UNIFIED THEORETICAL FRAMEWORK FOR UNIT ROOT AND FRACTIONAL UNIT ROOT

Ahmed BENSALMA

ahmed_bensalma@yahoo.fr

Ecole Nationale Supérieure de la Statistique et de l'Economie Appliquée

High National School of Statistic and Applied Economic

11, Chemin Doudou Mokhtar Ben-Aknoun, Algiers, Algeria

Abstract

To distinguish between purely fractionally integrated (FI) processes, we propose in this article a new and appropriate fractional Dickey-Fuller ($F-DF$) test. This new test extends the familiar Dickey-Fuller (1979) type tests for unit root ($I(1)$ against $I(0)$) by embedding the case $d=0$ and $d=1$ in continuum of memory properties. The new F-DF test is easy to implement and is based on two time domains properties of $FI(d)$ processes. First, if a time series follows $FI(d)$ process, than the $(-1+d)th$ differenced series follows an $I(1)$ process. Second, the purely $FI(d)$ processes are non explosive for $d \in \mathbb{R}$. These two properties will allow us to draw a bridge between the process $I(0)$ and $I(1)$ by testing the general hypotheses test $H_0 : d \geq d_0$ against $H_0 : d < d_0$, with $d \in \left]-\frac{1}{2}, \frac{3}{2}\right[$ and $d_0 \in [0,1]$.

Keywords: First autoregression, fractional integration, explosive processes, unit root, fractional unit root.

INTRODUCTION

Recently, extensively worked area in time series analysis has concerned to the design of appropriate test statistics to distinguish between $I(0)$, $I(d)$ $d \in]0,1[$ and $I(1)$ behavior. Some recent contributions on this topic include Dolado, Gonzalo and Mayoral (2002), Nilsen and Johansen (2010), Lobato and Velasco (2007). The fractional unit root distributions was first considered by Sowell (1990), who analyzed the behavior of the usual Dickey-Fuller type regression, when the errors are fractional. Specifically, Sowell (1990) considered the regression

$$y_t = y_{t-1} + \varepsilon_t, \text{ for } t = 1, 2, \cdots, n,$$

where $y_0 = 0$ and ε_t is a stationary fractionally integrated process that is, $I(\delta)$ process with $-0.5 < \delta < 0.5$. He showed that the ordinary least squares (OLS) estimator for ϕ (coefficient on y_{t-1}, hereafter $\widehat{\phi}_n$) has a nonzero density over the whole real line for the special case of $\delta = 0$, i.e., a unit root. For other values of δ, $\widehat{\phi}_n$ has different normalized constant depending on the magnitude of δ, i.e. $n^{\min[1,1+2\delta]}(\widehat{\phi}_n - 1) = O_p(1)$ and his asymptotic is function of two distributions which both depend on fractional Brownian motion. Furthermore, the standard t statistic of $\widehat{\phi}_n$ (hereafter $t_{\widehat{\phi}_n}$) converges to the well defined density when $\delta = 0$, for other values of δ, the asymptotic distribution of $t_{\widehat{\phi}_n}$ diverges to infinity. Sowell (1990) concluded that if his distributional theory is used to test the presence of unit roots in fractional ARMA models the implementation would requires tabulations of the percentiles of fractional Brownian motion conditionally on δ, for the statistic $n^{\min[1,1+2\delta]}(\widehat{\phi}_n - 1)$, and thus might suffer from misspecification. He concluded, also,he statistic $t_{\widehat{\phi}_n}$ are not useful.

Diebold and Rudebush (1991) examined the properties of Dickey-Fuller test under fractionally integrated alternatives and showed by Monte Carlo simulations that this test has quite low power and can lead to the incorrect conclusion that a time series has a unit root also when this is not true. They pointed out that a more appropriate testing procedure is needed to draw conclusions about the presence of the unit root. The lack of power of Dickey-Fuller test to distinguish between the $I(1)$ null hypothesis and the fractional integrated of order d ($FI(d)$) alternative, has motivated the development of new testing procedures that take this type of alternative explicitly into consideration. One approach for testing against fractional alternatives belongs to the Lagrange-multiplier (LM) framework studied in Robinson $(1991, 1994)$, Agiakloglou and Newbold (1994), Tanaka (1999), Breitung and Hassler (2002) and Nielsen (2004). The aim of this approach is adapting time domain procedures and embedding the models of interest in general $FI(d)$ framework, instead of the autoregressive alternatives typically considered in the literature. Robinson (1994) has developed a test for unit roots, that unlike the familiar Dickey-Fuller test, was not embedded in AR structures of form:

$$(1-\phi L)x_t = u_t,\ t = 1, 2, ..., \tag{1.1}$$

where L is the lag operator $(i.e.\ Lx_t = x_{t-1})$ and u_t is stationary $I(0)$ process, and with, the unit root null corresponds to:

$$H_0 : \phi = 1,$$

but based on fractional alternatives of form:

$$\Delta^d x_t = u_t,\ t = 1, 2, ..., \tag{1.2}$$

where d can be any real number, $\Delta^d = (1-L)^d$ is the fractional difference operator defined by its Maclaurin series (by its binomial expansion if d is an integer):

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)},$$

where

$$\Gamma(z) = \begin{cases} \int_0^{+\infty} s^{z-1}e^{-z}ds, & \text{if } z > 0, \\ \infty & \text{if } z = 0, \end{cases}$$

if $z < 0$, $\Gamma(z)$ is defined in terms of the above expressions and the recurrence formula $z\Gamma(z) = \Gamma(z+1)$ and u_t is stationary $I(0)$ process, and where the unit root null corresponds to:

$$H_0 : d = 1.$$

Robinson (1994) has also proposed a Lagrange Multiplier (LM) test where the null hypothesis is:

$$H_0 : d = d_0$$

in model given by (1.2) for any given real value d_0.

The major consequence of this change, in the set of alternative models considered, is that the the asymptotic distributions are standard. Nonetheless, the advantage of having a standard limit distribution, Tanaka (1999), showed with simulation experiments that the LM tests have the serious size distortion (see Tanaka 1999 fore more detail). Another critic addressed for the LM tests is that, by working under the null hypothesis, it does yield any direct information about the correct long-memory parameter d, when the null is rejected (Dolado, Gonzalo and Mayoral 2002, Bertrand candelon, Gil Alana 2003). In order to overcome the drawback, researches are directed towards a global procedure test, which embedded the AR structure form (1.1) and the fractional form (1.2) in the same model. The aim of this approach is to extends the well known Dickey and Fuller approach, originally designed for the $H_0 : d = 1$ against $H_0 : d = 0$, to the more general setup $H_0 : d = d_0$ against $H_0 : d < d_0$, with $0 \leq d_0 \leq 1$.

The fractional Dickey-Fuller (FD-F) unit root test was first considered by Dolado, Gonzalo and Mayoral (2002), (hereafter DGM). The (DGM) test for the null hypothesis $d = d_0$ against a simple alternative $d = d_1$, with $d_1 < d_0$, is based on the OLS estimation of the following model

$$\Delta^{d_0} x_t = \rho \Delta^{d_1} x_{t-1} + u_t, \tag{1.3}$$

where u_t is $I(0)$ stationary process. When $d_0 = 1$, the model (1.3) becomes

$$\Delta x_t = \rho \Delta^{d_1} x_{t-1} + u_t,$$

where $d_1 < 1$. When d_1 is not taken to be known a priory, a pre-estimation of it is needed to implement the test. Lobato and Velasco (2006) show that the DGM test is inefficient and the regression model (1.3) is misspecified because it does not include the data generating process defined by (1.1) and (1.2) like a particular case. The regression equation, proposed by DGM, is used only to suggest a test statistic, and we can not considered it as a generalization of familiar Dickey-Fuller test for unit root.

To design the appropriate test statistics to distinguish between $I(d)$ process, we propose in this article more flexible and more appropriate regression model for fractional unit root test. By using the non explosive feature of *ARFIMA(0,d,0)* processes, our testing procedure generalize the familiar unit root test in the most adequate way. Indeed, there is a significant difference between the autoregressive (AR) models of form (1.1) and the fractional alternatives of form (1.2). As noted by Gil-Alana and Robinson (1997) and Gil-Alana (2004), "fractional departures from (1.1) and (1.2) have very different long run implication. In (1.1) for $|\phi| > 1$, x_t is explosive, for $|\phi| < 1$ x_t is covariance stationary, and for $\phi = 1$ it is nonstationary but not explosive. In (1.2), x_t is nonstationary but non explosive for all $d \geq 0.5$. As d decreases beyond 0.5 and through 1, x_t can be viewed as becoming "more nonstationary" (in the sense, for example, that the variance of partial sums increases in magnitude) but it does so gradually, unlike in case of (1.1) around $\phi = 1$."

The non explosive feature of $ARFIMA(0, d, 0)$, often quoted in the literature, has neither been studied theoretically nor been used in practice, for the needs in statistical inference on the fractional unit root test. Intuitively, the non explosive feature of the process $ARFIMA(0, d, 0)$ for all $d \geq 0.5$ means that if a first order autoregression model is fitted to a sample of size n generated according (1.2) then the OLS estimator of the first order autoregression parameter will not exceed 1. Figure 1 below illustrates this fact in an obvious way.

