\}$, the value of $\ell(x_1) + \ell(x_2)$ — $(-5) + 6 = 1$ for example does not necessarily occur in $\Omega_\pm$.

In what follows, we validate Property 6 and 8 with two examples when $m = 2$ and $h = 1$. Especially assume that $\Omega_\pm = \{5, 6, \ldots, n + 4\}$ is selected to a turn.

Example 1:

It will illustrate the ineffectuality of continued fraction attack by (4).

Assume that the bit-length of a plaintext block is $n = 6$.

Let $\{A_i\} = \{11, 10, 3, 7, 17, 13\}$, and $\Omega_\pm = \{5, 6, 7, 8, 9, 10\}$.

Find $M = 510931 > 11 \times 10 \times 3 \times 7 \times 17 \times 13$.

Stochastically pick $W = 17797$, and

$$\ell(1) = 9,\ \ell(2) = 6,\ \ell(3) = 10,\ \ell(4) = 5,\ \ell(5) = 7,\ \ell(6) = 8.$$

From $C_i \equiv A_i W^{\ell(i)}$ (% M), we obtain

$\{C_i\} = \{113101, 79182, 175066, 433093, 501150, 389033\}$.

Stochastically pick $x_1 = 2$, $x_2 = 6$, and $y_1 = 5$.

Notice that there is $\ell(5) \ne \ell(2) + \ell(6)$.

Compute

$$G_z \equiv C_2 C_6 C_5^{-1} \equiv 79182 \times 389033 \times 434038 \equiv 342114 \ (\%\ 510931).$$

Presuppose that the power of W in $C_2 C_6$ is just counteracted by the power of W^{-1} in C_5^{-1}, and then

$$342114 \equiv A_2 A_6 A_5^{-1} \ (\%\ 510931).$$

According to (3),

$$342114 / 510931 - L / A_5 = A_2 A_6 / (510931 A_5).$$

It follows that the continued fraction expansion of 342114 / 510931 equals

$$1 / (1 + 1 / (2 + 1 / (37 + 1 / (1 + 1 / (2 + \dots + 1 / 4))))),$$

where the denominators $1 = a_1$, $2 = a_2$, $37 = a_3$, … .

Heuristically let

$$L / A_5 = 1/ (1 + 1 / 2) = 2 / 3,$$

which indicates it is probable that $A_5 = 3$. Further,

$$342114 / 510931 - 2 / 3 = 0.002922769 < 1 / (2^3 \times 3^2) = 0.013888889,$$

which satisfies (4). Then $A_5 = 3$ is deduced, which is in direct contradiction to factual $A_5 = 17$, so it is impossible that (4) may serve as a sufficient condition.

Meantime, in Example 1, we observe $a_2 = 2$ and $a_3 = 37$, and the increase from a_2 to a_3 should be sharp. However, even though the case is this, the continued fraction attack by (4) fails.

Example 2:

It will illustrate the ineffectuality of a continued fraction attack by a discriminant relevant to (4″).

The following Algorithm 4.3.1 which is evolved from the analysis task in [12] describes a continued fraction attack on a simplified REESSE1+ private key. The attack rests on the discriminant

$$q_s \Delta < q_{s+1} \text{ and } q_s < A_{\max}, \quad (5)$$

where q_s, q_{s+1}, Δ, and $A_{\max}$ are referred to Algorithm 4.3.1 for their meanings.

In terms of [12], (5) is derived from (4″). Seemingly, (5) is stricter than (4″), and intended to uniquely determine A_{y_1}.

Algorithm 4.3.1:

INPUT: a public key ($\{C_1, \dots, C_n\}$, M).

S1: Generate the first $2n$ primes $p_1, \dots, p_{2n}$ of the natural set.

S2: Set $\Delta \leftarrow (M / (2\prod_{i=n-2}^{u} p_i))^{1/2}$, $A_{\max} \leftarrow M / \prod_{i=1}^{n-1} p_i$,

where u meets $\prod_{i=1}^{u} p_i < M \le \prod_{i=1}^{u+1} p_i$.

