# Jacobians of singular matrix transformations: Extensions 

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#### Abstract

This article presents a unified approach to simultaneously compute the Jacobians of several singular matrix transformations in the real, complex, quaternion and octonion cases. Formally, these Jacobians are obtained for real normed division algebras with respect to the Hausdorff measure.


## 1 Introduction

A fundamental tool in statistical theory and in particular in distribution theory is the computation of Jacobians of matrix transformations. Many such Jacobians were first found in the real case and subsequently for complex cases; among many others, see Wooding (1956); James (1964, Sections 4 and 8); Khatri (1965), Muirhead (1982); Ratnarajah et al. (2005) and Mathai (1997) for a detailed review of this topic. Most of these Jacobians were obtained with respect to the Lebesgue measure. However, several recent articles have examined these Jacobian for the singular random matrix case. i.e., they study densities and then calculate the Jacobians of the matrix transformations with respect to the Hausdorff measure, see Khatri (1968), Uhlig (1994), Díaz-García et al. (1997), Díaz-García and Gutiérrez (1997), Srivastaval (2003), Díaz-García and Gutiérrez-Jáimez (2006), Díaz-García and González-Farías (2005a), Díaz-García and González-Farías (2005b), Ip et al. (2007) and Díaz-García (2007), among others. In addition, Ratnarajah and Villancourt (2005) studied some of these latter Jacobians in the singular complex case and Li and Xue (2010) considered the singular quaternion case.

Using results obtained from abstract algebra, it is possible to demonstrate a unified means of addressing the computation of Jacobians in the singular and nonsingular real, complex, quaternion and octonion cases, simultaneously. This approach has been used for

[^0]some time in random matrix theory in the nonsingular case, see Edelman and Rao (2005) and Forrester (2009).

For the sake of completeness, in the present work we consider the case of octonions, but it should be noted that many results for the octonion case can only be conjectured, because there remain many unresolved theoretical problems in this respect, see Dray and Manogue (1999). Furthermore, as stated by Baez (2002), the relevance of the octonion case for understanding the real world has yet to be clarified.

The rest of this paper is structured as follows: Section 2 provides some definitions and notation on real normed division algebras, showing some ideas about the explicit form of the Hasusdorff measure. The main results on the computation of Jacobians for singular matrix transformations are presented in Section 3. It is emphasised that all these results are obtained for real normed division algebras.

## 2 Notation and real normed division algebras

Let us introduce some notation and useful results. A detailed discussion of real normed division algebras may be found in Baez (2002). For convenience, we shall introduce some notation, although in general we adhere to standard forms.

For the purposes of this study, a vector space is always a finite-dimensional module over the field of real numbers. An algebra $\mathfrak{F}$ is a vector space that is equipped with a bilinear map $m: \mathfrak{F} \times \mathfrak{F} \rightarrow \mathfrak{F}$ termed multiplication and a nonzero element $1 \in \mathfrak{F}$ termed the unit such that $m(1, a)=m(a, 1)=1$. As usual, we abbreviate $m(a, b)=a b$ as $a b$. We do not assume $\mathfrak{F}$ associative. Given an algebra, we freely think of real numbers as elements of this algebra via the map $\omega \mapsto \omega 1$.

An algebra $\mathfrak{F}$ is a division algebra if given $a, b \in \mathfrak{F}$ with $a b=0$, then either $a=0$ or $b=0$. Equivalently, $\mathfrak{F}$ is a division algebra if the operation of left and right multiplications by any nonzero element is invertible. A normed division algebra is an algebra $\mathfrak{F}$ that is also a normed vector space with $\|a b\|=\|a\|\| \| b \|$. This implies that $\mathfrak{F}$ is a division algebra and that $\|1\|=1$.