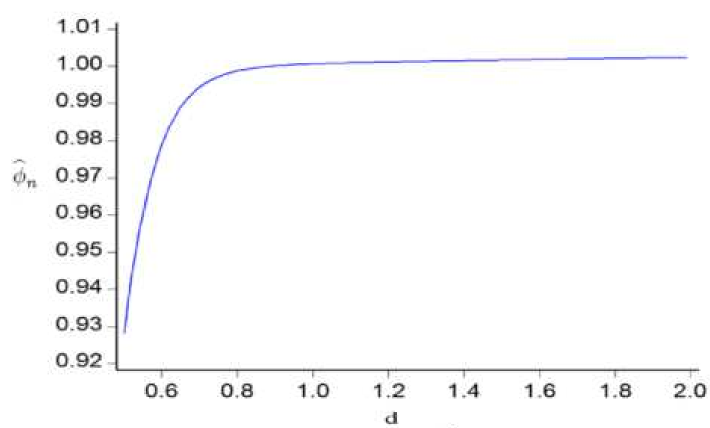

Fig.1. Relation between $\widehat{\phi}_n$ and d

This figure was made as follows: For a fixed sample $\{u_{1-n}, \cdots, u_0, \cdots, u_n\}$ generated from

$i.i.d.N(0,1)$, with $n = 1000$, samples of $ARFIMA(0,d,0)$ processes were generated for d varying between 0.5 and 2, with step of 0.01. For each sample $\{x_t, t = 1, \cdots, n\}$ a first order autoregression model

$$x_t = \widehat{\phi}_n x_{t-1} + \widehat{\epsilon}_t \tag{1.5}$$

is fitted and estimate of ϕ (hereafter $\widehat{\phi}_n$) are calculated. By plotting the parameter $\widehat{\phi}_n$ against the fractional differencing parameter d, one obtains the figure 1. This figure indicates that if the value of the parameter d is lower than 1, then the value of the parameter $\widehat{\phi}_n$ is lower than 1. While, when the parameter d is greater or equal to one, the value of the parameter $\widehat{\phi}_n$ converges to 1. Based on the relation between $\widehat{\phi}_n$ and d highlighted by figure 1, the non explosive feature of the process $ARFIMA(0,d,0)$ will allow us to build a unified theoretical framework for the unit root test and fractional unit root test. More precisely, to distinguish between $FI(d)$, $d \in \left]-\frac{1}{2}, \frac{3}{2}\right[$ processes, we present a new and appropriate Fractional Dickey-Fuller test in time domain that extends the familiar Dickey-Fuller (1979) type tests for unit root ($I(1)$ against $I(0)$), by embedding the $d = 0$ and $d = 1$ in continuum of memory properties. The proposed statistical test is based on the following idea: to test if a given process y_t is integrated of order d ($y_t \rightsquigarrow I(d)$), it is enough to test if the process $x_t = (1-L)^{-1+d} y_t$ is integrated of order 1. This simple idea associated with the non explosive feature of the $ARFIMA(0,d,0)$ processes will allow us to draw up a bridge between the process $I(0)$ and $I(1)$, by using a new fractional autoregression model (see section 3), which includes process (1.1) and (1.2) like particular case and which makes it possible to test the following general testing problem

$$H_0 : d \geq d_0 \text{ against } H_1 : d < d_0, \tag{1.6}$$

with $d \in \left]-\frac{1}{2}, \frac{3}{2}\right[$ and $d_0 \in [0,1]$.

The rest of the paper is organized as follows. The next section contains an introduction to some standard results and concepts of both fractional integrated series and the fractional space $D\,[0,1]$, that we need for this study. In section 3, to explain the relation between $\widehat{\phi}_n$ and d, highlighted by figure 1, we will study the behavior of the asymptotic distribution of $\widehat{\phi}_n$ when $d \in \left[\frac{1}{2}, \infty\right[$. In the same section, we discuss the consequences of the nonexplosiveness of $ARFIMA(0,d,0)$ process to deal with the unit root test and fractional unit root test. In section 4, by using a new auxiliary fractional autoregression model, we provide the main results on asymptotic null and alternative distributions for the testing problem (1.6). In section 5, through Monte-Carlo study, we show that the proposed test fare very well both in terms of power and size when we use the $t-$statistic. In section 6, we give some concluding remarks.

I. Preliminaries

In this section, the expression $D\left[0,1\right]$ denotes the space of functions on $\left[0,1\right]$ in which all elements are right continuous and have left-hand limits, endowed with the Skorohod topology (see Billingsley, 1968, $p.111$). Weak convergence of probability measures on $D\left[0,1\right]$ and convergence in probability are denoted by $\Longrightarrow$ and $\overset{p}{\longrightarrow}$, respectively.

The main technical tool, that we need for this study is in Sowell (1990) and Liu (1998). To begin with, we briefly outline some results, which will be used extensively, concerning the limit behavior of sample moments of long memory processes. Let $y_t = (1-L)^{-\delta} u_t$, with $\delta \in \left]-0.5, 0.5\right]$ and $u_{t,}\ t = 0, \pm 1, \cdots$, are $i.i.d.$ random variables with $E\left(u_t\right) = 0$ and $E\left|u_t\right|^a < \infty$ for $a \geq \max\left\{4, -8\delta / \left(1+2\delta\right)\right\}$. Since

$$E\left(\sum_{t=1}^{n} y_t\right)^2 = \begin{cases} O_p(n^2 \log n), & \text{for } \delta = 0.5 \\ O_p(n^{1+2\delta}), & \text{for } -0.5 < \delta < 0.5 \end{cases} \tag{2.1}$$

from the invariance principle of Davydov (1970) it follows, as $n \to \infty$

$$\frac{1}{\kappa(\delta,n) n^{1/2+\delta}} \sum_{t=1}^{[nr]} y_t \Rightarrow \begin{cases} r\mathbf{B}\left(1\right), & \text{if } \delta = \frac{1}{2}, \\ \mathbf{B}_{\delta,0}(r), & \text{if } -0.5 < \delta < 0.5, \end{cases} \tag{2.2}$$

where

$$\kappa^2(\delta,n) = \begin{cases} \frac{2\sigma_u^2}{\pi} \log n, & \text{if } \delta = \frac{1}{2}, \\ \frac{\sigma_u^2 \Gamma(1-2\delta)}{(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)}, & \text{if } \delta \neq \frac{1}{2}, \end{cases} \tag{2.3}$$

and $\mathbf{B}_{\delta,0}(r)$ is the type I fractional Brownian motion on $D\left[0,1\right]$ defined as follows,

$$\mathbf{B}_{\delta,0}(r)=\begin{cases}\frac{1}{A(\delta)}\int_{-\infty}^{0}\left[(t-s)^{\delta}-(-s)^{\delta}\right]d\mathbf{B}(s)+\int_{0}^{r}(t-s)^{\delta}\,d\mathbf{B}(s) & \text{if } \delta\in\left]-\frac{1}{2},\frac{1}{2}\right[,\\ r\mathbf{B}(1) & \text{if } \delta=0.5\end{cases}$$

where $\mathbf{B}(s)$ is standard Brownian motion and

$$A(\delta)=\left(\frac{1}{2\delta+1}+\int_{0}^{\infty}\left[(1+s)^{\delta}-s^{\delta}\right]^{2}ds\right)^{1/2}.$$

For the time series $y_t=(1-L)^{-\delta-m}u_t$, with $m\geq 1$ and $\delta\in\left]-\frac{1}{2},\frac{1}{2}\right]$ by combining the continuous mapping theorem, the results from Sowell (1990) and Theorem 2.2 of Liu (1998) we have the following useful results

$$\frac{1}{\kappa(\delta,n)n^{-1/2+\delta+m}}y_{[nr]}\Rightarrow\mathbf{B}_{\delta,m}(r), \tag{2.5}$$

$$\frac{1}{\kappa(\delta,n)n^{1/2+\delta+m}}\sum_{j=1}^{[nr]}y_j\Rightarrow\int_0^r\int_0^{r_m}\cdots\int_0^{r_2}\mathbf{B}_{\delta,0}(r_1)dr_2dr_3\cdots dr_m, \tag{2.6}$$

$$\frac{1}{\kappa^2(\delta,n)n^{2(m+\delta)}}\sum_{j=1}^{[nr]}y_j^2\Rightarrow\int_0^r\left[\mathbf{B}_{\delta,m}(s)\right]^2ds, \tag{2.7}$$

where $\kappa^2(\delta,n)$ is defined by (2.3) and

$$\mathbf{B}_{\delta,m}(r)=\begin{cases}\mathbf{B}_{\delta,0}(r) & \text{if } m=1\\ \int_0^r\int_0^{r_{m-1}}\cdots\int_0^{r_2}\mathbf{B}_{\delta,0}(r_1)dr_1dr_2dr_3\cdots dr_{m-1} & \text{if } m\geq 2.\end{cases} \tag{2.8}$$

II. Consequences of nonexplosiveness of the $ARFIMA(0,d,0)$ processes

Consider a purely fractionally integrated process $\{y_t\}$ defined by

$$(1-L)^{m+\delta}y_t=u_t,\ \text{for } t=1,2,\cdots n, \tag{3.1}$$

with $y_0=0$, and where $m\geq 0$ and $\delta\in\left]-\frac{1}{2},\frac{1}{2}\right]$, L is a lag operator and u_t is defined as in the section 2. We denote $m+\delta=d$.