S3: For ($x_1 = 1$, $x_1 \le n$, x_1++)

For ($x_2 = 1$, $x_2 \le n$, x_2++)

For ($y_1 = 1$, $y_1 \le n$, y_1++) {

Compute $G_z \leftarrow C_{x_1} C_{x_2} C_{y_1}^{-1}$ % M;

Get convergent sequence $\{r_0/q_0, r_1/q_1, \dots, r_t/q_t\}$

of continued fraction of G_z/M;

Get denominator sequence $\{q_1, q_2, \dots, q_t\}$

from the convergent sequence;

For ($s = 1$, $s \le t$, s++)

If ($q_s \Delta < q_{s+1}$) and ($q_s < A_{\max}$) then {

Let $A_{y_1} \leftarrow q_s$;

Return (A_{y_1}, (x_1, x_2, y_1)).

}

}

OUTPUT: entries (A_{y_1}, (x_1, x_2, y_1)).

Notice that z++ denotes $z \leftarrow z + 1$, where z is any arbitrary variable.

However, Algorithm 4.3.1 is ineffectual in practice. Please see the following example.

Assume that the bit-length of a plaintext block is $n = 10$.

Let $\{A_i\}$ = {437, 221, 77, 43, 37, 29, 41, 31, 15, 2}, and $\Omega_{\pm}$ = {5, 6, 7, 8, 9, 10, 11, 12, 13, 14}.

Find $M = 13082761331670077 > \prod_{i=1}^{n} A_i = 13082761331670030$.

Randomly select $W = 944516391$, and

$\ell(1) = 11$, $\ell(2) = 14$, $\ell(3) = 13$, $\ell(4) = 8$, $\ell(5) = 10$, $\ell(6) = 5$, $\ell(7) = 9$, $\ell(8) = 7$, $\ell(9) = 12$, $\ell(10) = 6$.

By $C_i \equiv A_i W^{\ell(i)}$ (% M), obtain

$\{C_1, \dots, C_{10}\}$ = {3534250731208421, 12235924019299910, 8726060645493642, 10110020851673707, 2328792308267710, 8425476748983036, 6187583147203887, 10200412235916586, 9359330740489342, 5977236088006743}.

On input of the public key ($\{C_i\}$, M), the program by Algorithm 4.3.1 will evaluate Δ = 506, $A_{\max}$ = 58642670, and output A_{y_1} and (x_1, x_2, y_1). Structure Table 1 with entries (A_{y_1}, (x_1, x_2, y_1)). On Table 1, the number of triples (x_1, x_2, y_1) is greater than 100.

A_{y_1}	Triple (x_1, x_2, y_1)
A_1 = 187125	(1, 1, 1)
A_1 = 121089	(2, 1, 1), (1, 2, 1)
A_1 = 77	(5, 3, 1), (3, 5, 1)
A_1 = 23	(8, 6, 1), (6, 8, 1), (10, 10, 1)
A_1 = 437	(10, 6, 1), (6, 10, 1)
A_2 = 1251	(1, 1, 2)
A_2 = 187125	(2, 1, 2), (1, 2, 2)
A_2 = 121089	(2, 2, 2)
A_2 = 17	(8, 4, 2), (6, 5, 2), (5, 6, 2), (10, 7, 2), (4, 8, 2), (7, 10, 2)
A_2 = 221	(10, 4, 2), (7, 6, 2), (6, 7, 2), (8, 8, 2), (4, 10, 2)
A_2 = 77	(9, 8, 2), (8, 9, 2)
A_2 = 4204	(10, 10, 2)
A_3 = 187125	(3, 1, 3), (1, 3, 3)
A_3 = 12	(7, 1, 3), (1, 7, 3)
A_3 = 121089	(3, 2, 3), (2, 3, 3)
A_3 = 77	(6, 4, 3), (4, 6, 3), (10, 8, 3), (8, 10, 3)
A_3 = 11	(10, 4, 3), (7, 6, 3), (6, 7, 3), (8, 8, 3), (4, 10, 3)
A_3 = 2113	(8, 7, 3), (7, 8, 3)
A_3 = 769	(9, 8, 3), (8, 9, 3)
A_4 = 187125	(4, 1, 4), (1, 4, 4)
A_4 = 121089	(4, 2, 4), (2, 4, 4)
A_4 = 76	(10, 6, 4), (6, 10, 4)
A_4 = 56	(10, 9, 4), (9, 10, 4)
A_5 = 187125	(5, 1, 5), (1, 5, 5)
A_5 = 630269	(6, 1, 5), (1, 6, 5)
A_5 = 121089	(5, 2, 5), (2, 5, 5)
A_5 = 41	(8, 2, 5), (2, 8, 5)
A_5 = 97	(4, 3, 5), (3, 4, 5)
A_5 = 37	(6, 6, 5), (10, 6, 5), (6, 10, 5)
A_6 = 187125	(6, 1, 6), (1, 6, 6)
A_6 = 121089	(6, 2, 6), (2, 6, 6)
A_7 = 187125	(7, 1, 7), (1, 7, 7)
A_7 = 121089	(7, 2, 7), (2, 7, 7)
A_7 = 3	(9, 3, 7), (3, 9, 7)
A_8 = 187125	(8, 1, 8), (1, 8, 8)
A_8 = 34945619	(6, 2, 8), (2, 6, 8)
A_8 = 121089	(8, 2, 8), (2, 8, 8)
A_9 = 187125	(9, 1, 9), (1, 9, 9)
A_9 = 121089	(9, 2, 9), (2, 9, 9)
A_9 = 5	(6, 4, 9), (4, 6, 9), (10, 8, 9), (8, 10, 9)
A_9 = 15	(8, 6, 9), (6, 8, 9), (10, 10, 9)
A_{10} = 259970	(4, 1, 10), (1, 4, 10)
A_{10} = 187125	(10, 1, 10), (1, 10, 10)
A_{10} = 121089	(10, 2, 10), (2, 10, 10)
A_{10} = 7629	(8, 3, 10), (3, 8, 10)

