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K Leschke

F Pedit

University of Massachusetts - Amherst, pedit@math.umass.edu

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SEQUENCES OF WILLMORE SURFACES

K. LESCHKE AND F. PEDIT

ABSTRACT. In this paper we develop the theory of Willmore sequences for Willmore surfaces in the 4-sphere. We show that under appropriate conditions this sequence has to terminate. In this case the Willmore surface either is the twistor projection of a holomorphic curve into $\mathbb{C}\mathbb{P}^3$ or the inversion of a minimal surface with planar ends in \mathbb{R}^4 . These results give a unified explanation of previous work on the characterization of Willmore spheres and Willmore tori with non-trivial normal bundles by various authors.

1. INTRODUCTION

The differential geometric transformation theory of special surfaces classes plays an important role in the study of their classification: initially, these transformations were used to construct more complicated examples of surfaces from “trivial” surfaces and, more recently, the existence of such transformations has been linked to the phenomenon of complete integrability. Classically known examples include the Bäcklund transformations of constant Gaussian or mean curvature surfaces and the Darboux transformations of isothermic surfaces. Both types of transformations satisfy what is known as Bianchi permutability and therefore generate an abelian group acting on those surfaces. Under appropriate circumstances orbits of this action can be seen as the “energy shells” of a completely integrable system. In analytical terms these transformations allow the construction of solutions of the nonlinear system of PDEs describing a special surface class from known, often trivial, solutions by solving auxiliary ODEs together with algebraic manipulations.

In general, of course, one cannot expect to obtain a complete classification of all surfaces of a particular type from successive applications of such transforms. But there are a number of interesting instances, most strikingly perhaps in the theory of harmonic maps, where this is the case. Physicists [8],[5] working on non-linear sigma models observed that taking the $(1, 0)$ -part of the derivative of a harmonic map from a Riemann surface M into complex projective space $\mathbb{C}\mathbb{P}^n$ yields a new harmonic map into $\mathbb{C}\mathbb{P}^n$, thereby generating what is now called the harmonic sequence. This construction turned out to be important for classifying harmonic maps: tracing the energies of the harmonic maps in the sequence one obtains, under appropriate degree assumptions, an increasing function in the sequence length. This function turns out to be bounded by the energy of the initial harmonic map. Thus the sequence has to terminate, in which case the last element of the sequence is a holomorphic map, and the initial harmonic map is part of the Frenet frame of a holomorphic curve in $\mathbb{C}\mathbb{P}^n$. In other words, under an appropriate degree assumption on a harmonic map into $\mathbb{C}\mathbb{P}^n$ such a map comes from a holomorphic curve by taking derivatives

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and projections [2]. This construction can be generalized to other target spaces such as Grassmannians [18] and Lie groups giving rise to the uniton transform of Uhlenbeck [17].

These advances in the theory of harmonic maps from a Riemann surface have implications to classical surface theory: in certain cases curvature conditions on a surface translate to harmonicity conditions on a ‘‘Gauss map’’, the classical example being that of constant mean curvature surfaces whose unit normal maps into the 2-sphere are harmonic. Applying the harmonic sequence arguments to this special case yields the following result by Eells and Wood [6]: if the degree of a harmonic map from a compact Riemann surface M into $\mathbb{C}\mathbb{P}^1$ (with the appropriate orientation) is at least the genus g of M then the map has to be holomorphic. For example, a constant mean curvature sphere has a holomorphic unit normal map and thus is a round sphere as first observed by Hopf [10]. On the other hand, immersed constant mean curvature tori have unit normal maps of degree zero and the harmonic sequence does not terminate. This case leads to the theory of spectral curves and the construction of constant mean curvature tori from algebraically completely integrable systems [16], [9], [1].