Theorem 1. *Let* $\{y_t\}$ *satisfy* (3.1). *If the first autoregression model* (1.4) *is fitted to a sample of size* n *then*

$$(\log n)\left(\widehat{\phi}_n-1\right)\Rightarrow\frac{-2}{\int_0^1\left[\mathbf{B}_{\frac{1}{2},0}(r)\right]^2dr}, \tag{3.2}$$

when $\delta = 0.5$ *and* $m = 0$.

$$n^{1+2\delta}\left(\widehat{\phi}_n - 1\right) \Rightarrow \frac{-\left[\frac{1}{2}+\delta\right]\frac{\Gamma(1+\delta)}{\Gamma(1-\delta)}}{\int_0^1 \mathbf{B}_{\delta,0}^2(r)dr}, \tag{3.3}$$

when $-0.5 < \delta < 0$ *and* $m = 1$.

$$n\left(\widehat{\phi}_n - 1\right) \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{B}^2(1) - 1\right\}}{\int_0^1 \mathbf{B}^2(r)dr}, \tag{3.4}$$

if $\delta = 0$ *and* $m = 1$.

$$n\left(\widehat{\phi}_n - 1\right) \Rightarrow \frac{\frac{1}{2}\mathbf{B}_{\delta,m}^2(1)}{\int_0^1 \mathbf{B}_{\delta,m}^2(r)\,dr)}, \tag{3.5}$$

if $(m = 1$ and $0 < \delta \leq 0.5)$ *or* $(-0.5 < \delta \leq 0.5$ and $m \geq 2)$. Here $B_{\delta,m}(r)$ *is defined by* (2.8).

Proof. See Appendix

Theorem 1 indicates that the least squares estimate is super consistent, for $d \geq 1$. The rate of convergence depends on the order of integration. It is well known that if x_t is $I(0)$, the OLS estimate converges at the rate $n^{1/2}$, and if x_t is $I(1)$ i.e. $\delta = 0$ and $m = 1$, convergence is at the rate n. This may suggest that the rate of convergence increases with the order of integration. However, this is not the case. If a series is $I(m+\delta)$ for $m \geq 1$ and $0 \leq \delta < 1/2$, $\widehat{\phi}_n$ converges at the rate n, the same for all m and δ. If a series is $I(1+\delta)$ for $-1/2 < \delta \leq 0$, $\widehat{\phi}_n$ converges at the rate $n^{1+2\delta}$. So for $-1/2 < \delta < -1/4$ the rate of convergence is slower than $n^{1/2}$. The figure 2 illustrates this fact clearly. Furtheremore, when $m = 1$, the rate at which $n^{\min[1,1+2\delta]}\left(\widehat{\phi}_n - 1\right)$ converges to its limiting distribution is slow for nonpositive values of δ. This implies that $\widehat{\phi}_n$ converges very slowly towards 1 for $d < 1$. Moreover, for $d < 1$ the limiting distribution of $\left(\widehat{\phi}_n - 1\right)$ has nonpositive support and then

$$\lim_{n\to\infty} P\left(\widehat{\phi}_n - 1 < 0\right) = 1.$$

When $d \geq 1$, $n^{\min[1,1+2\delta]}\left(\widehat{\phi}_n - 1\right)$ converges to its limiting distribution at a faster rate than they do in standard first autoregressions with stationary variables. This implies that $\widehat{\phi}_n$ has high speed convergence towards 1 for $d \geq 1$. Moreover, for $d > 1$ the limiting distribution of $\left(\widehat{\phi}_n - 1\right)$ has nonnegative support and then

$$\lim_{n\to\infty} P\left(\widehat{\phi}_n - 1 < 0\right) = 0.$$

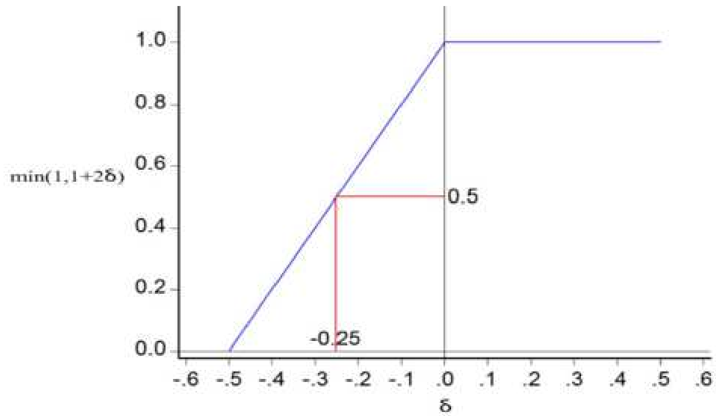

Fig.2. Convergence speed $n^{\min(1,1+2\delta)}$

The relation between the first autoregressive parameter $\widehat{\phi}_n$ and the fractional parameter d, highlighted by the results of Theorem 1 and illustrated by the figure 1, in the context of $ARFIMA(0, d, 0)$ process, suggests that when we deal with unit root test, we have $\phi = 1$ or $\phi < 1$, according to wether $d \geq 1$ or $d < 1$. In other words, the testing problem

$$H_0 : \phi = 1 \text{ against } H_1 : \phi < 1 \tag{3.6}$$

is equivalent to the testing problem

$$H_0 : d \geq 1 \text{ against } H_1 : d < 1 \tag{3.7}$$

It is obvious that the classical testing hypothesis of the Dickey-Fuller test is (3.6). If one would test the hypotheses (3.6), then in the familiar Dickey-Fuller test we have

$$H_0 : \phi = 1 \text{ implies } d = 1,$$

$$H_1 : \phi < 1 \text{ implies } d = 0,$$

whereas in the context of $ARFIMA(0, d, 0)$ processes we have

$$H_0 : \phi = 1 \text{ implies } d \geq 1,$$

$$H_1 : \phi < 1 \text{ implies } d < 1.$$

Now, we can expose our simple idea associated with the non explosive feature of the $ARFIMA(0,d,0)$ process which will allow us to draw up a bridge between the process $I(0)$ and $I(1)$, by using a new fractional autoregression model. The new fractional model includes process (1.1) and (1.2) like a particular case. For the testing problem (1.6), we propose to test the null hypothesis by means of the t-statistic of the coefficient of $(1-L)^{-1+d_0}y_{t-1}$ in the ordinary least squares (OLS) autoregression

$$\Delta^{-1+d_0}y_t = \phi\Delta^{-1+d_0}y_{t-1} + \epsilon_t,\ t = 1,2,\cdots,n, \tag{3.8}$$

or equivalently

$$\Delta^{d_0}y_t = \rho\Delta^{-1+d_0}y_{t-1} + \epsilon_t,\ t = 1,2,\cdots,n, \tag{3.9}$$

where $\rho = \phi - 1$ and ϵ_t the residuals. If $d = d_0$ and $\epsilon_t = u_t$ (recall that d is the true value of integration parameter and d_0 is the value specified under the null), in the fractional autoregressive model (3.8) and (3.9), according to wether

$$d_0 = 0 \text{ and } \phi = 1 \text{ i.e. } (\rho = 0)$$

$$d_0 \in \,]0,1[\, \text{ and } \phi = 1 \text{ i.e. } (\rho = 0)$$

$$d_0 = 1 \text{ and } \phi = 1 \text{ i.e. } (\rho = 0)$$

we have respectively

$$y_t = u_t \text{ i.e. } y_t \rightsquigarrow I(0)$$

$$(1-L)^{d_0}y_t = u_t,\ d_0 \in \,]0,1[\, \text{ i.e. } y_t \rightsquigarrow FI(d_0)$$

$$(1-L)y_t = u_t \text{ i.e. } y_t \rightsquigarrow I(1)$$

In order to grasp the intuition behind the fractional autoregressive model (3.9), suppose that $y_t \rightsquigarrow FI(d_0)$ and let us consider the relation between $\Delta^{d_0}y_t$ and $\Delta^{-1+d_0}y_{t-1}$. Note that, it is easy to check, that $\Delta^{d_0}y_t = (1-L)\left[\Delta^{-1+d_0}y_t\right]$ and $\Delta^{-1+d_0}y_t \rightsquigarrow I(1)$. Putting $\Delta^{-1+d_0}y_t = x_t$ we can rewrite (3.9) as follows

$$(1-L)x_t = \rho x_{t-1} + \epsilon_t,\ t = 1,2,\cdots,n. \tag{3.10}$$

The regression model (3.10) is the simple Dickey-Fuller framework to deal with the testing problem (3.6). Furthermore, suppose that $y_t \rightsquigarrow FI(d)$ then $x_t \rightsquigarrow FI(1+d-d_0)$ and we have the following (because of nonexplosiveness of $ARFIMA(0,d,0)$ processes)

$$\phi = 1 \text{ i.e. } \rho = 0,$$

$$\phi < 1 \text{ i.e. } \rho < 0,$$

according to wether

$$1 + d - d_0 \geq 1 \text{ i.e. } d \geq d_0,$$

$$1 + d - d_0 < 1 \text{ i.e. } d < d_0.$$

In other words, by using the regression model (3.9), the testing problem (1.6) is equivalent to the testing problem (3.6).

