Hindawi Publishing Corporation

International Journal of Quality, Statistics, and Reliability

International Journal of Quality, Statistics, and Reliab
Volume 2008 (2008), Article ID 594753, 10 pages
doi:10.1155/2008/594753

Research Article

A Unified Approach for Predicting Long- Term Capability Indices with Dependen Manufacturing Target Bias

Nikhil T. Satyala and R. J. Pieper

Department of Electrical Engineering, The University of Texas at Ty

Received 15 April 2008; Revised 25 September 2008; Accepted 14

Academic Editor: Satish Bukkapatnam

Abstract

It is shown that the exact solution for the capability index (CPI) fo
be expressed in terms of an unbiased CPI and a normalized ta
formulation is that it facilitates evaluation of the degradation of
process mean and the process target. It is shown how this formal
is readily extended to long-term process for which the distribution
the two long-term CPI degrading effects, namely, process ins
guarantee that long-term processes are distributed as Gaussian a
conditions, a new paradigm for a long-term locator ‘ ‘k’ ’ is
indicate that the exact CPI model is a less pessimistic predictor tha

1. Introduction

In 1979, Taguchi and Wu [1] introduced a viewpoint on estimatin
with lack of precision and accuracy in a manufacturing process
viewpoint introduced by Taguchi is the classical “goal-post model’
is whether the product parameters fall within the process sp

philosophy, the level of process control is typically characterized
(CPIs) [2]. Capability indices provide a numerical assessment of
specifications [3, 4].

A manufacturing process would commonly be described in terms
upper specification limit (USL), and a lower specification limit (LSL
for a certain specification exceeds the USL or falls below the LS
product characteristic is centered between the USL and LSL, th
capability index is of interest to the manufacturing community b
multifaceted manufacturing process down to one quantity which c
Typical capability index values can range from 0.7 to 2.0. In the j
correspond to a capability index of 1.0 while a much improved s
would correspond to a capability index of 2.0 [2]. The standard c
precision. The absolute value of the difference between the distrib
is indicative of the process accuracy. According to the Taguchi gu
easier to adjust the manufacturing process to improve accuracy t
[2]. The most commonly assumed probabilistic distribution for a
the normal distribution which can be defined in terms of mean μ, a

The primary situation for the application of asymmetric tolerances
exhibits a skewed distribution [6]. Historically, capability indices
mean of the process is on the target [2]. Target bias is at best ap
necessary to consider the impact of the distribution mean of the p
a variety of target-bias-dependent capability index models introd
such models has been assembled for purposes of this paper.

What follows first is a brief qualitative review of capability indices
capability index, C_p, is gauged within a relatively narrow window
be found in the literature to be applied in two ways. It could be
(e.g., measured over days or even hours) to long-term (e.g., mea
case, that process mean shifts around the target but on the avera
shifting around in the short-term process is accounted for with a
to a higher standard deviation. The long-term precision in the m
term and, therefore, the long-term capability index is lower than t

On the other hand, C_{pk} has also found utility as a capability inde
However, as pointed out in [6], this type of usage of C_{pk} to accou
measured paradigm for a capability index, C_{pm} paradigm [2, 7,
account for target bias. Because C_{pm} can be related to Taguchi lo
Taguchi index. One advantage of the C_{pm} approach is that it is no
on the underlying distribution of the specified product parameter d

It has been shown [5] that a probabilistic description of the manu
dependence for the fraction of rejected components and related to
normal (Gaussian) distribution for the process product and symme
bias was shown to be expressed in terms of four parameters, proc
and lower specification limit. Additionally, it has been demonst
reparameterized solution expressed in terms of appropriately defin

A target bias-dependent capability index (CPI) for the symmetric
proposed and tested. It is shown that various exact expressio

proposed short-term CPI model dependent on only two parameter bias. One advantage of this particular formulation is that it fa capability of the process due to an offset between the mean advantage of this parameterization is that it allows for a convenien commonly used industry models which also estimate the CPI proposed formalization facilitates setting up a CPI model for th methodology unified in approach with that of the proposed short- term CPI degrading effects, namely, long-term process instability that the long-term processes are distributed as Gaussian are conditions, a new paradigm for a long-term locator “*k*” is propos

Two implementation schemes for the proposed model are discu numerical built-in mathematical routines for the error function and in functions used in the first scheme with recently reported ana process, the results indicate that the exact CPI model is a less models tested.