In the present paper we consider Willmore surfaces in the 4-sphere S^4 . These surfaces are also characterized by a harmonicity condition, this time on their conformal Gauss maps or mean curvature sphere congruences. Bryant’s [3], Ejiri’s [7], and later Montiel’s [15], classification of Willmore spheres in S^3 and S^4 indicate that there ought to be a harmonic sequence type explanation for these results. But the conformal Gauss map takes values in the space of oriented 2-spheres in S^4 , a Grassmannian of space-like planes, for which the harmonic sequence and uniton transform constructions are much less developed. To remedy this situation and to stay conceptually close to the $\mathbb{C}\mathbb{P}^n$ case, we describe the 4-sphere by the quaternionic projective line $\mathbb{H}\mathbb{P}^1$ so that a Willmore surface $f: M \rightarrow S^4$ is given by the line sub-bundle $L \subset V$, $L_p = f(p)$, where $V = M \times \mathbb{H}^2$ is the trivial bundle over M . The conformal Gauss map then is a harmonic complex structure $S \in \Gamma(\text{End}(V))$, $S^2 = -1$, on V with the property that the 2-spheres $S(p)$, given by the quaternionic eigenlines of $S(p)$, touch the surface f and have the same mean curvature then f at $p \in M$. In terms of the type decomposition

$$\frac{1}{2}SdS = A + Q$$

of the derivative of S these conditions imply

$$L \subset \ker Q \quad \text{or equivalently,} \quad \text{im } A \subset L$$

and, denoting by $*$ the complex structure on M ,

$$d * A = d * Q = 0,$$

which is the harmonicity of S . If A and Q are not identically zero they both have rank one and we can define [4] two more line sub-bundles $\hat{L} \subset V$ and $\tilde{L} \subset V$ satisfying

$$\text{im } Q \subset \hat{L}, \quad \text{and} \quad \tilde{L} \subset \ker A.$$

Due to the harmonicity equations these line sub-bundles extend smoothly across the isolated zeros of A and Q . Moreover, since A and Q have type the maps $\hat{f}, \tilde{f}: M \rightarrow S^4$ given by the line bundles \hat{L} and \tilde{L} are conformal and thus have at most isolated branch points. Since A and Q anti-commute with S the bundles \hat{L} and \tilde{L} are stable under S which means that the surfaces \hat{f} and \tilde{f} lie for each $p \in M$ on the mean curvature sphere $S(p)$ of f . Unless $\hat{f} = \tilde{f}$, in which case the mean curvature sphere congruence S of f has a second envelope, the mean curvature spheres of \hat{f} and \tilde{f} are distinct from S . By using the theory of 1-step Bäcklund transforms [4], [12], we show that the mean curvature sphere

congruences of \hat{f} and \tilde{f} extend smoothly across the possible branch points. Since $\hat{A} = Q$ and $\tilde{Q} = A$ the conformal Gauss maps \hat{S} and \tilde{S} are again harmonic and therefore \hat{f} and \tilde{f} are Willmore surfaces. Continuing this construction we obtain, starting from $f_0 = f$ and putting $f_{-1} = \hat{f}$, $f_1 = \tilde{f}$, a sequence

$$\cdots \rightarrow f_{-2} \rightarrow f_{-1} \rightarrow f_0 \rightarrow f_1 \rightarrow f_2 \rightarrow \cdots$$

of (possibly branched) Willmore surfaces $f_i: M \rightarrow S^4$ with smooth conformal Gauss maps $S_i \in \Gamma(\text{End}(V))$, $S_i^2 = -1$. Not unlike in the case of the harmonic sequence, tracing the Willmore energies of f_i along the sequence provides a function in the sequence length bounded by the Willmore energy of f . In case M has genus zero, or M has genus one and f has non-trivial normal bundle degree, this function in the sequence length is strictly increasing and therefore the sequence has to terminate.

If the Willmore sequence terminates, it can only happen in two ways: either some $A_i = 0$ in which case already $A = 0$ or $Q = 0$ and f is the twistor projection of a holomorphic curve into $\mathbb{C}\mathbb{P}^3$; or one of the maps f_i is constant in which case already $f_{-1} = f_i = f_1$ and stereographic projection of f from this point gives a minimal surface in \mathbb{R}^4 . Thus, if the Willmore sequence of a Willmore surface $f: M \rightarrow S^4$ terminates, f is either the twistor projection of a holomorphic curve into $\mathbb{C}\mathbb{P}^3$ or comes from a minimal surface in \mathbb{R}^4 via stereographic projection. This provides a unifying point of view of the results in [3], [7], [15], and [14]. As already conjectured and partially verified in [14], the analogue for Willmore surfaces of the result by Eells and Wood [6] should be the following: if the normal bundle degree of f (with the appropriate orientation) is at least $4g - 3$, where g is the genus of M , then the Willmore sequence of f terminates.