Table 1: A_{y_1} and the Triple (x_1, x_2, y_1)

On Table 1, we observe that

A_{y_1} relevant to 5 triples is $A_2 = 221$ or $A_3 = 11$,

A_{y_1} relevant to 4 triples is $A_3 = 77$ or $A_9 = 5$,

A_{y_1} relevant to 3 triples is $A_1 = 23$, $A_5 = 37$, or $A_9 = 15$,

A_{y_1} relevant to 2 triples is $A_1 = 77$, $A_2 = 77$, $A_3 = 12$, $A_4 = 56$, $A_5 = 41$, or $A_7 = 3$ etc,

A_{y_1} relevant to 1 triple is $A_1 = 187125$, $A_2 = 1251$, $A_2 = 121089$, or $A_2 = 4204$.

Among these A_{y_1}′s, there exist at least 2^{n-5} compatible selections from which some elements of the coprime sequence $\{A_i\}$ can be obtained.

For instance, randomly select compatible A_{y_1}′s: $A_3 = 11$, $A_9 = 5$, $A_1 = 23$, $A_5 = 41$, and $A_2 = 1251$, and work out $\ell(y_1)$′s: $\ell(3) = 14$, $\ell(9) = 13$, $\ell(1) = 12$, $\ell(5) = 11$, and $\ell(2) = 10$ according to the rule that the number of the triples (x_1, x_2, y_1) tied to A_{y_1} equals $(\ell(y_1) - 9)$ when $\ell(y_1) \geq 10$ [12].

Obviously, such A_1, A_2, A_3, A_5, A_9 are not original elements, which indicates (5) derived from (4″) is essentially insufficient even if a concrete $\Omega_{\pm} = \{5, 6, \ldots, n + 4\}$ is selected and known.

4.3.2 Case of $h \neq 1$

The $h \neq 1$ means $\bar{A}_y = A_{y_1} \ldots A_{y_h}$. It is well known that any composite $\bar{A}_y \neq p^k$ (p is a prime) can be factorized into some prime multiplicative factors, and many coprime sequences of the same length can be obtained from a prime factor set.

For instance, let $h = 3$ and $\bar{A}_y = 210$ with the prime factor set $\{2, 3, 5, 7\}$. We can obtain the coprime sequences $\{5, 6, 7\}$, $\{6, 5, 7\}$, $\{3, 7, 10\}$, $\{10, 3, 7\}$, $\{2, 15, 7\}$, $\{3, 2, 35\}$, etc. Which is the original?

Property 4 makes it clear that due to the indeterminacy of $\ell(.)$, no matter whether the power of W and W^{-1} counteract each other or not, in some cases, one or several values of $\bar{A}_y$ which may be written as the product of h coprime integers, and satisfy (4) can be found out from the convergents of the continued fraction of G_z / M when the interval $[1, n]$ is traversed respectively by $x_1, \ldots, x_m, y_1, \ldots, y_h$. Thus, "whether $\bar{A}_y$ can be written as the product of h coprime integers" may not be regarded as a criterion for verifying that the powers of W and W^{-1} counteract each other.