2. Background

2.1. Background on CPI Model

In general, the measurements for the process parameter to m distribution having a mean μ and a standard deviation σ [2]. The lower specification limit (LSL). The distance between the USL specifications are considered to be symmetric if the target satisfie recipe for the capability index intended for situations for which the symmetric limits is

$$C_{p_o} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma} =$$

The subscript “*o*” in (1) indicates that it does not account fo asymmetry in tolerances and nonnormal distributed parameters index is a direct measure of the process control and relates C_{p_o} to

$$p_o = 2\Phi(-3C_{p_o}$$

where $\Phi(z)$ is the standard normal cumulative density function (C normal distribution, no target bias, and symmetric limits.

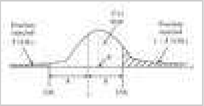


Figure 1: PDF showing manufacturing specific

2.2. Distribution Independent Observation

Independent of whether the distribution is normal with symmetric the “component of nonconformity” [6] can be computationally distribution at selected points. As suggested by Figure 1, this predi

$$p = F(\mathrm{LSL}) + (1 - F($$

This combines the parts that do not meet the specifications, t characteristic “*x*” which is either lower than the LSL or higher process capability index should be consistent with the rule

$$C_p = -\frac{1}{3}\Phi^{-1}\left(\frac{p}{2}\right)$$

where for the normal distribution with symmetric limits, $p_o = p$

From Figure 1, it follows that

$$\mathrm{USL} = \tau + \Delta,$$
$$\mathrm{LSL} = \tau + \Delta .$$

Consistent with (3) and (4), it can be shown that [5]

$$p = \Phi\left(\frac{\mathrm{LSL}-\mu}{\sigma}\right) + \Phi(-(\mathrm{U}$$

However, (6) conjunction with (4) produces an exact short-term c
[5] yield index model. Demonstration details are provided in Apper

3. Model-A: Standard Normal Version and Computer

3.1. Model-A Analysis

The PDF represented in Figure 1 can be transformed to the standa
(5), it can be stated that

$$\frac{\mathrm{LSL}-\mu}{\sigma} = \frac{(\tau-\mu)-\Delta}{\sigma} = -$$
$$\frac{\mathrm{USL}-\mu}{\sigma} = \frac{(\tau+\Delta)-\mu}{\sigma} =$$

After defining a normalized target bias,

$$\delta_b = \left(\frac{\tau-\mu}{\sigma}\right).$$

Therefore, for normal process PDFs with symmetric limits and tar
[11] has the following rejection fraction after substituting from (1)

$$p_A = \Phi(-3C_{p_o} + \delta_b) + \Phi(-$$

Consistent with the general approach (4),

$$C_{p_A} = -\frac{1}{3}\Phi^{-1}\left(\frac{p_A}{2}\right.$$

A check with $\delta_b = 0$ from (9) yields $p_A = p_o$, and consistent with (10

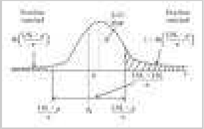


Figure 2: Transformed PDF showing the specif

3.2. Model-A Implementation Using Built-In Error Function

Noting that the standard normal arguments “z” needed in (9)
target bias, the following conversion rule valid for $z \le 0$ is useful wi

$$\Phi(z) = 0.5 - 0.5\,\mathrm{erf}($$

and for inspection of (11), it follows that

$$z = \sqrt{2}\ \mathrm{erf}^{-1}\left(\frac{0.5-}{0.5}\right.$$

Appendix 2 describes the definitions and the approximations cons

error function (erf^{-1}) in (11) and (12), respectively. However, (
inverse error functions of MATLAB. An alternative to using built-in
employ approximate analytic expressions described in the following

3.3. Model-A Implementation Using Analytic Approximation

The built-in error function routines, available in MATLAB [12],
predicting the bias-dependent capability indices. After defining $a =$
10]

$$\mathrm{erf}(x) \cong \sqrt{1 - \exp\left(-x^2\left(\frac{(4/}{1}\right.\right.}$$