The simple fractional autoregression model (3.9) can be easily implemented for practical settings and is flexible enough to account for broad family of long memory specification of the fractional parameter d. The OLS estimator of ρ (hereafter $\widehat{\rho}_n$ and its t-ratio (hereafter $t_{\widehat{\rho}_n}$) for the regression model (3.9), are given by the usual squares expressions

$$\widehat{\rho}_n = \frac{\sum_{t=1}^{n}\left[\Delta^{d_0} y_t\right]\left[\Delta^{-1+d_0} y_{t-1}\right]}{\sum_{t=1}^{n}\left[\Delta^{-1+d_0} y_{t-1}\right]^2},$$

$$t_{\widehat{\rho}_n} = \frac{\sum_{t=1}^{n}\left[\Delta^{d_0} y_t\right]\left[\Delta^{-1+d_0} y_{t-1}\right]}{\left\{s_n^2 \sum_{t=1}^{n}\left[\Delta^{-1+d_0} y_{t-1}\right]^2\right\}^{1/2}},$$

where the variance of the residuals, s_n^2 is given by

$$s_n^2 = \frac{\sum_{t=1}^{n}\left[\Delta^{d_0} y_t - \widehat{\rho}\Delta^{-1+d_0} y_{t-1}\right]}{n}.$$

We call the testing procedure based both on the regression model (3.9) and the hypothesis test (1.6) the ”Fractional Dickey and Fuller” ($F-DF$) test.

III. Asymptotic null and alternative distribution and consistency of F-DF test

A. Asymptotic null and alternative distribution

The asymptotic distribution of the statistic’s $\widehat{\rho}_n$ (appropriately standardized) and $t_{\widehat{\rho}_n}$, under the hypothesis $H_0 : d \geq d_0$ and $H_1 : d < d_0$, is given in the following theorems.

Theorem 2. *Let y_t satisfy (3.1). For $\delta = d - d_0 \in \left]-\frac{1}{2}, \frac{1}{2}\right[$, if the regression model (3.9) is fitted to a sample of size n then, as $n \to \infty$*

$$n^{1+2\delta}\widehat{\rho}_n \Rightarrow -\frac{\left[\frac{1}{2}+\delta\right]\frac{\Gamma(1+\delta)}{\Gamma(1-\delta)}}{\int_0^1 \mathbf{B}_\delta^2(r)dr}, \quad \text{if } -0.5 < \delta < 0, \tag{4.1}$$

$$n\widehat{\rho}_n \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{B}^2(1)-1\right\}}{\int_0^1 \mathbf{B}^2(r)dr}, \quad \text{if } \delta = 0, \tag{4.2}$$

$$n\widehat{\rho}_n \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{B}_\delta^2(1)\right\}}{\int_0^1 \mathbf{B}_\delta^2(r)dr}, \quad \text{if } 0.5 \geq \delta > 0, \tag{4.3}$$

where $B_\delta(r)$ and $B(r)$ are respectively the type I fractional Brownian Motion and the standard Brownian Motion.

Theorem 2 indicates that the rate of convergence depends on the difference between the unknown true value of the parameter d and the value specified under the null d_0 i.e. $\delta = d - d_0$. The rate of convergence increases with δ for $-\frac{1}{2} < \delta < 0$ until the rate n, and converges at the same rate n for all $\delta \geq 0$ (See figure 2 above). The asymptotic distributions of $\widehat{\rho}_n$ depends also on the $\delta = d - d_0$. If $\delta = 0$ i.e. $(d = d_0)$, then the asymptotic distribution of $\widehat{\rho}_n$ is reduced to that deduced by Dickey and Fuller (1979) for the particular case $H_0 : d = 1$. In the case $\delta \neq 0$, then the distribution of $\widehat{\rho}_n$ are reduced to those deduced by Sowell (1990) for particular case $H_0 : d = 1$ and $d - 1 = \delta$ with $\delta \neq 0$.

Theorem 3. *Let y_t satisfy (3.1). For $\delta = d - d_0 \in \left]-\frac{1}{2}, \frac{1}{2}\right[$, if the regression model (3.9) is fitted to a sample of size n then, as $n \to \infty$*

$$t_{\widehat{\rho}_n} \xrightarrow{p} -\infty, \quad \text{if } -0.5 \leq \delta < 0, \tag{4.5}$$

$$t_{\widehat{\rho}_n} \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{B}^2(1)-1\right\}}{\left[\int_0^1 \mathbf{B}^2(r)dr\right]^{1/2}}, \quad \text{if } \delta = 0, \tag{4.6}$$

$$t_{\widehat{\rho}_n} \xrightarrow{p} +\infty, \quad \text{if } 0.5 \geq \delta > 0, \tag{4.7}$$

where $B(r)$ is the standard Brownian Motion.

Theorem 3 indicates that the asymptotic distribution of $t_{\widehat{\rho}_n}$ depends also on $\delta = d - d_0$. If $\delta = 0$ the asymptotic distribution of $t_{\widehat{\rho}_n}$ is reduced to that deduced by Dickey and Fuller (1979) for the particular case $H_0 : d = 1$. In the case $\delta > 0$ and $\delta < 0$ the distribution of $t_{\widehat{\rho}_n}$ diverge respectively to $(+\infty)$ and $(-\infty)$. Similar results, for the particular case $H_0 : d = 1$ with $d - 1 = \delta \neq 0$ are derived by Sowell (1990).

Corollary. *Let* $\{y_t\}$ *be generated by* (3.1). For $\delta = d - d_0 \in \left]-\frac{1}{2}, \frac{1}{2}\right[$. *If regression model* (3.9) *is fitted to a sample of size* n *then, as* $n \to \infty$*, we have for the* t-statistic

$$n^{-\delta}\left(t_{\widehat{\rho}_n}\right) \Rightarrow C_\delta \frac{\frac{1}{2}\left[\mathbf{B}_\delta^2(r)\right]}{\left(\int_0^1 \mathbf{B}_\delta^2(r)dr\right)^{1/2}} \text{ if } \delta > 0,$$

and

$$n^{\delta}\left(t_{\widehat{\rho}_n}\right) \Rightarrow -C_\delta \; \frac{\left(\frac{1}{2}+\delta\right)\Gamma(1+\delta)}{\Gamma(1-\delta)\left(\int_0^1 \mathbf{B}_\delta^2(r)dr\right)^{1/2}} \text{ if } \delta < 0,$$

where $C_\delta = \left[\frac{\Gamma(1-2\delta)}{(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)}\right]^{1/2}$ and *where* $B_\delta(r)$ *and* $B(r)$ *are respectively the type I fractional Brownian Motion and the standard Brownian Motion.*

The limiting distribution of $n^{-|\delta|}\left(t_{\widehat{\rho}_n}\right)$ has nonnegative support if $\delta > 0$ and a nonpositive support if $\delta < 0$.

The standardized least squares estimate $\widehat{\rho}_n$ and the corresponding t-statistic, $t_{\widehat{\rho}_n}$ will be noted hereafter respectively by τ_ρ and τ_t. By using these notations, we explain now, how one can exploit the results of theorem (3) to statistical inference in time series. Our interest is to show that the F-DF tests, based on τ_t statistics are consistent whereas the F-DF tests based on τ_ρ statistics are not consistent.

B. Consistency of F-DF test based on τ_t statistic

Consider the problem of hypothesis test (1.6) in sample of size n. It is convenient to introduce the nonrandomized test defined by a function $\mathbf{\Phi}_n$ on the sample space of the observations τ_t, with critical region C. The $\mathbf{\Phi}_n$ test for a region C is its indicator function

$$\mathbf{\Phi}_n\left(\tau_t\right) = \begin{cases} 1 \text{ if } \tau_t \in C \\ 0 \text{ if } \tau_t \notin C \end{cases} \tag{4.8}$$

Let α and β respectively the type I error and the type II error of the test $\mathbf{\Phi}_n$. Since H_0 and H_1 are composite, we have

$$\alpha = Sup_{d \geq d_0} \pi_0(d)$$

$$\beta = Inf_{d < d_0} \pi_1(d)$$

where $\pi_0(d) = P_{H_0}(\text{reject } H_0)$ and $\pi_1(d) = P_{H_1}(\text{accept } H_0)$. Practically, this entails computing a statistic τ_t from a sample, whose distribution P_{H_0} when the null hypothesis H_0 is true can be tabulated, and used to fix the probability of rejection when H_0 is true (a Type I error) not exceed a chosen value α. Since, the alternative hypothesis is $H_1 : d < d_0$, it is natural to consider one sided critical regions of the form

$$C = \{\tau_t < c_n(\alpha)\}, \tag{4.9}$$

where α is the level of the test. With these settings, the power function of the test $\mathbf{\Phi}_n$, denoted by $\pi_{\mathbf{\Phi}_n}(d)$ is :

$$\pi_{\mathbf{\Phi}_n}(d) = 1 - Inf_{d < d_0} \pi_1(d).$$

Theorem 4. *A sequence of tests* $\{\mathbf{\Phi}_n\}$ *defined by* (4.8), *with critical region* (4.9) *each of given size* α *is consistent i.e.*

$$\lim_{n \to \infty} \pi_{\mathbf{\Phi}_n}(d) \to 1 \text{ for } d < d_0.$$

Proof. If $\delta < 0$, from theorem 3 and corollary 1 it follows

$$\begin{aligned}
\lim_{n \longrightarrow \infty} P_{H_1}(\tau_t < c_n(\alpha)) &= \lim_{n \longrightarrow \infty} P_{H_1}\left(n^{\delta}\tau_t < n^{\delta} c_n(\alpha)\right) \\
&= \lim_{n \longrightarrow \infty} P_{H_1}\left(-C_\delta \, \frac{\left(\frac{1}{2}+\delta\right)\Gamma(1+\delta)}{\Gamma(1-\delta)\left(\int_0^1 \mathbf{B}_\delta^2(r)dr\right)^{1/2}} < 0\right) \\
&= 1, \text{ because } n^{\delta}\tau_t \text{ has nonpositive support.}
\end{aligned}$$

As n becomes infinite, $\pi_{\mathbf{\Phi}_n}(d)$ tends to 1, if $\delta < 0$. Moreover, for each n, $\pi_{\mathbf{\Phi}_n}(d)$ is an increasing function, with larger absolute value on $\delta < 0$ than $\delta = 0$. Hence, for testing $d \geq d_0$ against $d < d_0$, the region $\tau_t < c_n(\alpha)$ is unbiased (for any n) and defines a consistent family of tests.