2. THE WILLMORE SEQUENCE

In the construction of the Willmore sequence of a Willmore surface in the 4-sphere S^4 one generally encounters special branched conformal immersions $f: M \rightarrow S^4$ from a Riemann surface M whose mean curvature spheres extend smoothly across the branch points of f . At a regular point $p \in M$ of f the mean curvature sphere is the oriented 2-sphere $S(p) \subset S^4$ touching f and having the same mean curvature as f at the point $p \in M$. Even though this description seems to depend on Euclidean quantities it is invariant under the Möbius group of S^4 . The map which assigns to $p \in M$ the mean curvature sphere $S(p)$ is called the conformal Gauss map or the mean curvature sphere congruence and plays a pivotal role in the theory of Willmore surfaces.

In what follows, we present the conformal 4-sphere S^4 by the quaternionic projective line $\mathbb{H}\mathbb{P}^1$ on which $\mathbf{G}\mathbf{l}(2, \mathbb{H})$ acts by Möbius transformations. A round 2-sphere $S \subset S^4$ is given by the quaternionic eigenlines of a complex structure $S \in \mathbf{G}\mathbf{l}(2, \mathbb{H})$, $S^2 = -1$, on \mathbb{H}^2 and $-S$ is the sphere with the opposite orientation. The conformal map $f: M \rightarrow S^4$ is described as the line sub-bundle $L \subset V$ of the trivial bundle $V = M \times \mathbb{H}^2$ over M , namely $L_p = f(p)$ for $p \in M$. As always with maps into projective spaces, under the identification $f^*TS^4 = \text{Hom}(L, V/L)$, the derivative of f is expressed by the 1-form

$$\delta = \pi d|_L \in \Omega^1(\text{Hom}(L, V/L)).$$

Here d denotes the trivial connection on V and $\pi: V \rightarrow V/L$ is the projection into the quotient bundle. A congruence of touching 2-spheres S along the conformal map $f: M \rightarrow S^4$ thus is a complex structure S on the bundle V stabilizing $L \subset V$ and satisfying

$$*\delta = S\delta = \delta S,$$

where $*$ denotes the complex structure on 1-forms on M . The first equality expresses the conformality of f and the second equality assures that, at regular points $p \in M$ of f , the tangent spaces to f coincide with the tangent spaces of the spheres $S(p)$ at $f(p)$. Since S stabilizes L there are well-defined induced complex structures, again denoted by S , on L and V/L . The existence of a touching sphere congruence along f allows us to decompose the pulled back tangent bundle of the 4-sphere,

$$f^*TS^4 = \text{Hom}(L, V/L) = \text{Hom}_+(L, V/L) \oplus \text{Hom}_-(L, V/L),$$

into the two complex line bundles $\text{Hom}_\pm(L, V/L)$ which agree with the tangent and normal bundles of f along the regular points of f . The subscript \pm indicates complex linear respectively anti-linear homomorphisms with respect to the complex structures induced by S . Note that the complex structure on a Hom-bundle is always given by the target complex structure so that, for instance, $\text{Hom}_-(L, V/L) = \text{Hom}_+(\bar{L}, V/L)$, where \bar{L} indicates that we use the opposite complex structure $-S$ on L . A quaternionic vector bundle with a complex structure is just the double of a complex vector bundle [4, Sec. 11.1]: therefore $V = V_+ \oplus V_-$ where V_\pm are the $\pm i$ eigen-spaces of S and V_- is complex isomorphic to V_+ via multiplication by j . Thus $\text{Hom}_+(L, V/L) = \text{Hom}_\mathbb{C}(L_+, (V/L)_+)$ is indeed a complex line bundle.

If M is compact the degree of a quaternionic bundle with complex structure is the degree of the underlying complex bundle: for example, the normal bundle degree of f is

$$(1) \quad v := \deg \text{Hom}_-(L, V/L) = \deg \text{Hom}_+(\bar{L}, V/L) = \deg(V, S).$$

What distinguishes the mean curvature sphere congruence S among touching sphere congruences is a second order touching condition [4, Thm. 2]. In terms of the type decomposition of the derivative $dS = dS' + dS''$ with respect to S on V and $*$ on M this condition is

$$L \subset \ker dS'' \quad \text{or equivalently,} \quad \text{im } dS' \subset L.$$