There are exactly four normed division algebras: real numbers ( $\Re$ ), complex numbers $(\mathfrak{C})$, quaternions $(\mathfrak{H})$ and octonions $(\mathfrak{O})$, see Baez $(2002)$. We take into account that $\Re$, $\mathfrak{C}, \mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras; moreover, they are the only alternative division algebras, and all division algebras have a real dimension of $1,2,4$ or 8 , which is denoted by $\beta$, see Baez (2002, Theorems 1, 2 and 3). In other branches of mathematics, parameters $\alpha=2 / \beta$ and $t=\beta / 4$ are used, see Table 1 and Edelman and Rao (2005) and Kabe (1984).

Table 1: Values of $\beta=2 / \alpha$ and $t=\beta / 4$ parameters.

| $\beta$ | $\alpha$ | $t$ | Normed divison algebra |
| :--- | :--- | :--- | :--- |
| 1 | 2 | $1 / 4$ | real $(\Re)$ |
| 2 | 1 | $1 / 2$ | complex $(\mathfrak{C})$ |
| 4 | $1 / 2$ | 1 | quaternionic $(\mathfrak{H})$ |
| 8 | $1 / 4$ | 2 | octonion $(\mathfrak{O})$ |

Let $\mathfrak{L}_{m, n}^{\beta}$ be the linear space of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{F}$ with $m$ distinct positive singular values, where $\mathfrak{F}$ denotes a real finite-dimensional normed division algebra. Let $\mathfrak{F}^{n \times m}$ be the set of all $n \times m$ matrices over $\mathfrak{F}$. The dimension of $\mathfrak{F}^{n \times m}$ over $\Re$ is $\beta m n$. Let $\mathbf{A} \in \mathfrak{F}^{n \times m}$, then $\mathbf{A}^{*}=\overline{\mathbf{A}}^{T}$ denotes the usual conjugate transpose.

The set of matrices $\mathbf{H}_{1} \in \mathfrak{F}^{n \times m}$ such that $\mathbf{H}_{1}^{*} \mathbf{H}_{1}=\mathbf{I}_{m}$ is a manifold denoted $\mathcal{V}_{m, n}^{\beta}$, and is termed the Stiefel manifold $\left(\mathbf{H}_{1}\right.$ is also known as semi-orthogonal $(\beta=1)$, semiunitary $(\beta=2)$, semi-symplectic $(\beta=4)$ and semi-exceptional type $(\beta=8)$ matrices, see Dray and Manogue (1999)). The dimension of $\mathcal{V}_{m, n}^{\beta}$ over $\Re$ is $[\beta m n-m(m-1) \beta / 2-m]$. In particular, $\mathcal{V}_{m, m}^{\beta}$ with dimension over $\Re,[m(m+1) \beta / 2-m]$, is the maximal compact subgroup $\mathfrak{U}^{\beta}(m)$ of $\mathfrak{L}_{m, m}^{\beta}$ and consists of all matrices $\mathbf{H} \in \mathfrak{F}^{m \times m}$ such that $\mathbf{H}^{*} \mathbf{H}=\mathbf{I}_{m}$. Therefore, $\mathfrak{U}^{\beta}(m)$ is the real orthogonal group $\mathcal{O}(m)(\beta=1)$, the unitary group $\mathcal{U}(m)$ $(\beta=2)$, compact symplectic group $\mathcal{S p}(m)(\beta=4)$ or exceptional type matrices $\mathcal{O} o(m)$ $(\beta=8)$, for $\mathfrak{F}=\Re, \mathfrak{C}, \mathfrak{H}$ or $\mathfrak{O}$, respectively.