We have also

$$\lim_{n\longrightarrow\infty} P_{H_0}\left(\tau_t > c_n\left(\alpha\right)\right) = 1-\alpha, \text{ if } \delta = 0,$$

and

$$\lim_{n\longrightarrow\infty} P_{H_0}\left(\tau_t > c_n\left(\alpha\right)\right) = 1, \text{ if } \delta > 0.$$

For the case where $\delta = 0$, the exact critical points $c_n\left(\alpha\right)$ are given by Fuller (1976, pp. 373 and 375). These critical points are also asymptotically valid (with in the case where the u_t's are not normal). This implies that the proposed test can be understood and implemented exactly as the Dickey and Fuller test for unit root by using the usual table statistics.

In practice, the statistician or econometrician, would wish to know the answer to the following question : Is the series, under study, stationary or not ? To answer this question, we propose the following downward testing procedures where the highest integration level is tested first.

1) Reject $H_0 : d \geq 1$ (against $H_1 : d < 1$) and go to step 2 if $\tau_t < c_n\left(\alpha\right)$; otherwise, conclude that $H_0 : d \geq 1$ is true.
2) Reject $H_0 : d \geq 0.5$ (against $H_1 : d < 0.5$) if $\tau_t < c_n\left(\alpha\right)$; otherwise, conclude that $H_1 : d \geq 0.5$ is true and $0.5 \leq d < 1$.

IV. Power of F-DF test in finite sample

After the theoretical analysis of the F-DF test based on τ_t statistic, we now conduct a Monte-Carlo experiment to examine the finite sample performance of F-DF test based on τ_t and the F-DF test based on τ_ρ.

We consider the data generating processes,

$$(1-L)^d y_t = u_t. \tag{5.1}$$

The regression model will be used for estimation and inference to deal with the hypothesis test

$$H_0 : d \geq d_0 \text{ against } H_0 : d < d_0, \qquad (I)$$

is

$$(1-L)^{d_0} y_t = \rho_n (1-L)^{-1+d_0} y_{t-1} + \epsilon_t. \qquad (5.2)$$

Before giving the main results of these experiments, we examine initially, the two following simple hypothesis tests

$$H_0 : d = d_0 \text{ against } H_1 : d < d_0 \qquad (II)$$

and

$$H_0 : d = d_0 \text{ against } H_1 : d > d_0 \qquad (III)$$

In the experiments reported in this section, a general procedure for generating a stationary fractionally integrated series of length n is to apply, for $t = 1, \cdots, n$ and some fixed m, the formula

$$x_t = \sum_{j=0}^{m+t-1} \frac{\Gamma(d+j)}{\Gamma(d)\,\Gamma(j+1)} u_{t-j}, \qquad (5.3)$$

where $\{u_{1-m}, \cdots, u_n\}$ is a random sequence of $i.i.d.N(0,1)$. This simulation strategie for type I fractional Brownian processes has suggested by Davidson and Hashimzade (2008). They argued that, by choosing m large we should be able to approximate the type I processes to any desired degree of accuracy. For the non-stationary parameter confugarations $d = 0.6$, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3 and 1.4, we use the following:

$$x_t = x_{t-1} + \sum_{j=0}^{m+t-1} \frac{\Gamma(\delta+j)}{\Gamma(\delta)\,\Gamma(j+1)} u_{t-j}, \qquad (5.4)$$

where $\delta \in \left]-\frac{1}{2}, \frac{1}{2}\right[$. For the particular case $d = 0.5$ we can use (5.3) or (5.4) and the value of $m = 500$. All computations were done in Eviews 4.0.

A. The size of the hypothesis tests (II) and (III).

Trough Monte Carlo study we show that the F-DF test, based on the fractional first autoregression model (5.2), fare very well in terms of size when we use τ_ρ or τ_t statistics. To investigate the size of the hypothesis test (II) and (III), 10000 samples of $FI(d)$ Gaussian processes (5.1) are generated and the regression model (5.2) is used for estimate τ_ρ and τ_t. The samples sizes considered are $n = 50$; 100; 150; 200 and 250. Samples of n observations were generated for five values of $d : 0$; 0.2; 0.5; 0.8; 1 and for each value d, we specify under the null only one value d_0 equal to d.

For the case where u_t's are normal, the exact critical points for τ_ρ and τ_t, under the null are given by Fuller (1976, pp. 373 and 375: Table 8.5.1 for τ_ρ and Table 8.5.2 for τ_t). For the testing problem (II) we consider one-sided critical regions of the form $\tau_\rho < b_n(\alpha)$ and $\tau_t < c_n(\alpha)$. For the testing problem (III) we consider one sided critical regions of the form $\tau_\rho > b_n(1-\alpha)$ and $\tau_t > c_n(1-\alpha)$, where α is the level of the test.

Table 1 reports the acceptation frequencies at, respectively the $\alpha = 1\%$, 5%, 10% significance level, for the testing problem (II) when we use the dickey Fuller *t*-statistic τ_t. It is noted that the estimated frequencies are very close to the theoretical frequencies ($1-\alpha = 0.99$, 0.95, 0.90). Similar results, not reported here, are found for the test hypothesis (III) when we use the dickey Fuller *t*-statistic τ_t and for the tests hypothesis (II) and (III) when we use the Dickey-Fuller 'normalized biais' statistic τ_ρ.

Table 1.

	$T \setminus d$	0	0.2	0.5	0.8	1
	50	98.96	98.93	98.91	98.90	99.12
	100	98.94	98.91	98.96	98.93	99.12
$\alpha = 0.01$	150	99.05	98.98	98.96	98.81	99.04
	200	99.05	98.90	99.11	99.01	99.08
	250	99.01	98.83	99.18	99.11	98.89
	50	94.05	95.22	95.22	95.10	95.13
	100	94.48	95.05	94.97	94.95	95.17
$\alpha = 0.05$	150	95.09	95.09	94.79	94.59	94.91
	200	95.04	95.20	95.32	94.83	94.88
	250	94.68	94.90	95.44	94.93	94.87
	50	90.34	89.79	90.00	90.25	90.02
	100	89.73	99.76	90.21	89.97	90.24
$\alpha = 0.10$	150	90.17	90.34	89.86	89.72	89.95
	200	90.39	90.23	89.84	89.66	89.99
	250	89.49	90.04	90.49	90.11	90.15

Although these results are very significant we have also, used the samples of 10000 observations to estimate the densities (following Sowell (1990)) of τ_ρ and τ_t under $\delta = 0$ by the kernel estimator

$$\widehat{f}(x) = \frac{1}{10000h} \sum_{j=1}^{10000} \Psi \left(\frac{x - \widehat{\theta}_j}{h} \right),$$

where $\widehat{\theta}_j$ are the estimated values of τ_ρ under $(\delta = 0)$ or τ_t under $(\delta = 0)$ for the 10000 samples and $\Psi(y)$ is the density defined by $\left(\frac{15}{16}\right)\left((1-y^2)^2\right)$ for $-1 < y < 1$ and zero elsewhere. The value of h was chosen to minimize the integrated mean square error (see Tapia and Thompson (1978, pp 67)). The estimated densities are presented in Fig. 3. and Fig.4.