Moreover, even if the k values $v_1, \ldots, v_k$ of the product $A_{y_1} A_{y_2} \ldots A_{y_h}$ are obtained, where y_1 is fixed, and $y_2, \ldots, y_h$ are varied, $\gcd(v_1, \ldots, v_k)$ can not be judged to be A_{y_1} in terms of the definition of a coprime sequence.

If take $m = 2$ and $h = 2$, in terms of Property 4.2 and $P_{\forall x_1, x_2, y_1, y_2 \in [1, n]}$, the number of rationals formed as G_z / M which leads (4) to hold is roughly $n^4 / 2^{n-5}$ when the interval $[1, n]$ is traversed by x_1, x_2, y_1, y_2 respectively. What is most pivotal is that the value of $\ell(x_1) + \ell(x_2)$ or $\ell(y_1) + \ell(y_2)$ $\forall x_1, x_2, y_1, y_2 \in [1, n]$ does not necessarily occur in a concrete $\Omega_{\pm}$.

4.4 Time Complexities of Two Attacks

The continued fraction attack and indeterministic intersection attack on $C_i \equiv A_i W^{\ell(i)}$ ($\%\, M$) are most efficient at present.

4.4.1 Time Complexity of Continued Fraction Attack

It can be seen from section 4.1 that continued fraction attack is based on the assumption that $\ell(x_1) + \ldots + \ell(x_m) = \ell(y_1) + \ldots + \ell(y_h)$. For convenience, usually let $m = 2$ and $h = 1$.

If $\Omega_{\pm}$ is determined as $\{5, 6, \ldots, n + 4\}$, the continued fraction attack by (4), (4′), (4″) or (5) contains five steps dominantly.

Note that it is known from Example 2 that $\Omega_{\pm} = \{5, 6, \ldots, n + 4\}$ does not mean that the continued fraction attack will succeed.

Algorithm 4.4.1:

INPUT: a public key $(\{C_1, \ldots, C_n\}, M)$;
the set $\Omega_{\pm} = \{5, 6, \ldots, n + 4\}$.

S1: Structure Table 2 according to $\Omega_{\pm}$.

S2: Get entries $(A_{y_1}, (x_1, x_2, y_1))$ by calling Algorithm 4.3.1.

S3: Structure Table 1 with entries $(A_{y_1}, (x_1, x_2, y_1))$.

S4: Find coprime A_{y_1} according to Table 1 and Table 2.

S5: Find pairwise different $\ell(y_1)$ according to A_{y_1} and Table 2.

OUTPUT: coprime values of A_{y_1} and pairwise different values of $\ell(y_1)$.

$\ell(y_1)$	10	11	……	$n+4$
$\ell(x_1)+\ell(x_2)$	$5+5$	$5+6, 6+5$	……	$5+(n-1), \ldots, (n-1)+5$
Number of $\ell(y_1)=\ell(x_1)+\ell(x_2)$	1	2	……	$n-5$

Table 2: Number of $\ell(x_1)+\ell(x_2)=\ell(y_1)$ over $\Omega_\pm = \{5, 6, \ldots, n+4\}$

At S4, finding coprime values of A_{y_1} will probably take $O(2^{n-5})$ running time.

At S1, when $\Omega_\pm$ is indeterminate (in fact, $\Omega_\pm$ is one of 2^n potential sets), an adversary must firstly determine all the elements of $\Omega_\pm$, which will take $O(2^n)$ running time.

4.4.2 Time Complexity of Indeterministic Intersection Attack

Due to $C_i \equiv A_i W^{\ell(i)}$ (% M) with $A_i \in \Lambda = \{2, \ldots, Ᵽ\}$ and $\ell(i) \in \Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n+4)\}$ for $i = 1, \ldots, n$, and elements in the sets Λ and $\Omega_\pm$ being small, an adversary may attempt the following attack with indeterminacy.