Moreover, after taking $r = 2/(\pi a)$ and $t(x) = \ln(1 - x^2)$, the inverse
10]

$$\mathrm{erf}^{-1}(x) \approx \left(-r - \frac{t(x)}{2} + \sqrt{\left(r + \frac{t}{}\right.}\right.$$

A comparison of (13) and (14) with the MATLAB built-in routines
the error function and 0.004% for its inverse. Hence, (13) and (14

To demonstrate this approach, the built-in error function-based M
Model-A predictions for $C_{p_o} = 1.0, C_{p_o} = 1.4, C_{p_o} = 1.6$, and $C_{p_o} =$
numerical and the analytic approaches are serving as approxim
analytic approximation approach compared to that of the built-in
with the maximum percentage error of 0.91% at $C_{p_o} = 1$*.*

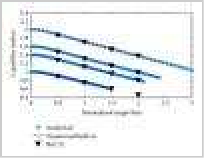


Figure 3: Model-A comparisons for numerical

Figure 3 also shows the comparison of Model-A with a well-estab
predictions at various target bias values have been considered to
pattern of the proposed model with the already existing industry m

The comparison of Model A with Boyle’ s exact model [5] require
USL = 58 and LSL = 26) [5]. For symmetrical specif
value. The USL and LSL values when taken in combination with th
(1) and the normalized target bias (8), lead to the process mean
quantities{USL, LSL, μ, and σ}, numerically determined applicatio
target-bias-dependent CPI which for comparison purposes has bee

4. Alternative Popular Methods

4.1. Model-X: The AMT Model

Model-X is based on incorporating the target bias with capability in

$$k_b = \frac{|\mu - \tau|}{\Delta}.$$

The subscript b indicates that this model includes the target bias. I
capability index C_{p_o} to define the Model-X capability index rule as

$$C_{pX}^{b} \equiv C_{p_o}(1 - k_b$$

From (15), it follows that

$$C_{pX}^{b} = C_{p_o}\Big(1 - \Big(\frac{\mu -}{\sigma}$$

This can be simplified via (1) and (8) [11]:

$$C_{pX}^{b} = C_p - \frac{|\delta_b|}{3}$$

and hence the corresponding fraction of rejection is

$$p_X = 2\Phi(-3C_{pX}^{b})$$

4.2. Model-Y: C_{pm} Model

The C_{pm} model was modeled to include the impact of the bias of
process parameter. As in the similarly defined Taguchi loss functio
The capability index in this model is defined using the variance of t

$$C_{pm} \equiv \frac{\Delta}{3\sigma_m}.$$

In (20), the variance is given by $\sigma_m = \sqrt{\sigma^2 + (\mu - \tau)^2}$. Hence, the M

$$C_{pY}^{b} \equiv \frac{\Delta}{3\sigma_m} = \left(\frac{\Delta}{3\sigma}\right)\frac{1}{\sqrt{1 + \delta_b^2}}$$

and the corresponding fraction of rejection predicted by

$$p_Y = 2\Phi(-3C_{pY}^{b})$$

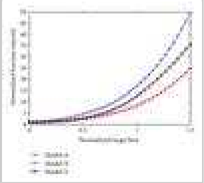
4.3. Computer Tests for Various Capability Index Models

The normalized fraction rejected for models A, X, and Y are define
conditions given in (2) as

$$\left(p_{A_N}, p_{X_N}, p_{Y_N}\right) \equiv \frac{p_{A,}}{,}$$

In Figure 4, the fraction rejected obtained from (23) is plotted ver
(19), and (22) for models A, X, and Y, respectively. It should b
pessimistic (i.e., higher fraction is rejected) than that predicted fro

Figure 4: Plot showing fraction rejected for m



The dependence of bias-inclusive capability indices on short-term
models A, X, and Y were taken from (10), (18), and (21), respe
target bias as shown in Figure 5.

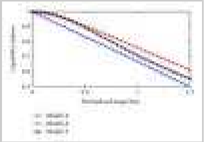


Figure 5: Plot showing capability indices for
target bias.

These final results indicate that the Model-X and Model-Y capabilit
lower in value). Of the two industry standard models Model-Y, usi
that it is closer in prediction to Model-A.