The trivial connection d on V can be written in terms of the complex connection $\hat{\nabla}$, $\hat{\nabla}S = 0$, as

$$(2) \quad d = \hat{\nabla} + \frac{1}{2}SdS = \hat{\nabla} + A + Q,$$

where $A \in \Gamma(K \text{End}_-(V))$ and $Q \in \Gamma(\bar{K} \text{End}_-(V))$ are the type decomposition of $\frac{1}{2}SdS = A + Q$ and K denotes the canonical bundle of M . With this notation

$$dS = 2*(Q - A)$$

and the mean curvature sphere condition can be rewritten as

$$L \subset \ker Q \quad \text{or equivalently,} \quad \text{im } A \subset L.$$

The Willmore energy [4, Def. 8] of the conformal map $f: M \rightarrow S^4$ with mean curvature sphere congruence $S \in \Gamma(\text{End}(V))$ is the same as the Dirichlet energy

$$(3) \quad W(f) = 2 \int_M \langle A \wedge *A \rangle$$

of S . At regular points of f the integrand $\langle A \wedge *A \rangle$ is given by the usual Willmore integrand $(|H|^2 - K - K^\perp) \text{vol}_{f^*h}$, where the mean curvature H , Gaussian curvature K , and normal bundle curvature K^\perp are computed with respect to a conformally flat metric h on S^4 .

The conformal map $f: M \rightarrow S^4$ with mean curvature sphere congruence S is said to be a *branched Willmore surface* if f is a critical point for the Willmore energy under compactly supported variations of f by maps which have a mean curvature sphere congruence. Note

that we also allow the conformal structure of M to change. As in the unbranched case the resulting Euler-Lagrange equation [12] is the harmonic map equation

$$(4) \quad d * A = d * Q = 0$$

for S . The complex linear part of the last equation gives

$$(5) \quad d^{\hat{\nabla}} A = d^{\hat{\nabla}} Q = 0$$

which says that $A \in H^0(K \text{Hom}_+(\bar{V}, V))$ and $Q \in H^0(K \text{Hom}_+(V, \bar{V}))$ are complex holomorphic bundle maps. Assuming that A and Q are not identically zero they have rank one and we obtain two smooth line sub-bundles $\tilde{L}, \hat{L} \subset V$, namely

$$\tilde{L} \subset \ker A \quad \text{and} \quad \text{im } Q \subset \hat{L},$$

extending the kernel of A and the image of Q across their zeros. The corresponding smooth maps $\tilde{f}, \hat{f}: M \rightarrow S^4$ are called *Bäcklund transforms* of f . This construction already appeared in [4] but it was unknown whether the Bäcklund transforms admitted smooth mean curvature spheres. Their existence is crucial for the continuation of this construction to obtain a sequence of Willmore surfaces.

Theorem 2.1. *Let $f: M \rightarrow S^4$ be a branched Willmore surface with mean curvature sphere congruence S . Then the Bäcklund transforms $\tilde{f}, \hat{f}: M \rightarrow S^4$ are again branched Willmore surfaces with mean curvature sphere congruences \tilde{S} and \hat{S} . Moreover, $\tilde{S} = -S$ on V/\tilde{L} , $\hat{S} = -S$ on \hat{L} , $\tilde{Q} = A$ and $\hat{A} = Q$.*

This theorem is a consequence of a more fundamental result, Theorem 4.1, about the smoothness of the mean curvature sphere congruences of 1-step Bäcklund transforms. Since this result is interesting in its own right and relies on the theory of 1-step Bäcklund transforms developed in [4],[12],[13], and [11], we postpone its proof to the last section of this paper.

From Theorem 2.1 and $\text{im } A \subset L \subset \ker Q$ we see that

$$\tilde{\tilde{f}} = f = \hat{\hat{f}}.$$

Therefore we obtain a sequence, the *Willmore sequence*,

$$\cdots \rightarrow f_{-2} \rightarrow f_{-1} \rightarrow f_0 = f \rightarrow f_1 \rightarrow f_2 \rightarrow \cdots$$

of branched Willmore surfaces $f_i: M \rightarrow S^4$, $f_{i+1} = \tilde{f}_i$, with mean curvature spheres S_i provided that the A_i do not vanish and that the f_i are not constant. The difference of the Willmore energies of successive sequence elements is given by the normal bundle degree (1) of f_i ,

$$(6) \quad W(f_i) - W(f_{i-1}) = 4\pi \deg(V, S_i) = 4\pi v_i.$$

This follows from

$$W(f) - W(\hat{f}) = 2 \int_M \langle A \wedge *A \rangle - 2 \int_M \langle Q \wedge *Q \rangle = 2 \int_M SR^{\hat{\nabla}} = 4\pi \deg(V, S) = 4\pi v,$$

where we used (3) and the complex linear part of the zero curvature equation for $d = \hat{\nabla} + A + Q$.