For each statistics τ_ρ or τ_t under $\delta = 0$ (i.e. $d = d_0$) and for a given size of sample n, the

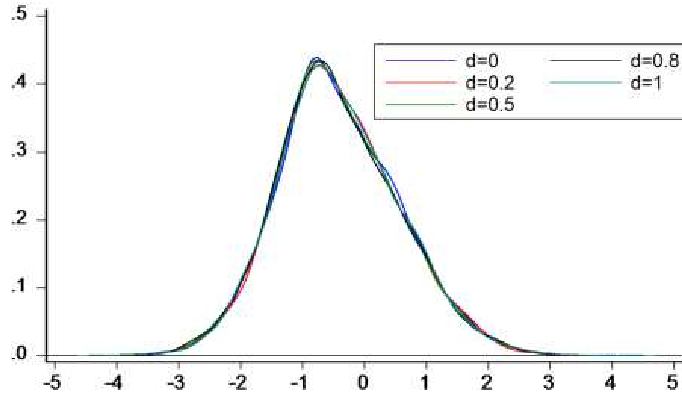

Fig.3. Kernel densities estimate of τ_t statistic under the null H_0: $d = d_0$, using 10000 samples of size $n = 250$

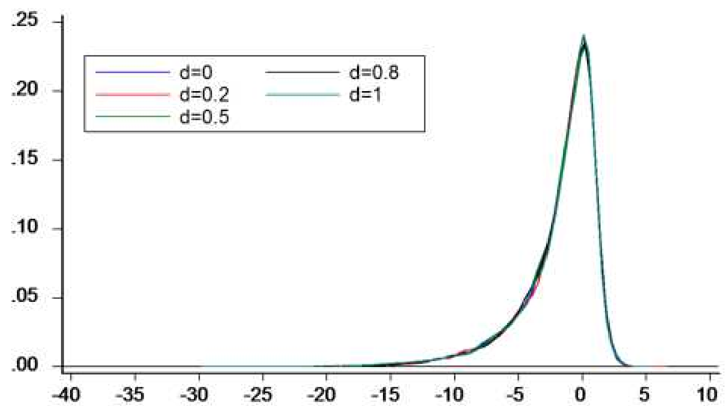

Fig.4. Kernel densities estimate of τ_ρ statistic under the null H_0: $d = d_0$, using 10000 samples of size $n = 250$

estimated densities for different values of d are represented on the same graph. The figures 3 and 4 shows that by fitting the regression model (5.2) to the sample generated according (5.1), one obtains the same distribution that those used by Dickey-Fuller (1979,1981).

B. The power of the hypothesis tests (II) *and* (III)

To study the power of fractional unit root test, we have generated 10000 samples of $FI(d)$ Gaussian process according (5.1) for $d = 1$. To evaluate the power of the test (II) we specify different values of d_0 under the null : 1, 1.1, 1.2, 1.3 and 1.4. The figure below represent the different densities of τ_ρ under $(\delta \leq 0)$ for the case where the true value of d is equal 1 and $\delta \in \{-0.4;\ -0.3;\ -0.2;\ -0.1;\ 0\}$. Let $b_n(\alpha)$ the critical point at the significance level α for τ_ρ

under $(\delta < 0)$, we have

$$P\left[\tau_\rho > b_n(\alpha) \mid \delta < 0\right] = \beta'.$$

If the test (II) has good performance in terms of power then the probability β' should decrease when δ decreases (i.e. d_0 increases) and moves away from zero. β' should be always less than β. The figure 5 below show the opposite i.e. β' increases when δ decreases, because, as noted by Sowell (1990), the density appears to be converging to unit mass at zero when δ approaches (-0.5). Consequently, we can not use the statistic τ_ρ for the hypotheses test (II).

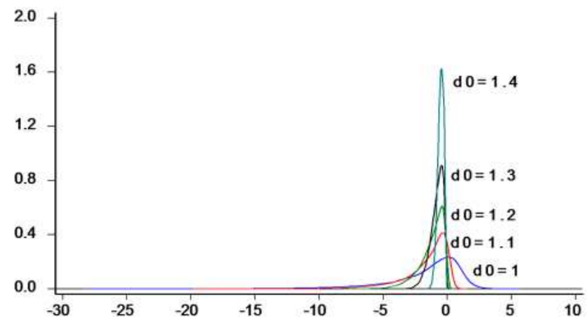

Fig.5. Kernel densities estimate of $n\left(\hat{\delta}_n - 1\right)$ statistic using 10000 samples of size $n = 250$, when the true value of $d = 1$ and the values of d_0 are respectively 1, 1.1, 1.2, 1.3 and 1.4.

Now considering the test (III) in the case where $d = d_0 + \delta$ is the true value of fractional integration of the process y_t with $\delta \in \{0; 0.1; 0.2; 0.3; 0.4\}$. The figure below represents the estimated densities of τ_ρ under $(\delta \geq 0)$ for a size of sample $n = 250$, if the true value of the fractional parameter of integration is equal to 1, whereas the values specified under the null are $d_0 \in \{1; 0.9; 0.8; 0.7; 0.6\}$. The two graphs show that the various densities τ_ρ under $(\delta \geq 0)$, with $\delta \in \{0; 0.1; 0.2; 0.3; 0.4\}$ are all to fit together. In other terms, when $H_1 : d > d_0$ is true, the distribution of τ_ρ under $(\delta \geq 0)$ does not make it possible to have a powerful test. Contrary to the preceding test, the reason of disappointing results of the third test is not in the distribution of τ_ρ, but in the parameter ρ. Indeed, the parameter ρ takes the value 0 if the null is true, i.e. $d = d_0$ and it takes also value 0 if the alternative is true i.e. $d > d_0$.

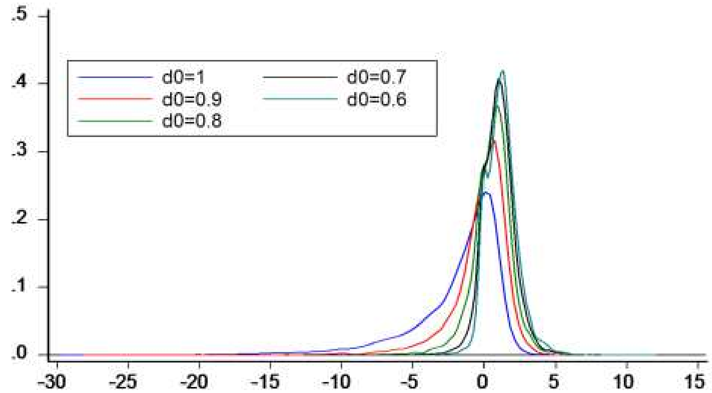

Fig.6. Kernel densities estimate of $n\left(\widehat{\phi}_n-1\right)$ statistic using 10000 samples of size $n=250$, when the true value of $d=1$ and the values of d_0 are respectively 1, 0.9, 0.8, 0.7 and 0.6.

C. Size and power of the F-DF test based on hypothesis test (I) and τ_t

In this subsection, in Monte Carlo study we show that the proposed hypotheses test (I) based on the DGP (5.1) and in the auxiliary regression model (5.2) fare very well both in terms of power and size when we use the τ_t statistic. To investigate the size and power of the hypotheses test (I), 10000 samples of $FI(d)$ Gaussian processes (5.1) are generated and the regression model (5.2) is used to estimate τ_t. The sample size considered is $n = 50$ and $n = 250$. We will use, as true values of the fractional parameter of integration of the process y_t, three values $d : 0$; 0.5; 1 and for each value, we specify various values for d_0. If the set of the values of d_0 for a given value of d is denotes by $S_d(d_0)$ then the sets which will be used for the three values of d are respectively

$$S_0(d_0) = \{-0.4;\ -0.3;\ -0.2;\ -0.1;\ \mathbf{0};\ 0.1;\ 0.2;\ 0.3;\ 0.4\},$$

$$S_{0.5}(d_0) = \{0;\ 0.1;\ 0.2;\ 0.3;\ 0.4;\ \mathbf{0.5};\ 0.6;\ 0.7;\ 0.8;\ 0.9\},$$

$$S_1(d_0) = \{0.5;\ 0.6;\ 0.7;\ 0.8;\ 0.9;\ \mathbf{1};\ 1.1;\ 1.2;\ 1.3;\ 1.4\}.$$

The tables 2 and 3 contains the simulations results on the size of the F-DF test for the hypotheses test (I). The tables 2 and 3 shows that the testing problem (I) has good performances in terms of size, since we have

$$P\left(\tau_t \geq c_n(\alpha) \mid \delta \geq 0\right) \geq 1 - \alpha.$$

The table 5 and Table 6 contains the simulations results on the power of the F-DF test for the hypotheses test (I). There are some conclusions to be drawn from it. First, the power of the F-DF test increases with the increase of sample size and $\delta = d - d_0$. For examples, for $\alpha = 5\%$, $d = 1$ and $\delta = -0.1$, power is 12.36% for $n = 50$, 20.76% for $n = 250$and for $\alpha = 5\%$, $d = 1$ and $\delta = -0.3$, power is 48.5% for $n = 50$, 86.05% for $n = 250$. Second, as shown in table 5 and 6, for $n = 250$, the power of F-DF test is below 50% for ($\delta = -0.1$) and for ($\alpha = 1\%$, $\delta = -0.2$). Third, for a given n, α and δ, the power for $d = 0$, $d = 0.5$ and $d = 1$ are approximately similar because the asymptotic under the alternative does not depend on d but depend on $\delta = d - d_0$.