Algorithm 4.4.2:
INPUT: a public key ($\{C_1, \ldots, C_n\}$, M); the set Λ.
S1: Let $\ell(i)$ traverse $\{5, \ldots, n+4, -5, \ldots, -(n+4)\}$,
  and A_i traverse Λ for every i:
  S1.1: Compute W such that $W^{\ell(i)} \equiv C_i A_i^{-1}$ (% M)
    for every possible value of (A_i, $\ell(i)$).
  S1.2: Place the triple (W, A_i, $\ell(i)$) into the set Θ_i
    for every possible value of (A_i, $\ell(i)$).
S2: Seek the intersection $\Theta = \Theta_1 \cap \ldots \cap \Theta_n$ on W.
S3: If W unique in Θ, and relevant (A_i, $\ell(i)$) unique in every Θ_i, then
    a private key ($\{A_i\}$, $\{\ell(i)\}$, W) is extracted;
  else (W nonunique in Θ, or relevant (A_i, $\ell(i)$) nonunique in some Θ_i)
    check whether every possible $\{A_1, \ldots, A_n\}$ is a coprime sequence,
    and whether every possible $\{\ell(1), \ldots, \ell(n)\}$ is a lever function.
OUTPUT: private keys ($\{A_i\}$, $\{\ell(i)\}$, W).

When the number of private keys is larger than 1, the original private key need to be verified.

Note that at S1.1, may compute W by the Moldovyan root finding method [13], and the time complexity of the method is $O(\ell(i)^{1/2}\lceil \lg M \rceil) \approx O(n^{1/2}\lceil \lg M \rceil)$.

The size of every Θ_i is about $O(|\Lambda||\Omega_\pm|^2) \approx O(Ᵽn^2)$ due to $q^2 \mid \bar{M}\ \forall\ q$ (prime) $\in |\Omega_\pm|$.

At S2, seeking the intersection Θ will take $O(Ᵽn^3)$ running time which is polynomial in n.

At S3, seeking a coprime sequence will take $O(n)$ running time in the best case with low probability, but it will take $O(2^n)$ running time in a worse case. The low probability can be guaranteed through the selection of some private parameters in the process of a key pair generation.

Thus, the adversary cannot extract a simplified REESSE1+ private key in determinate polynomial time.

4.5 Relation between a Lever Function and a Random Oracle

4.5.1 What Is a Random Oracle

An oracle is a mathematical abstraction, a theoretical black box, or a subroutine of which the running time may not be considered [11][14]. In particular, in cryptography, an oracle may be treated as a subcomponent of an adversary, and lives its own life independent of the adversary. Usually, the adversary interacts with the oracle but cannot control its behavior.

A random oracle is an oracle which answers to every query with a completely random and unpredictable value chosen uniformly from its output domain, except that for any specific query, it outputs the same value every time it receives that query if it is supposed to simulate a deterministic function [14].

Random oracles are utilized in cryptographic proofs for relpacing any irrealizable function so far which can provide the mathematical properties required by the proof. A cryprosystem or a protocol that is proven secure using such a proof is described as being secure in the random oracle model, as opposed to being secure in the standard model where the integer factorization problem, the discrete logarithm problem etc are assumed to be hard. When a random oracle is used within a security proof, it is made available to all participants, including adversaries. In practice, random oracles producing a bit-string of infinite length which can be truncated to the length desired are typically used to model cryptographic hash functions in schemes where strong randomness assumptions of a hash function′ s output are needed.

In fact, it draws attention that certain artificial signature and encryption schemes are proven secure in the random oracle model, but are trivially insecure when any real hash function such as MD5 or SHA-1 is substituted for the random oracle [15][16]. Nevertheless, for any more natural protocol, a proof of security in the random oracle model gives very strong evidence that an attacker have to discover some unknown and undesirable property of the hash function used in the protocol.

A function or algorithm is regarded random if its output depends not only on the input but also on some random ingredients, namely if its output is not uniquely determined by the input. Hence, to a function or algorithm, randomness contains indeterminacy.

4.5.2 Design of a Random Oracle

Correspondingly, the indeterminacy of the $\ell(i)$ may be expounded in terms of a random oracle.

Suppose that $\bar{O}_d(y, g)$ is an oracle on solving $y \equiv g^x$ (% M) for x, and $\bar{O}_\ell$ is an oracle on solving $C_i \equiv A_i W^{\ell(i)}$ (% M) for $\ell(i)$, where M is prime, and i is from 1 to n.