We are now in the position to give a variation of the arguments used in the harmonic sequence construction [18],[2]. Eventually, this will give a characterization of Willmore surfaces under some assumptions on their normal bundle degrees. First we note that $*\delta = S\delta = \delta S$ implies that the holomorphic structure $\bar{\partial} = \hat{\nabla}''$ stabilizes L and that $\bar{\partial}\delta = 0$.

This means that the derivative δ of $f: M \rightarrow S^4$ is a holomorphic section of the complex line bundle $K \operatorname{Hom}_+(L, V/L)$. We already have seen that $A \in H^0(K \operatorname{Hom}_+(\bar{V}, V))$ is a complex holomorphic bundle map (5). But the anti-holomorphic structure $\partial = \hat{\nabla}'$ stabilizes \tilde{L} since, given $\psi \in \Gamma(\tilde{L})$ and using $*A = -AS$, we have

$$A \wedge \partial\psi = A \wedge \hat{\nabla}'\psi = -d^{\hat{\nabla}'}(A\psi) = 0.$$

Therefore, A can be regarded as a holomorphic section of the complex line bundle $K \operatorname{Hom}_+(\bar{V}/\tilde{L}, L)$. If none of the sections δ , $\tilde{\delta}$ and A are trivial, we obtain

$$\begin{aligned} \deg K + \deg(V, S) - 2 \deg L &\geq 0, \\ \deg K + \deg(V, \tilde{S}) - 2 \deg \tilde{L} &\geq 0, \\ \deg K + \deg L - \deg(V, \tilde{S}) + \deg \tilde{L} &\geq 0, \end{aligned}$$

and thus

$$4 \deg K + v - \tilde{v} \geq 0,$$

with v and \tilde{v} the degree of the normal bundles of f and \tilde{f} . Telescoping this last relation over the sequence f_0, \dots, f_{i-1} yields

$$v - v_i + 4i \deg K \geq 0.$$

But $4\pi v_i = W(f_i) - W(f_{i-1})$ is a difference of Willmore energies (6) so that by summing over $i = 1, \dots, n$ we get the following relation between the sequence length and the Willmore energy of f :

$$nv + \frac{1}{4\pi}W(f) + 2n(n+1) \deg K \geq 0.$$

As an immediate consequence we obtain a criterion when the Willmore sequence has finite length.

Theorem 2.2. *Let $f: M \rightarrow S^4$ be a branched Willmore surface such that either M has genus zero or M has genus one and f has non-trivial normal bundle. Then the Willmore sequence of f terminates.*

Proof. If M has genus zero then $\deg K = -2$ which gives a contradiction to the finiteness of $W(f)$. If M has genus one then $\deg K = 0$ and we again contradict the finiteness of $W(f)$, since we always may choose our orientation so that the normal bundle degree is negative. \square

3. FINITE WILLMORE SEQUENCES

We have seen that in certain circumstances the Willmore sequence of a compact branched Willmore surface $f: M \rightarrow S^4$ terminates. As it turns out, a finite Willmore sequence can at most have length two and f is either the twistor projection of a holomorphic curve into $\mathbb{C}\mathbb{P}^3$ or comes from a minimal surface in \mathbb{R}^4 by inversion at infinity.

From the previous section we know that the Willmore sequence can be continued past a branched Willmore surface $f: M \rightarrow S^4$ as long as A and the derivative $\tilde{\delta}$ of \tilde{f} are not zero. We first discuss what happens when $\tilde{\delta} = 0$ and hence \tilde{f} is a point, say $\infty \in S^4$. Since the mean curvature sphere congruence S of f stabilizes \tilde{L} the spheres $S(p)$ for $p \in M$ all contain the point ∞ . Therefore, by inverting f at this point we obtain a surface in \mathbb{R}^4 whose mean curvature spheres become its tangent planes and hence this surface is minimal. Since there are no compact minimal surfaces in \mathbb{R}^4 the point $\infty = f(q)$ for some

$q \in M$, i.e., ∞ must be lying on the surface f . Moreover, if f is immersed at $q \in M$ then the corresponding minimal surface in \mathbb{R}^4 has planar ends.