We denote by $\mathfrak{S}_{m}^{\beta}$ the real vector space of all $\mathbf{S} \in \mathfrak{F}^{m \times m}$ such that $\mathbf{S}=\mathbf{S}^{*}$. Let $\mathfrak{P}_{m}^{\beta}$ be the cone of positive definite matrices $\mathbf{S} \in \mathfrak{F}^{m \times m}$; then $\mathfrak{P}_{m}^{\beta}$ is an open subset of $\mathfrak{S}_{m}^{\beta}$. Over $\Re, \mathfrak{S}_{m}^{\beta}$ consist of symmetric matrices; over $\mathfrak{C}$, Hermitian matrices; over $\mathfrak{H}$, quaternionic Hermitian matrices (also termed self-dual matrices) and over $\mathfrak{O}$, octonionic Hermitian matrices. Generically, the elements of $\mathfrak{S}_{m}^{\beta}$ are termed Hermitian matrices, irrespective of the nature of $\mathfrak{F}$. The dimension of $\mathfrak{S}_{m}^{\beta}$ over $\Re$ is $[m(m-1) \beta+2] / 2$.
Let $\mathfrak{D}_{m}^{\beta}$ be the diagonal subgroup of $\mathcal{L}_{m, m}^{\beta}$ consisting of all $\mathbf{D} \in \mathfrak{F}^{m \times m}, \mathbf{D}=\operatorname{diag}\left(d_{1}, \ldots, d_{m}\right)$ and let $\mathfrak{T}_{\mathrm{U}_{m, n}^{+\beta}}$ be the semi-upper-triangular subgroup of $\mathcal{L}_{m, n}^{\beta}$ consisting of all $\mathbf{T} \in \mathfrak{F}^{n \times m}$, with $t_{i i}>0$.

Now, let $\mathfrak{L}_{m, n}^{+\beta}(q)$ be the linear space of all $n \times m$ matrices of $\operatorname{rank} q \leq \min (n, m)$ with $q$ distinct singular values and let $\mathfrak{S}_{m}^{+\beta}(q)$, the $(\beta m q-\beta q(q-1) / 2)-q$-dimensional manifold of rank $q$ of positive semidefinite matrices $\mathbf{S} \in \mathfrak{F}^{m \times m}$ with $q$ distinct positive eigenvalues.

For any matrix $\mathbf{X} \in \mathfrak{F}^{n \times m}, d \mathbf{X}$ denotes the matrix of differentials ( $d x_{i j}$ ), and we denote the measure or volume element as $(d \mathbf{X})$ when $\mathbf{X} \in \mathfrak{F}^{m \times n}$, see Dimitriu (2002).

If $\mathbf{X} \in \mathfrak{F}^{n \times m}$ then $(d \mathbf{X})$ (the Lebesgue measure in $\mathfrak{F}^{n \times m}$ ) denotes the exterior product of the $\beta m n$ functionally independent variables

$$
(d \mathbf{X})=\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} d x_{i j} \quad \text { where } \quad d x_{i j}=\bigwedge_{k=1}^{\beta} d x_{i j}^{(k)}
$$

If $\mathbf{S} \in \mathfrak{S}_{m}^{\beta}\left(\right.$ or $\left.\mathbf{S} \in \mathfrak{T}_{\mathrm{U}_{m, n}^{+\beta}}^{+\beta}\right)$ then $(d \mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}_{m}^{\beta}$ or in $\mathfrak{T}_{\mathrm{U}_{m, n}^{+}}^{+\beta}$ ) denotes the exterior product of the $m(m+1) \beta / 2$ functionally independent variables (or denotes the exterior product of the $m(m-1) \beta / 2+n$ functionally independent variables, if $s_{i i} \in \Re$ for all $i=1, \ldots, m$ )

$$
(d \mathbf{S})=\left\{\begin{array}{l}
\bigwedge_{i \leq j}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)}, \\
\bigwedge_{i=1}^{m} d s_{i i} \bigwedge_{i<j}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)}, \quad \text { if } s_{i i} \in \Re
\end{array}\right.
$$

The context generally establishes the conditions on the elements of $\mathbf{S}$, that is, if $s_{i j} \in \Re$, $\in \mathfrak{C}, \in \mathfrak{H}$ or $\in \mathfrak{O}$. It is considered that

$$
(d \mathbf{S})=\bigwedge_{i \leq j}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)} \equiv \bigwedge_{i=1}^{m} d s_{i i} \bigwedge_{i<j}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)}
$$

Observe, too, that for the Lebesgue measure ( $d \mathbf{S}$ ) defined thus, it is required that $\mathbf{S} \in \mathfrak{P}_{m}^{\beta}$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian positive definite matrix).