5. Extension to Long-Term Process

The standard approach [2, 13] for predicting the long-term capabil

$$C_{pk_o} \equiv \frac{\Delta}{3\sigma_k} .$$

The suffix “o” in (24) indicates that there is no target bias and
deviation σ_k . Consistent with the qualitative description in Section
(24)

$$\sigma_k \geq \sigma .$$

This is consistent with a long-term process being less precise, tha
with the corresponding short-term processes. By assumption, tl
deviation. The claim here is that under certain restrictions, to be
with standard deviation σ_k, and mean $\bar{\mu}$, the latter is given by

$$\bar{\mu} = \text{avg}\{\mu_i\}, \quad i = 1,2$$

where avg{ } is the average operator and i indexes the M short-t
bias, $\bar{\mu} \neq \tau$, it can be accounted for with Model-A type analysis by d

$$\delta_{bk} = \frac{\tau - \bar{\mu}}{\sigma_k} .$$

The long-term capability index representation equivalent to (24) is

$$C_{pk_o} = C_{p_o}(1-k) \leq C_{p_o},$$

Unlike the locator index defined by (15), this representation for
accounts for the long-term effective spread. The target bias is ac
index which excludes target bias effect is [11]

$$k_{\text{revised}} \equiv \frac{\text{avg}\{|\overline{\mu} - \mu_i|\}}{\Delta},$$

It should be noted that in this revised form of locator (29), the av
τ in (15). If the long-term process is “on target” (i.e., if $\overline{\mu} = \tau$),
used (15) [2]. As discussed in what follows, the restrictive math
term Gaussian process from the superposition of short-term Gaus
than commonly seen model in (15) or (29).

The long-term (LT) process distribution PDF can be viewed as be
multiple short-term PDFs as represented in the classic Harris a
Appendix 3, it is formally shown that a long-term Gaussian PDF
PDFs,

$$f_{\text{LT}}(y) = \frac{1}{M}\sum_{i=1}^{M} f_{\text{ST}}(y$$

In this case, the short-term process PDF is given by

$$f_{\text{ST}}(y,\mu_i) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-\mu_i}$$
$$-\infty \le y \le \infty, \quad i$$

The often-cited mean-finding rule [2], taking discrete-term expec
PDF for y on both sides of (30), would lead to

$$\overline{\mu} = \text{E}\{y\} = \frac{1}{M}\sum_{i=1}^{M} \text{E}\{y f_{\text{ST}}(y,\mu$$

which confirms (26). No claim is made that $f_{\text{LT}}(y)$ is guarantee
Gaussians with same standard deviation σ but with different me
would not be a Gaussian. Nonetheless, the assumption which is o
is that the overriding long-term distribution in (30) is approximat
[2, 13]. This assumption is shown to be justified in Appendix
associated with the (short-term) mean, μ, is approximately distrib

$$f_{\mu}(\mu) = \frac{1}{\sqrt{2\pi}\,\sigma_{\mu}} e^{-(\mu-\overline{\mu})^2/2\sigma_{\mu}^2}$$

with an average mean $\overline{\mu}$ and a standard deviation σ_{μ}. Consider fi
are identical (i.e., same constant mean $\mu = \overline{\mu}$). Then, the mean μ
This is accounted for in (33) by taking a limit $\sigma_{\mu} \to 0$ and in that lin

$$\lim_{\sigma_{\mu}\to 0} f_{\mu}(\mu) \to \delta(\mu -$$

For large-enough sampling of the short-term PDF (see (30)) (i.e.,
to averaging over the continuous random variable, μ, with an integ

$$f_{\text{LT}}(y) = \text{E}\{f_{\text{ST}}(y,\mu)\} = \int_{-\infty}^{\infty} f_{\text{S}}$$