The fiber $f^{-1}(\infty) \subset M$ is finite because f is a branched conformal immersion. Thus, away from those points the bundle V decomposes as a direct sum $V = L \oplus \underline{L}_q$ of L and the trivial bundle $\underline{L}_q = M \times L_q$. But S stabilizes \underline{L}_q and hence also $Q\underline{L}_q \subset \underline{L}_q$ which, together with $QL = 0$, implies that $\text{im } Q \subset \underline{L}_q$. Therefore, also \hat{f} is the point ∞ and the original Willmore sequence consisted of only one element, f , and terminated on either side to the same constant map. It is easy to see that the converse also holds: a minimal surface in $\mathbb{R}^4 \subset S^4$ is a Willmore surface whose forward and backward Bäcklund transforms are the point at infinity $S^4 = \mathbb{R}^4 \cup \{\infty\}$.

We now come to the second possibility, namely $A = 0$, for the Willmore sequence to terminate. In this case it follows from [4, Thm. 4] that $f : M \rightarrow S^4$ is the composition of a holomorphic curve $h : M \rightarrow \mathbb{C}\mathbb{P}^3$ followed by the twistor projection $\mathbb{C}\mathbb{P}^3 \rightarrow S^4$. If in addition $Q = 0$, then $dS = 0$ by (2), and S is constant implying that $f(M) \subset S^4$ is itself a round 2-sphere. Excluding this possibility we show that the branched Willmore surface $\hat{f} : M \rightarrow S^4$ has $\hat{Q} = 0$ and is therefore the twistor projection of a holomorphic curve into $\mathbb{C}\mathbb{P}^3$, after changing the orientation on S^4 . To see that $\hat{Q} = 0$ we show that $\hat{S} = -S$ which implies $\hat{Q} = -A = 0$. For this it suffices to observe that

$$(7) \quad *\hat{\delta} = -\hat{\delta}S = -S\hat{\delta},$$

meaning that $-S$ is a touching sphere congruence along \hat{f} . Due to $A = 0$ this touching sphere congruence is in fact the mean curvature sphere congruence of \hat{f} . The first equality in (7) follows from Theorem 2.1 whereas the second equality follows from the definition of $\hat{\delta}$, the fact that $A = 0$, $\text{im } Q \subset \hat{L}$ and the first equality in (7):

$$\delta = \hat{\pi}d|_{\hat{L}} = \hat{\pi}(\partial + \bar{\partial} + A + Q)|_{\hat{L}} = \hat{\pi}\bar{\partial}|_{\hat{L}}.$$

Thus, if f does not come from a minimal surface the Willmore sequence has exactly two elements both of which are twistor projections of holomorphic curves in $\mathbb{C}\mathbb{P}^3$.

Strictly speaking, we have only discussed what happens if the sequence terminates in the forward direction. To deal with the reversed direction, we note that we have a dual branched Willmore surface f^\perp obtained from the point-point duality of $\mathbb{H}\mathbb{P}^1$. Its mean curvature sphere congruence is given by $S^\perp = S^*$ and then $\ker A^\perp = (\text{im } Q)^\perp$ which reverses the direction of the Willmore sequence.

This discussion, together with Theorem 2.2, gives a unified explanation of results by various authors [3], [7], [15], [4],[14] on the characterization of Willmore surfaces:

Theorem 3.1. *Let $f : M \rightarrow S^4$ be a branched Willmore surface and assume that M has genus zero or that M has genus one and the normal bundle of f is non-trivial. Then the Willmore sequence of f has at most two elements and f is obtained from a minimal surface in \mathbb{R}^4 via inversion at infinity, or f is the twistor projection of a holomorphic curve in $\mathbb{C}\mathbb{P}^3$.*

4. SMOOTH MEAN CURVATURE SPHERE CONGRUENCES

In this last section we provide a proof that the mean curvature spheres \tilde{S} of a Bäcklund transform $\tilde{f} : M \rightarrow S^4$ of a branched Willmore surface $f : M \rightarrow S^4$ extend smoothly across the branch locus of \tilde{f} , which is the main content of Theorem 2.1. Rather than computing the mean curvature spheres for the Bäcklund transform directly, we show that

an intermediate transform, the 1-step Bäcklund transform, has smooth mean curvature spheres. We then apply a result of [4, Lemma 10] which expresses our Bäcklund transform \tilde{f} as a composition of two 1-step transforms.