If $\boldsymbol{\Lambda} \in \mathfrak{D}_{m}^{\beta}$ then $(d \boldsymbol{\Lambda})$ (the Lebesgue measure in $\left.\mathfrak{D}_{m}^{\beta}\right)$ denotes the exterior product of the
$\beta m$ functionally independent variables

$$
(d \boldsymbol{\Lambda})=\bigwedge_{i=1}^{n} \bigwedge_{k=1}^{\beta} d \lambda_{i}^{(k)}
$$

In addition, observe that, if $\mathbf{X} \in \mathfrak{L}_{m, n}^{+\beta}(q)$, we can write $\mathbf{X}$ as

$$
\mathbf{X}_{1}=\left(\begin{array}{cc}
\mathbf{X}_{11} & \underset{q \times \mathbf{X}_{12}}{ } \\
\underset{q \times m-q}{ } \\
\mathbf{X}_{21} & \underset{n-q \times m-q}{ }
\end{array}\right)
$$

such that $r\left(\mathbf{X}_{11}\right)=q$. This is equivalent to the right product of the matrix $\mathbf{X}$ with a permutation matrix $\boldsymbol{\Pi}$, see Golub and Van Loan (1996, section 3.4.1, 1996), that is $\mathbf{X}_{1}=$ $\mathbf{X \Pi}$. Note that the exterior product of the elements from the differential matrix $d \mathbf{X}$ is not affected by the fact that we multiply $\mathbf{X}$ (right or left) by a permutation matrix, that is, $\left(d \mathbf{X}_{1}\right)=(d(\mathbf{X \Pi}))=(d \mathbf{X})$, since $\boldsymbol{\Pi} \in \mathfrak{U}^{\beta}(m)$, see Muirhead (1982, Section 2.1, 1982) and James (1954). Then, without loss of generality, ( $d \mathbf{X}$ ) will be defined as the exterior product of the differentials $d x_{i j}$, such that $x_{i j}$ are mathematically independent. It is important to note that we will have $\left(n q+m q-q^{2}\right) \beta$ mathematically independent elements in the matrix $\mathbf{X} \in \mathcal{L}_{m, n}^{+\beta}(q)$, corresponding to the elements of $\mathbf{X}_{11}, \mathbf{X}_{12}$ and $\mathbf{X}_{21}$. Explicitly,

$$
\begin{equation*}
(d \mathbf{X}) \equiv\left(d \mathbf{X}_{11}\right) \wedge\left(d \mathbf{X}_{12}\right) \wedge\left(d \mathbf{X}_{21}\right)=\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{q} \bigwedge_{k=1}^{\beta} d x_{i j}^{(k)} \bigwedge_{i=1}^{q} \bigwedge_{j=q+1}^{m} \bigwedge_{k=1}^{\beta} d x_{i j}^{(k)} \tag{1}
\end{equation*}
$$

Similarly, given $\mathbf{S} \in \mathfrak{S}_{m}^{+\beta}(q)$, we define $(d \mathbf{S})$ as

$$
(d \mathbf{S})=\left\{\begin{array}{l}
\bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)}, \\
\bigwedge_{i=1}^{q} d s_{i i} \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)}, \quad \text { if } s_{i i} \in \Re
\end{array}\right.
$$

The context generally establishes the conditions on the elements of $\mathbf{S}$, that is, if $s_{i j} \in \Re$, $\in \mathfrak{C}, \in \mathfrak{H}$ or $\in \mathfrak{O}$. It is considered that

$$
(d \mathbf{S})=\bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)} \equiv \bigwedge_{i=1}^{q} d s_{i i} \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} \bigwedge_{k=1}^{\beta} d s_{i j}^{(k)}
$$