In the special case where all the short-term Gaussian distributi
random variable μ reduces to a Dirac delta function (34) and subs
long-term PDF that is identical to the time-wise stable short-term
the integrated (35) is distributed Gaussian with mean $\overline{\mu}$ and standa

$$\sigma_k = \sqrt{\sigma^2+\sigma_\mu^2} = \sigma\sqrt{1+}$$

In Appendix 4, it is shown that

$$\sigma_\mu = \sqrt{\frac{\pi}{2}}\,\mathrm{avg}\{|\bar{\mu}-$$

It follows from (24) and (36) that the capability index is given by

$$C_{pk_o} = \frac{C_{p_o}}{\sqrt{1+(\sigma_\mu/\sigma)^2}} = \frac{}{\sqrt{1+(\sqrt{\pi/2}}}$$

and as expected C_{pk_o} is reduced with increasing temporal instal

Consistent with (28), the effective locator, *k*, would be given by

$$k_{\text{eff}} = 1-\frac{1}{\sqrt{1+(\sigma_\mu/\sigma)^2}}$$

$$= 1-\frac{1}{\sqrt{1+(\sqrt{\pi/2}(\mathrm{avg}\{|}}$$

For relatively small ratios of $x=\sigma_\mu/\sigma$, a two-term expansion can b

$$\frac{1}{\sqrt{1+x^2}} \approx \frac{1}{(1+x^2/2)} \approx (1-$$

Hence, it facilitates for setting up an approximation for (39),

$$k_{\text{eff}} \approx \frac{1}{2}\left(\frac{\sigma_\mu}{\sigma}\right)^2 = \frac{1}{2}\left(\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{avg}\{|\bar{\mu}-}{\sigma}\right.$$

As expected, $k_{\text{eff}}=0$ for stable short-term processes (i.e., when σ_μ
instability in the short-term process. If there is no target bias, ther

Under the restriction (33), a unified approach, which includes in
mathematical thread-of-logic (Section 3). A long-term Model-A-ty
given by

$$p_{kA} = \Phi(-3C_{pk_o}+\delta_{bk})+\Phi(-$$

Following (8), (9), and (10) after the substitutions,

$$C_{p_o}\to C_{pk_o},\qquad \delta_b\to\delta_{bk},$$
$$p_A\to p_{kA},\qquad \sigma\to$$

6. Conclusion

It is has been shown that a short-term CPI model dependent on o
normalized target bias is equivalent to various exact CPI expres
principal advantage of this specific formulation is that it facilitates
of the process due to an offset between the mean and the target o
predicting short-term CPI is applicable to the long-term CPI, p
distributed Gaussian. Sufficient conditions to guarantee that the lo

discussed. Within the context of these assumed conditions, a new

Two implementation schemes for the proposed reformulation for dependent on the availability of a built-in error function and its approximation for the error function and its inverse.

The second scheme supplants the built-in functions used in th approximations. For a three sigma process, the results indicate tha than both of the industry CPI models tested. Our results indicate (Model-X) were more pessimistic than the exact model (Model-A) i

In the literature, methods have been reported to account for nonr would be interesting to demonstrate that any distribution can be the Model-A approach would again be applicable as long as a co equivalent Gaussian parameters are appropriately defined.

Appendices

A. Comparison of C_{pA} to Yield Index S_{pk}

In 1994, R. A. Boyles proposed a yield-based capability index [5] The current Model-A approach can be compared to that of Boyles'

$$S_{pk} = S(3C_{pl}, 3C_{p}$$

where C_{pl} and C_{pu} are the capability indices with mean μ, standar upper specification limit (USL) and are given by [5]

$$C_{pl} = \frac{\mu - \text{LSL}}{3\sigma}, \qquad C_{pu} =$$

The operator S in (A.1) for a standard normal cumulative distributi

$$S(x,y) = \frac{1}{3}\Phi^{-1}\left(\frac{\Phi(x)+}{2}\right.$$

Hence, the yield index in (A.1) can be represented as,

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\{\Phi(3C_{pl})+\right.$$

Substituting C_{pl} and C_{pu} from (A.2) (into (A.4)) gives the yield in standard deviation, and the specification limits:

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\left\{\Phi\left(\frac{\mu - \text{LSL}}{\sigma}\right)+\right.\right.$$

From the basic properties of normalized distribution functions

$$\Phi(x) = 1 - \Phi(-x)$$

and after noting that

$$\Phi^{-1}(x) = \Phi^{-1}[1 - \Phi\{\Phi^{-1}($$

The use of (A.6) leads to

$$\Phi^{-1}(x) = -\Phi^{-1}(1 -$$

Applying (A.8) on (A.5) leads to

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[1 - \frac{1}{2}\Phi\left(\frac{\mu - \mathrm{LSL}}{\sigma}\right)\right.$$

hence,

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\left(1 - \Phi\left(\frac{\mu - \mathrm{LSL}}{\sigma}\right)\right) + \right.$$