First note that \tilde{f} is a conformal map and thus has isolated branch points: for $\psi \in \Gamma(\tilde{L})$ we have

$$0 = d(*A\psi) = - * A \wedge d\psi = - * A \wedge \tilde{\pi}d\psi = - * A \wedge \tilde{\delta}\psi$$

where we used $\tilde{L} \subset \ker A$. Since $*A = A(-S)$ and we assume that A is non-trivial this gives $*\delta = -S\delta$ which implies conformality of \tilde{f} . Therefore, away from its isolated branch points \tilde{f} has a mean curvature sphere congruence \tilde{S} which is given by $-S$ on V/\tilde{L} . It is shown in [4, Thm. 7] that $\tilde{Q} = A$ which implies that \tilde{S} is harmonic and \tilde{f} an immersed Willmore surface (away from its branch locus). Thus it suffices to show that \tilde{S} can be extended smoothly across the branch points of \tilde{f} to obtain a harmonic mean curvature sphere congruence for \tilde{f} which then is a branched Willmore surface. This is a purely local problem and so we may assume that M is a simply connected Riemann surface.

To define a 1-step Bäcklund transform of the branched Willmore surface f , we choose a point at infinity $\infty = e\mathbb{H}$ in S^4 which is not contained in the image of f . After applying a Möbius transformation, we may assume $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so that $f = \begin{pmatrix} g \\ 1 \end{pmatrix} \mathbb{H}$. Let $\alpha \in (\mathbb{H}^2)^*$ be the projection onto the second component so that $\langle \alpha, e \rangle = 0$. Then, due to the harmonicity (4) of the mean curvature sphere congruence S of f , the \mathbb{H} -valued 1-form

$$(8) \quad dg^\sharp = \langle \alpha, *Ae \rangle$$

is closed and thus exact since M is simply connected. The map $g^\sharp: M \rightarrow \mathbb{H}$, respectively the map $f^\sharp: M \rightarrow S^4$ given by $f^\sharp = \begin{pmatrix} g^\sharp \\ 1 \end{pmatrix} \mathbb{H}$, is called a *1-step Bäcklund transform* of the branched Willmore surface f .

Theorem 4.1. *Let $f: M \rightarrow S^4$ be a branched Willmore surface and let $f^\sharp: M \rightarrow S^4$ be a 1-step Bäcklund transform. Then f^\sharp is again a branched Willmore surface, that is to say, the mean curvature sphere congruence S^\sharp extends smoothly across the branch locus of f^\sharp .*

Proof. Since $\infty \in S^4$ is not on the surface f the trivial bundle $V = M \times \mathbb{H}^2$ splits as

$$V = L \oplus \underline{e\mathbb{H}}$$

and $V/L = \underline{e\mathbb{H}}$. We choose the adapted framing (ψ, e) of $V = L \oplus \underline{e\mathbb{H}}$ with $\psi = \begin{pmatrix} g \\ 1 \end{pmatrix}$ a smooth section of L . In this frame the mean curvature sphere congruence S of f is given by [4, Sec. 7.2]

$$S\psi = -\psi R \quad \text{and} \quad Se = eN - \psi H$$

with $R, N, H: M \rightarrow \mathbb{H}$ satisfying $R^2 = N^2 = -1$ and $RH = NR$. The conformality and touching conditions $*\delta = S\delta = \delta S$ translate to

$$*dg = Ndg = -dgR$$

so that N and R are the left and right normals of $g: M \rightarrow \mathbb{H}$. That S is the mean curvature sphere congruence along f is expressed by [4, Sec. 7.2]

$$(9) \quad 2dgH = dN - N * dN.$$

From [4, Thm. 6] we know that the 1-step Bäcklund transform f^\sharp is a branched conformal immersion which is Willmore away from its branch points. Therefore it suffices to show