Again, we should note that, for this case, the matrix $\mathbf{S}$ can be written as

$$
\mathbf{S} \equiv\left(\begin{array}{cc}
\mathbf{S}_{11} & \mathbf{S}_{12} \\
q \times q & q \times m-q \\
\mathbf{S}_{21} & \mathbf{S}_{22} \\
m-q \times q & m-q \times m-q
\end{array}\right) \quad \text { with } \quad r\left(\mathbf{S}_{11}\right)=q
$$

such that the number of mathematically independent elements in $\mathbf{S}$ is $\beta m q-q(q+1) \beta / 2-q$ corresponding to the mathematically independent elements of $\mathbf{S}_{12}$ and $\mathbf{S}_{11}$. Recall that $\mathbf{S}_{11} \in \mathfrak{P}_{q}^{\beta}$, in such a way that $\mathbf{S}_{11}$ has $q(q-1) \beta / 2+q$, therefore,

$$
(d \mathbf{S}) \equiv\left(d \mathbf{S}_{11}\right) \wedge\left(d \mathbf{S}_{12}\right)
$$

Observe that an explicit form for $(d \mathbf{X})$ and $(d \mathbf{S})$ depends on the factorisation (base and coordinate set) employed to represent $\mathbf{X}$ or $\mathbf{S}$, that is, they depend on the measure
factorisation of $(d \mathbf{X})$ and $(d \mathbf{S})$. For example, by using the nonsingular part of the decomposition in singular values and the nonsingular part of the spectral decomposition for $\mathbf{X}$ and $\mathbf{S}$, then we can find an explicit form for $(d \mathbf{X})$ and $(d \mathbf{S})$ (see Propositions 3.1 and 3.2 , respectively), which are not unique, see Khatri (1968), Díaz-García et al. (1997), Uhlig (1994) and Díaz-García and Gutiérrez (1997). Alternatively, an explicit form for ( $d \mathbf{X}$ ) and $(d \mathbf{S})$ can be found in terms of the QR and Cholesky decompositions, see Propositions 3.3 and 3.4, respectively.

A singular random matrix $\mathbf{X}$ in $\mathfrak{L}_{m, n}^{+\beta}(q)$ or $\mathfrak{S}_{m}^{+\beta}(q)$ does not have a density with respect to Lebesgue's measure in $\mathfrak{F}^{n \times m}$, but it does possess a density on a subspace $\mathcal{M} \subset \mathfrak{F}^{n \times m}$; see Khatri (1968), Rao (1973, p. 527), Díaz-García et al. (1997), Uhlig (1994) and Cramér (1999, p. 297). Formally, $\mathbf{X}$ has a density with respect to Hausdorff's measure, which coincides with Lebesgue's measure, when the latter is defined on the subspace $\mathcal{M}$; see Billingsley (1986, p. 247), Uhlig (1994), Díaz-García et al. (1997), Díaz-García and Gutiérrez-Jáimez (2005) and Díaz-García and Gutiérrez-Jáimez (2009).

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m, n}^{\beta}$ is

$$
\begin{equation*}
\operatorname{Vol}\left(\mathcal{V}_{m, n}^{\beta}\right)=\int_{\mathbf{H}_{1} \in \mathcal{V}_{m, n}^{\beta}}\left(\mathbf{H}_{1}^{*} d \mathbf{H}_{1}\right)=\frac{2^{m} \pi^{m n \beta / 2}}{\Gamma_{m}^{\beta}[n \beta / 2]} \tag{2}
\end{equation*}
$$

where $\mathbf{H}=\left(\mathbf{H}_{1} \mid \mathbf{H}_{2}\right)=\left(\mathbf{h}_{1}, \ldots, \mathbf{h}_{m} \mid \mathbf{h}_{m+1}, \ldots, \mathbf{h}_{n}\right) \in \mathfrak{U}^{\beta}(m)$. It can be proved that this differential form does not depend on the choice of the $\mathbf{H}_{2}$ matrix. When $m=1 ; \mathcal{V}_{1, n}^{\beta}$ defines the unit sphere in $\mathfrak{F}^{n}$. This is, of course, an $(n-1) \beta$ - dimensional surface in $\mathfrak{F}^{n}$. When $m=n$ and denoting $\mathbf{H}_{1}$ by $\mathbf{H},\left(\mathbf{H}^{*} d \mathbf{H}\right)$ is termed the Haar measure on $\mathfrak{U}^{\beta}(m)$. Also, $\Gamma_{m}^{\beta}[a]$ denotes the multivariate Gamma function for the space $\mathfrak{S}_{m}^{\beta}$, and is defined by