Table 2. Size of the hypotheses test (I) when we use τ_t $(n = 50)$

	$\alpha \setminus \delta$	0.4	0.3	0.2	0.1	**0**
$d_0 = 0$	1%	100	99.99	99.93	99.73	**98.96**
	5%	99.98	99.85	99.39	98.17	**94.05**
	10%	99.85	99.48	98.26	95.78	**90.34**
$d_0 = 0.5$	1%	99.98	99.97	99.90	99.74	**98.91**
	5%	99.91	99.78	99.28	98.14	**95.22**
	10%	99.78	99.42	98.25	95.63	**90.00**
$d_0 = 1$	1%	100	99.99	99.97	99.79	**99.12**
	5%	99.95	99.92	99.55	98.34	**95.13**
	10%	99.83	99.54	98.72	96.03	**90.02**

Table 3. Size of the hypotheses test (I) when we use τ_t $(n = 250)$

	$\alpha \setminus \delta$	0.4	0.3	0.2	0.1	0
$d = 0$	1%	100	100	99.99	99.92	$\mathbf{99.01}$
	5%	100	99.99	99.95	99.29	$\mathbf{94.68}$
	10%	100	99.97	99.73	97.87	$\mathbf{89.49}$
$d = 0.5$	1%	100	99.98	99.8	99.91	$\mathbf{99.18}$
	5%	100	100	99.97	99.27	$\mathbf{95.44}$
	10%	100	100	100	97.66	$\mathbf{90.49}$
$d = 1$	1%	100	100	100	99.92	$\mathbf{98.89}$
	5%	100	100	99.96	99.32	$\mathbf{94.87}$
	10%	99.98	99.97	99.80	98.00	$\mathbf{90.15}$

Table 5. power of the hypotheses test (I) when we use τ_t $(n = 50)$

	$\alpha \setminus \delta$	-0.1	-0.2	-0.3	-0.4
$d = 0$	1%	3.61	9.53	22.23	45.82
	5%	12.66	26.68	47.16	73.03
	10%	21.97	39.56	62.14	84.92
$d = 0.5$	1%	2.87	9.36	22.59	45.16
	5%	11.9	25.17	48.34	72.74
	10%	21.17	39.58	63.23	84.39
$d = 1$	1%	3.2	9.39	22.91	46.23
	5%	12.36	26.1	48.50	73.47
	10%	21.63	39.24	63.69	84.99

Table 6. Power of the hypotheses test (I) when we use τ_t $(n = 250)$

	$\alpha \setminus \delta$	-0.1	-0.2	-0.3	-0.4
	1%	7.59	30.07	68.00	95.48
$d = 0$	5%	19.98	52.53	85.90	$99;16$
	10%	30.62	64.77	92.95	99.87
	1%	7.3	30.06	68.54	95.60
$d = 0.5$	5%	20.35	52.40	85.80	99.39
	10%	31.44	65.10	92.50	99.89
	1%	7.71	29.90	68.36	95.39
$d = 1$	5%	20.76	51.77	86.05	99.35
	10%	31.71	64.29	92.41	99.87

V. Discussion and concluding Remarks

In this paper, to distinguish between $FI(d)$ processes with $d \in \left]-\frac{1}{2}, \frac{3}{2}\right[$, we have proposed a new and appropriate testing procedure in time domain that extends the familiar Dickey-Fuller (1979) types tests for unit root ($I(1)$ against $I(0)$), by embedding the case $d = 0$ and $d = 1$ in continuum of memory properties. The main idea of our test procedure is the following: in order to test if the process y_t is fractionally integrated of order d_0, it suffices to test if the process $x_t = (1 - L)^{-1+d_0} y_t$ is integrated of order 1. We have referred to the test based on this original idea as the Fractional Dickey-Fuller (FD-F) test. The proposed test is based on the OLS estimator ($\widehat{\rho}_n$) or its t-ratio in the autoregression model

$$\Delta^{d_0} y_t = \rho \Delta^{-1+d_0} y_{t-1} + \epsilon_t, \ t = 1, 2, \cdots, n$$

With this regression model associated with the non explosive feature of $FI(d)$ processes, we have showed that the testing problem $H_0 : d \geq d_0$ against $H_1 : d < d_0$ with $d \in \left]-\frac{1}{2}, \frac{3}{2}\right[$ and $d_0 \in [0, 1]$, is equivalent to $H_0 : \rho = 0$ against $H_1 : \rho < 0$. We have also, showed that the asymptotic distributions for ordinary least squares (OLS) and its t-ratio under the null simple hypothesis $H_0 : d = d_0$ are identical to those derived by Dickey and Fuller (1979,1981) for the simple case (without drift and trend). This implies that the proposed test can be understood and implemented exactly as the Dickey-Fuller test for unit root by using the usual tables statistics. It worthnoting that the new F-DF test proposed in this paper is a generalization of the particular case $H_0 : d = 1$ against $H_1 : d < 1$ (or $d > 1$) studied by Sowell (1990) for $d \in \left]\frac{1}{2}, \frac{3}{2}\right[$, to the general case $H_0 : d \geq d_0$ against $H_1 : d < d_0$ with $d \in \left]-\frac{1}{2}, \frac{3}{2}\right[$ and $d_0 \in [0, 1]$. For the particular case, Sowell (1990) have concluded that the asymptotic distributions of $\widehat{\rho}_n$ and $t_{\widehat{\rho}_n}$ can not be used to test the presence of unit root in fractional ARMA models, since the implementation of the test require tabulations of the percentiles of fractional Brownian motion conditionally on $\delta = d - 1$ and thus suffer from misspecification. These disappointing conclusions originate from an ill defined statistical problem and from an inappropriate use of asymptotic distributional theory. In fact, Sowell have focused his attention on the parameter ρ by considering the test $H_0 : \rho = 0$ against $H_1 : \rho < 0$ and considered that under the null there are three asymptotic distributions conditionally on $\delta = d - 1$ $(\delta < 0, \delta = 0, \delta > 0)$. In this paper, by using the non explosiveness of $ARFIMA(0, d, 0)$ processes, we have showed, for the general case, that under the null $H_0 : \rho = 0$ there are only two possible asymptotic distributions conditionally on $\delta = d - d_0$ ($\delta = 0$, $\delta > 0$) and under the alternative there is only one asymptotic distribution ($\delta < 0$).

The theoretical framework above, is the unified framework for the unit root test and fractional unit root test. Furthermore, in order to test if a given process y_t is stationary we can perform the downward F-DF testing procedures where the highest integration level is tested first (in our case $d_0 = 1$). We can easily extend our results, by using the same framework, to the sequential testing procedure advocate by Dickey and Pantula, allowing the analysis to cover the case $d \in \left]-1.5,\ 2.5\right[$ and $d_0 \in [0, 2]$.

Further research is currently being undertaken towards generalizing the F-DF testing approach along similar directions as the D-F test has been extended in the unit root literature accounting for time series which may exhibit a trending behavior and for general ARFIMA case.

VI. Appendix

Proof of Theorem 1. If $\{x_t\}$ satisfy (2.1) and if the first autoregression model $x_t = \phi x_{t-1} + \omega_t$ or equivalently $\Delta x_t = (\phi - 1)\, x_{t-1} + \omega_t$ is fitted to a sample of size n then, the least squares slope estimate have the following expression:

$$\widehat{\phi}_n = \frac{\sum_{t=1}^{n} x_{t-1} x_t}{\sum_{t=1}^{n} x_{t-1}^2} = 1 + \frac{\sum_{t=1}^{n} x_{t-1} \Delta x_t}{\sum_{t=1}^{n} x_{t-1}^2}. \qquad (A1)$$

When $d = m + \delta = 0.5$ i.e. $m = 0$, $\delta = 0.5$ and $\Delta x_t \sim FI\,(-0.5)$ it follows from (2.7), that

$$\kappa^{-2}(\delta, n)\, n^{-1} \sum_{t=1}^{n} x_{t-1}^2 \Longrightarrow \int_0^1 \left[\mathbf{B}_{\frac{1}{2},0}(r)\right]^2 dr. \qquad (A.2)$$

When $m = 1$ for $-1/2 < \delta \leq 1/2$ and $\Delta x_t \sim FI\,(\delta)$, it follows from (2.7), that

$$\kappa^{-2}(\delta, n)\, n^{-2\delta-2} \sum_{t=1}^{n} x_{t-1}^2 \Longrightarrow \int_0^1 \mathbf{B}_{\delta,0}^2(r)\, dr, \qquad (A.3)$$

and when $(m > 1$ for $0 < \delta \leq 1/2)$ and $\Delta x_t \sim FI\,(m - 1 + \delta)$, it follows from (2.7), that

$$\kappa^{-2}(\delta)\, n^{-2\delta-2m} \sum_{t=1}^{n} x_{t-1}^2 \Longrightarrow \int_0^1 \mathbf{B}_{\delta,m}^2(r)\, dr. \qquad (A.4)$$

As regards the $\sum_{t=1}^{n} x_{t-1}\Delta x_t$ term, we can rewrite it as

$$\tfrac{1}{2}\sum_{t=1}^{n}\left\{x_t^2 - x_{t-1}^2 - (x_t - x_{t-1})^2\right\} = \tfrac{1}{2}\sum_{t=1}^{n} x_n^2 - \tfrac{1}{2}\sum_{t=1}^{n}(\Delta x_t)^2.$$

When $m = 0$ for $\delta = 0.5$ and $\Delta x_t \sim FI\,(-0.5)$, the first term when multiplied by $\kappa^{-2}(\delta, n) = \frac{2\sigma_u^2}{\pi}\log n$ converges in distribution to $\frac{1}{2}\left[\mathbf{B}(1)\right]^2$ because relation (2.5), whilst the limiting distribution of the second term follows from Lemma 2.1 of Ming Liu (1998) result 2, we have

$$n^{-1}{\sum}_{t=1}^{n}(\Delta x_t)^2 \xrightarrow{p} \frac{4\sigma_\varepsilon^2}{\pi}. \qquad (A.5)$$