that the mean curvature sphere congruence S^\sharp of f^\sharp extends across the branch points which is a purely local argument. To calculate the left and right normals for the 1-step transform g^\sharp , we need to split the trivial bundle V with respect to the Bäcklund transform \tilde{f} into $V = \tilde{L} \oplus \underline{e\mathbb{H}}$. This is possible since the point $\infty \in S^4$ can be chosen in the complement of the images of f and \tilde{f} , after possibly restricting to a sufficiently small neighborhood of a branch point of f^\sharp . Let $\beta \in \Gamma(V^*)$ be the unique smooth section satisfying $\langle \beta, e \rangle = 1$ and $\beta|_{\tilde{L}} = 0$. We now express $Se = \tilde{\phi} + e \langle \beta, Se \rangle$ in the splitting $V = \tilde{L} \oplus \underline{e\mathbb{H}}$ and recall that $\tilde{L} \subset \ker A$ and $*A = -AS$. Then, from the definition (8) of g^\sharp , we see that its right normal is given by

$$\psi * dg^\sharp = S * Ae = - * ASe = - * Ae \langle \beta, Se \rangle = -\psi dg^\sharp \langle \beta, Se \rangle .$$

On the other hand, from $*A = SA$ we obtain

$$\psi * dg^\sharp = -Ae = S\psi dg^\sharp = -\psi R dg^\sharp .$$

Therefore, the left and right normals of g^\sharp are given by

$$N^\sharp = -R \quad \text{and} \quad R^\sharp = \langle \beta, Se \rangle$$

which are both smooth on M . Hence, if the mean curvature sphere congruence S^\sharp exists it is expressible in the adapted frame (ψ^\sharp, e) of $V = L^\sharp \oplus \underline{e\mathbb{H}}$ by

$$S^\sharp \psi^\sharp = -\psi^\sharp R^\sharp \quad \text{and} \quad S^\sharp e = e N^\sharp - \psi^\sharp H^\sharp ,$$

with $\psi^\sharp = \begin{pmatrix} g^\sharp \\ 1 \end{pmatrix}$ a smooth section of L^\sharp . In particular, S^\sharp has to satisfy

$$(10) \quad 2dg^\sharp H^\sharp = dN^\sharp - N^\sharp * dN^\sharp$$

by (9), which can be used to calculate H^\sharp : since $N^\sharp = -R$ we have

$$dN^\sharp - N^\sharp * dN^\sharp = -dR - R * dR = -4 \langle \alpha, *A\psi \rangle ,$$

with the last equality following from [4, Prop. 12]. Decomposing $\psi = \tilde{\phi} + e \langle \beta, \psi \rangle$ in the splitting $V = \tilde{L} \oplus \underline{e\mathbb{H}}$, and recalling once more $\tilde{L} \subset \ker A$ and the definition (8) of dg^\sharp , we obtain from (10)

$$dg^\sharp (H^\sharp + 2 \langle \beta, \psi \rangle) = 0$$

and hence $H^\sharp = -2 \langle \beta, \psi \rangle$ is smooth on M . For S^\sharp to actually be a sphere congruence we need $(S^\sharp)^2 = -1$. But $(R^\sharp)^2 = (N^\sharp)^2 = -1$ so it remains to verify $R^\sharp H^\sharp = H^\sharp N^\sharp$ which immediately follows from the explicit expression of H^\sharp . \square

As already mentioned it is proven in [4, Lemma 10] that the Bäcklund transform \tilde{f} of f is given by two successive 1-step transforms. Thus, applying the previous theorem twice we see that the mean curvature sphere congruence \hat{S} of \hat{f} is smooth. To obtain the result for the Bäcklund transform \hat{f} , we apply the above argument to the dual surface f^\perp thereby concluding the proof of Theorem 2.1.

Remark 4.2. There is a more conceptual way to prove Theorem 4.1: using the theory of envelopes of Frenet curves [12] in $\mathbb{H}\mathbb{P}^n$ one can show that a 1-step Bäcklund transform is, up to a suitable projection, given by an envelope [13]. A Frenet curve is the natural generalization to $\mathbb{H}\mathbb{P}^n$ of a conformal map into S^4 allowing a smooth mean curvature sphere congruence and the enveloping construction preserves the Frenet property. Using this generalized setup an analogue of Theorem 2.1 for Willmore surfaces in $\mathbb{H}\mathbb{P}^n$ can be found in [11].

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KATRIN LESCHKE, INSTITUT FÜR MATHEMATIK, LEHRSTUHL FÜR ANALYSIS UND GEOMETRIE, UNIVERSITÄT AUGSBURG, 86135 AUGSBURG, GERMANY

FRANZ PEDIT, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA 01003, USA, AND, MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, 72076 TÜBINGEN, GERMANY

E-mail address: katrin.leschke@math.uni-augsburg.de, pedit@math.umass.edu