Applying (A.6) on (A.10) leads to

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[\frac{1}{2}\Phi\left(\frac{\mathrm{LSL} - \mu}{\sigma}\right) + \right.$$

Using (6) described in Section 2, it can be shown that (A.11) is

paper:

$$S_{pk} = -\frac{1}{3}\Phi^{-1}\left[\frac{p_A}{2}\right] =$$

B. Error Function Approximation

The error function is usually encountered while integrating the nor
of the Gaussian distribution [14]:

$$\mathrm{erf} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}$$

The error function has the values of 0 and 1 for $x = 0$ and $x =$
equation [2]:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\xi^2}$$

and substituting $y = \xi / 2$ and separating the integral for positive an

$$\Phi(z) = 0.5 + \left(\frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{2}}\right.$$

For negative values of *z*, (B.3) can be written as

$$\Phi(z) = 0.5 - \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_0^{|z|/}\right.$$

where $t = -y$. Now, using the definition of the error function in (B.1

$$\Phi(z) = 0.5 - 0.5\,\mathrm{erf}\left(\right.$$

Solving *z* in (B.5) gives

$$z = \sqrt{2}\,\mathrm{erf}^{-1}\left(\frac{0.5 -}{0.}\right.$$

where erf^{-1} is the inverse error function. Figure 6 shows the p
MATLAB function. The plot also depicts a comparison of the built-i
(13) [9, 10] described in Section 4. It can be observed from the
close proximity to the built-in error function. This confirms that th
built-in function as necessary.

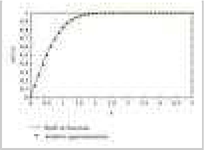


Figure 6: Comparison of built-in error functior

The inverse error function can also be implemented using the bui (14). Figure 7 confirms the compatibility of the analytic formula foı built-in routine.

Figure 7: Comparison of built-in inverse error

C. Histogram Approach to Long Term PDF Process

The heuristic gateway to the probabilistic approach for analysis is t being generated from the limit case of a histogram of mea: Specifically, the limit is for large number of measurements, *N*, a converted to an approximate PDF $f(y)$ at the value of process parǎ conversion rule [2] is

$$f(y_i) \approx \frac{n_i}{(N\Delta y)},$$

where n_i is the bin count in the $i\text{th}$ bin.

To demonstrate the linking of short-term (ST) PDFs to long-te processes are to be combined. Each of the $j = 1, 2, 3, \ldots, M$-associa have the same number of measurements, *N*, and bin size Δy. The summing the ST bin count numbers $(n_{ij})_{\text{ST}}$, that is,

$$(n_i)_{\text{LT}} = \sum_{j=1}^{M} (n_{ij})_{\text{S}}$$

with a total requisite LT number of measurements:

$$N' = N \times M.$$

Following the rule (C.1), the long term approximate PDF is given b

$$f_{\text{LT}}(y_i) \approx \frac{(n_i)_{\text{LT}}}{(N'\Delta y)} = \frac{\Sigma_{j=1}^{M} (n_{ij})_{\text{ST}}}{(NM\Delta y)}$$

The limits of large *N* and infinitesimal bin size lead to (30). This cc parameter PDF can be constructed from the average of the PDFs fc

D. Random Variable Analysis of Long Term PDF

The steps leading to (36) main text are summarized in (D.1) - (D. into (35), the PDF for the long term process is

$$f_{LT}(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{\sigma_\mu \sigma} \int_{-\infty}^{\infty} e^{-[(\mu-\bar{\mu})/}$$

The substitution

$$t = \mu - y,$$

$$g = \bar{\mu} - y$$

will facilitate completing-the-square process for the argument of th
associated with the Gaussian *t* variable is given by

$$\sigma_t^2 = \frac{\sigma^2 \sigma_\mu^2}{\sigma^2 + \sigma_\mu^2}.$$

An additional substitution,

$$g' \equiv g\left(\frac{\sigma_t^2}{\sigma_\mu^2}\right),$$

will then lead to a more compact form given by

$$f_{LT}(g) = \left(\frac{1}{\sqrt{2\pi}}\right) \frac{\sigma_t}{\sigma_\mu \sigma} e^{-g^2/2\sigma}$$

$$\times\left[\frac{1}{\sqrt{2\pi}\,\sigma_t} \int_{-\infty}^{\infty} e^{-((t-}\right.$$