When $m = 1$ for $-1/2 < \delta \leq 0$ and $\Delta x_t \sim FI\,(\delta)$, the first term when multiplied by $\kappa^{-2}(\delta, n)\, n^{-1-2\delta}$ converges in distribution to $\frac{1}{2}\left[\mathbf{B}_\delta(1)\right]^2$ because of the (2.5), whilst the limiting distribution of the second term follows by using the ergodic theorem

$$n^{-1}\sum_{t=1}^{n}\left(\Delta x_t\right)^2 \xrightarrow{p} \frac{\sigma_u^2\Gamma(1-2\delta)}{\Gamma^2(1-\delta)}. \tag{A.6}$$

Therefore, when $\delta = 0$, i.e.,when $d = 1$, $\kappa^2(\delta) = \sigma_u^2$, $\mathbf{B}_\delta(r) = \mathbf{B}(r)$ for $r \in [0,1]$ and

$$\kappa^{-2}(\delta,n)\, n^{-1}\sum_{t=1}^{n} x_{t-1}\Delta x_t \Longrightarrow \tfrac{1}{2}\left[\mathbf{B}^2(1)-1\right], \tag{A.7}$$

whereas when $d = m+\delta$ for $(m \geq 2$ and $-1/2 < \delta \leq 1/2)$ or $(m = 1$ and $0 < \delta \leq 1/2)$

$$\kappa^{-2}(\delta,n)\, n^{1-2m-2\delta}\sum_{t=1}^{n} x_{t-1}\Delta x_t \Longrightarrow \frac{1}{2}\mathbf{B}_{\delta,m}^2(1), \tag{A.8}$$

and when $d = 1+\delta$ for $-1/2 < \delta \leq 0$

$$\kappa^{-2}(\delta,n)\, n^{-1}\sum_{t=1}^{n} x_{t-1}\Delta x_t \Longrightarrow -\frac{\sigma_u^2\Gamma(1-2\delta)}{2\Gamma^2(1-\delta)}, \tag{A.9}$$

and when $d = 0.5$

$$\kappa^{-2}(\delta,n)\, n^{-1}\sum_{t=1}^{n} x_{t-1}\Delta x_t \Longrightarrow -2. \tag{A.10}$$

Hence, using $(A2)-(A10)$ and the continuous mapping theorem, we obtain that

$$\log n\left(\widehat{\phi}_n - 1\right) = \frac{\kappa^{-2}(\delta,n)\, n^{-1}\log n\sum_{t=1}^{n} x_{t-1}\Delta x_t}{\kappa^{-2}(\delta,n)\, n^{-1}\sum_{t=1}^{n} x_{t-1}^2} \Longrightarrow \frac{-2}{\int_0^1\left[\mathbf{B}_{\frac{1}{2},0}(r)\right]^2 dr}, \tag{A.11}$$

when $d = 0.5$.

$$n^{1+2\delta}\left(\widehat{\phi}_n - 1\right) = \frac{\kappa^{-2}(\delta,n)\, n^{-1}\sum_{t=1}^{n} x_{t-1}\Delta x_t}{\kappa^{-2}(\delta,n)\, n^{-2\delta-2}\sum_{t=1}^{n} x_{t-1}^2} \Longrightarrow \frac{-\left[\frac{1}{2}+\delta\right]\frac{\Gamma(1+\delta)}{\Gamma(1-\delta)}}{\int_0^1 \mathbf{B}_{\delta,0}^2(r)\, dr}, \tag{A.12}$$

when $d = 1+\delta$ for $-1/2 < \delta \leq 0$.

$$n\left(\widehat{\phi}_n - 1\right) = \frac{\kappa^{-2}(\delta,n)\, n^{-1}\sum_{t=1}^{n} x_{t-1}\Delta x_t}{\kappa^{-2}(\delta,n)\, n^{-2}\sum_{t=1}^{n} x_{t-1}^2} \Longrightarrow \frac{\frac{1}{2}\left[\mathbf{B}^2(1)-1\right]}{\int_0^1 \mathbf{B}^2(r)\, dr} \tag{A.13}$$

if $d = 1$, i.e., $m = 1$ and $\delta = 0$.

$$n\left(\widehat{\phi}_n - 1\right) = \frac{\kappa^{-2}(\delta,n)\, n^{1-2\delta-2m}\sum_{t=1}^{n} x_{t-1}\Delta x_t}{\kappa^{-2}(\delta,n)\, n^{-2\delta-2m}\sum_{t=1}^{n} x_{t-1}^2} \Longrightarrow \frac{\frac{1}{2}\mathbf{B}_{\delta,m}^2(1)}{\int_0^1 \mathbf{B}_{\delta,m}^2(r)\, dr)}, \tag{A.14}$$

if $d = m+\delta$ for $(m \geq 2$ and $-1/2 < \delta \leq 1/2)$ or $(m = 1$ and $0 < \delta \leq 1/2)$.

Proof of Theorem 2. The proof is omitted because of its similarity with Theorem 1. Indeed, if we denote $\Delta^{-1+d_0}y_t = x_t$ then we can rewrite the regression model (3.9) as $\Delta x_t = \rho x_{t-1} + \omega_t$ with $x_t \sim FI\,(1+\delta)$ and $-0.5 < \delta < 0.5$.

Proof of Theorem 3. If y_t satisfy (3.1) with $d \in \left]-\frac{1}{2}, \frac{3}{2}\right[$ and $d - d_0 = \delta$ with $-0.5 < \delta < 0.5$ and if the regression model (3.9) is fitted to sample of size n then, by denoting $\Delta^{-1+d_0}y_t = x_t \sim FI(1+\delta)$, the t student statistic have the following expression

$$t_{\widehat{\rho}_n} = \frac{\sum_{t=1}^{n} x_{t-1}\Delta x_t}{s_n\left\{\sum_{t=1}^{n} x_{t-1}^2\right\}^{1/2}}.$$

We can rewrite $t_{\widehat{\rho}_n}$ as follows

$$\begin{aligned} t_{\widehat{\rho}_n} &= \frac{\kappa\,(\delta)^{-1}\,n^{-1-\delta}\sum_{t=1}^{n} x_{t-1}\Delta x_t}{s_n\left\{\kappa\,(\delta)^{-2}\,n^{-2-2\delta}\sum_{t=1}^{n} x_{t-1}^2\right\}^{1/2}} \\ &= \frac{\frac{1}{2}\kappa\,(\delta)^{-1}\,n^{-1-\delta}\,(x_n)^2 - \frac{1}{2}\kappa\,(\delta)^{-1}\,n^{-1-\delta}\sum_{t=1}^{n}(\Delta x_t)^2}{s_n\left\{\kappa\,(\delta)^{-2}\,n^{-2-2\delta}\sum_{t=1}^{n} x_{t-1}^2\right\}^{1/2}}. \end{aligned}$$

First notice that

$$s_n = \frac{1}{n}\sum_{t=1}^{n}\left(\Delta x_t - \widehat{\rho}_n x_{t-1}\right)^2 = \frac{1}{n}\sum_{t=1}^{n}(\Delta x_t)^2 + \frac{1}{n}\widehat{\rho}_n^2\sum_{t=1}^{n} x_{t-1}^2 - \frac{2}{n}\widehat{\rho}_n\sum_{t=1}^{n} x_{t-1}\Delta x_t$$

Hence, by using $A3$, $A6$, $A7$, $A9$, Theorem 2 and the continuous mapping theorem, it follows that

$$s_n \xrightarrow{p} \frac{\Gamma(1-2\delta)}{\Gamma^2(1-\delta)}\sigma_u^2, \qquad (A.15)$$

for all $\delta \in \left]-0.5, 0.5\right[$.

Consider now, the numerator, N, of the $t_{\widehat{\rho}_n}$ statistic, which can rewrite as

$$N = \frac{1}{2}\kappa\,(\delta)\,n^{\delta}\left[\kappa\,(\delta)^{-2}\,n^{-1-2\delta}\,(x_n)^2\right] - \frac{1}{2}\kappa\,(\delta)^{-1}\,n^{-\delta}\left[n^{-1}\sum_{t=1}^{n}(\Delta x_t)^2\right].$$

By using (2.5) and (2.7) and $A.7$ we obtains

$$N \Longrightarrow \begin{cases} -\infty & \text{if } \delta < 0 \\ \frac{1}{2}\left\{(\mathbf{B}(1)^2 - 1)\right\} & \text{if } \delta = 0 \\ +\infty & \text{if } \delta > 0. \end{cases} \qquad (A.12)$$

Hence, using $A.3$, $A.11$ and $A.12$ and the continuous mapping theorem, we obtain that

$$t_{\widehat{\rho}_n} \xrightarrow{p} -\infty \quad \text{if } -0.5 \leq \delta < 0,$$

$$t_{\widehat{\rho}_n} \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{B}^2(1)-1\right\}}{\left[\int_0^1 \mathbf{B}^2(r)dr\right]^{1/2}} \quad \text{if } \delta = 0,$$

$$t_{\widehat{\rho}_n} \xrightarrow{p} +\infty \quad \text{if } 0.5 \geq \delta > 0 \ \cdot$$

Proof: **Proof of Corollary 1.** Direct consequence of Theorem 3. ■

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