Let $\underline{D}$ be a database which stores records $(\{C_1, \ldots, C_n\}, M, \{\ell(1), \ldots, \ell(n)\})$ computed already. If the arrangement order of some C_i's is changed, $\{C_1, \ldots, C_n\}$ is regarded as a distinct sequence.

The structure of $\bar{O}_\ell$ is as Algorithm 4.5.2:

INPUT: a public key $(\{C_1, \ldots, C_n\}, M)$.

S1: If find $(\{C_1, \ldots, C_n\}, M)$ in $\underline{D}$ then
  retrieve $\{\ell(1), \ldots, \ell(n)\}$, goto S6.

S2: Randomly produce a coprime sequence $A_1, \ldots, A_n$
  with each $A_i \le \text{Ƀ}$ and $\prod_{i=1}^{n} A_i < M$.

S3: Randomly pick a generator $W \in \mathbb{Z}_M^*$.

S4: Evaluate $\ell(i)$ by calling $\bar{O}_d(C_i A_i^{-1}, W)$ for $i = 1, \ldots, n$.

S5: Store $(\{C_1, \ldots, C_n\}, M, \{\ell(1), \ldots, \ell(n)\})$ to $\underline{D}$.

S6: Return $\{\ell(1), \quad, \ell(n)\}$, and end.

OUTPUT: a sequence $\{\ell(1), \ldots, \ell(n)\}$.

Of course, $\{A_i\}$ and W as side results may be outputted.

Obviously, for the same input $(\{C_1, \ldots, C_n\}, M)$, the output is the same, and for a different input, a related output is random and unpredictable.

Since $C_i A_i^{-1}$ is pairwise distinct, and W is a generator, the result $\{\ell(1), \ldots, \ell(n)\}$ will be pairwise distinct. Again according to Definition 2, every $\ell(i) \in [1, \bar{M}]$ may be beyond $\Omega_\pm$. Thus, $\{\ell(1), \ldots, \ell(n)\}$ is a lever function although it is not necessarily the original.

The $\bar{O}_\ell$ is perhaps strange to some people because they have never met any analogous oracle in classical cryptosystems.

Section 4.5 explains further why the continued fraction attack by (4), (4′), (4″), or (5) and the indeterministic intersection attack is ineffectual on $C_i \equiv A_i W^{\ell(i)}$ (% M).

5 Conclusion

Indeterminacy is ubiquitous. For example, for $x + y = z$, given $x = -122$ and $y = 611$, computing $z = 489$ is easy, and contrarily, given $z = 489$, seeking the original x and y is intractable since there exists indeterminacy in $x + y = z$. Indeterminacy in $C_i \equiv A_i W^{\ell(i)}$ (% M) is similar, and triggered by the lever function $\ell(.)$.

Inequation (4) is stricter than (4″) although both (4) and (4″) are only necessary but insufficient for $\ell(x_1) + \ell(x_2) = \ell(y_1)$. Property 4 and 8 show that attack by (4) is more effectual than attack by (4″) theoretically. However, Section 4.3 shows that when $\Omega_\pm = \{+/-5, +/-6, \ldots, +/-(n+4)\}$ is indeterminate, the continued fraction attack by (4), (4′), (4″), or (5) will take $O(2^n)$ running time, and is practically infeasible.

Section 4.4.2 manifests that the indeterministic intersection attack cannot extract a private key in determinate polynomial time although it unveils some lowly probabilistic risk.

Therefore, the lever function $\ell(.)$ from $\{1, 2, \ldots, n\}$ to $\{+/-5, +/-6, \ldots, +/-(n+4)\}$ is necessary and sufficient for resisting the continued fraction attack and the indeterministic intersection attack.

Resorting to $C_i \equiv A_i W^{\ell(i)}$ (% M), we expound theoretically the effect of the lever function with indeterminacy. In practice, to strictly assure the security of a private key and to decrease the length of modulus of the cryptosystem, the key transform should be strengthened to $C_i \equiv (A_i W^{\ell(i)})^{\delta}$ (% M) with $\delta \in [2, \bar{M}]$, $A_i \in \Lambda = \{2, 3, \dots, \bar{P}\}$, and $\ell(i) \in \Omega_{\pm} = \{+/-5, +/-6, \dots, +/-(n + 4)\}$ for $i = 1, \dots, n$ [6][17].

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