Additional simplifications are possible after two observations. First,
the maximum limit of a Gaussian CDF [2]. Second, it follows from

$$\frac{\sigma_t}{\sigma_\mu \sigma} = \frac{1}{\sqrt{\sigma^2 + \sigma_\mu^2}}$$

Combining observations and returning to “*y*” dependence via (D.

$$f_{LT}(y) = \left(\frac{1}{\sqrt{2\pi}}\right) \frac{1}{\sigma_k} e^{-(y-}$$

where the variance of the long-term distribution is

$$\sigma_k^2 = \sigma^2 + \sigma_\mu^2,$$

and this confirms (36) in the main text. The steps leading to (36)
using a transformation

$$z = \frac{\mu - \bar{\mu}}{\sigma_\mu},$$

the Gaussian PDF in random variable μ (33) can be converted to th

$$f_z(z) = \left(\frac{1}{\sqrt{2\pi}}\right) e^{-z^2/2}, \quad -$$

Again, making use of (D.10) the expectation of $|\mu - \bar{\mu}|$ is given by

$$E\{|\mu - \bar{\mu}|\} = \int_{-\infty}^{\infty} f_\mu(\mu)\,|\mu - \bar{\mu}|\,d\mu =$$

because the PDF $f_z(z)$ is by inspection (D.12) an even function. Th
be converted to $[0, \infty)$ by including a multiplicative factor of two. A
given by

$$E\{|\mu-\overline{\mu}|\}=\frac{2\sigma_\mu}{\sqrt{2\pi}}\int_0^\infty e^-$$

because $|z|=z$ on the new domain of integration. Evaluation of (D.

$$E\{|\mu-\overline{\mu}|\}=\text{avg}\{|\mu-\overline{\mu}|$$

which leads to (37).

Acknowledgments

This work was supported in part by Office of Sponsored Researcl appreciate the helpful remarks and suggestions provided by the pointed the thread-of-logic, summarized in Appendix 1, relatin previously appearing in the literature [5]. They are also grateful to his approximate analytic models for the error function and its inver

References

1. G. Taguchi and Y. Wu, *Introduction to Off-Line Quality Contr* Nagoya, Japan, 1979.
2. E. E. Lewis, *Introduction to Reliability Engineering*, John Wile
3. S. Kotz, W. L. Pearn, and N. L. Johnson, “Some process cap think,” *Applied Statistics*, vol. 42, no. 1, pp. 55 - 62, 1993.
4. E. Kureková, “Measurement process capability—trends and 1, no. 1, pp. 43 - 46, 2001.
5. R. A. Boyles, “Process capability with asymmetric tolerance *Computation*, vol. 23, no. 3, pp. 615 - 635, 1994.
6. J. P. Chen and C. G. Ding, “A new process capability index *and Reliability Management*, vol. 18, no. 6-7, 762 pages, 20
7. L. K. Chan, S. W. Cheng, and F. A. Spring, “A new measure *Technology*, vol. 20, pp. 162 - 175, 1998.
8. “A guide to Using Cpk,” The Association for Manufacturing http://www.amtonline.org/document_display.cfm/document aprocesscapabilityindex.
9. S. Winitzki, “A handy approximation for the error function a private communication.
10. S. Winitzki, “Uniform approximations for transcendental fur *Conference on Computational Science and Its Applications (* *Science*, pp. 780 - 789, Montreal, Canada, May 2003.
11. R. J. Pieper and N. T. Satyala, “An improved characterizatic on manufacturing target bias,” in *Proceedings of the 40th A* *(SSST '08)*, pp. 113 - 117, New Orleans, La, USA, March 200
12. “MATLAB Software,” The Math Works Inc., www.mathwork

13. M. L. Harry and J. R. Lawson, *Six Sigma Productivity Analys*
Wesley, Reading, Mass, USA, 1992.

14. M. R. Spiegel and J. Liu, *Mathematical Handbook of Formula*
Hill, New York, NY, USA, 2nd edition, 1999.