$$
\begin{aligned}
\Gamma_{m}^{\beta}[a] & =\int_{\mathbf{A} \in \mathfrak{P}_{m}^{\beta}} \operatorname{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1) \beta / 2-1}(d \mathbf{A}) \\
& =\pi^{m(m-1) \beta / 4} \prod_{i=1}^{m} \Gamma[a-(i-1) \beta / 2]
\end{aligned}
$$

where $\operatorname{etr}(\cdot)=\exp (\operatorname{tr}(\cdot)),|\cdot|$ denotes the determinant and $\operatorname{Re}(a)>(m-1) \beta / 2$, see Gross and Richards (1987).

## 3 Jacobians

We now consider several Jacobians of singular matrix transformations in terms of the $\beta$ parameter. For a detailed discussion of related issues see Uhlig (1994), Díaz-García and Gutiérrez (1997), Díaz-García et al. (1997), Ratnarajah and Villancourt (2005), Ip et al. (2007), Díaz-García and Gutiérrez-Jáimez (2012) and Li and Xue (2010).

Propositions 3.1 and 3.2 and Corollary 3.1 generalise the results in Díaz-García et al. (1997), Ratnarajah and Villancourt (2005) and Li and Xue (2010) obtained for the real, complex and quaternion cases, respectively.

Proposition 3.1 (Singular value decomposition, $S V D)$. Let $\mathbf{X} \in \mathfrak{L}_{m, n}^{+\beta}(q)$, such that $\mathbf{X}=\mathbf{V}_{1} \mathbf{D} \mathbf{W}_{1}^{*}$ be the nonsingular part of the $S V D$ with, $\mathbf{V}_{1} \in \mathcal{V}_{q, n}^{\beta}, \mathbf{W}_{1} \in \mathcal{V}_{q, m}^{\beta}$ and $\mathbf{D}=\operatorname{diag}\left(d_{1}, \cdots, d_{q}\right) \in \mathfrak{D}_{m}^{1}, d_{1}>\cdots>d_{q}>0$. Then

$$
\begin{equation*}
(d \mathbf{X})=2^{-q} \pi^{\tau} \prod_{i=1}^{q} d_{i}^{\beta(n+m-2 q+1)-1} \prod_{i<j}^{q}\left(d_{i}^{2}-d_{j}^{2}\right)^{\beta}(d \mathbf{D}) \wedge\left(\mathbf{V}_{1}^{*} d \mathbf{V}_{1}\right) \wedge\left(\mathbf{W}_{1}^{*} d \mathbf{W}_{1}\right) \tag{3}
\end{equation*}
$$

where

$$
\tau=\left\{\begin{aligned}
0, & \beta=1 \\
-\beta q / 2, & \beta=2,4,8
\end{aligned}\right.
$$

Proposition 3.2 (Spectral decomposition, $S D$ ). Let $\mathbf{S} \in \mathfrak{S}_{m}^{+\beta}(q)$. Then the nonsingular part of the spectral decomposition can be written as $\mathbf{S}=\mathbf{W}_{1} \boldsymbol{\Lambda} \mathbf{W}_{1}^{*}$, where $\mathbf{W}_{1} \in \mathcal{V}_{q, m}^{\beta}$ and $\boldsymbol{\Lambda}=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right) \in \mathfrak{D}_{q}^{1}$, with $\lambda_{1}>\cdots>\lambda_{q}>0$. Then

$$
\begin{equation*}
(d \mathbf{S})=2^{-q} \pi^{\tau} \prod_{i=1}^{q} \lambda_{i}^{\beta(m-q)} \prod_{i<j}^{q}\left(\lambda_{i}-\lambda_{j}\right)^{\beta}(d \boldsymbol{\Lambda}) \wedge\left(\mathbf{W}_{1}^{*} d \mathbf{W}_{1}\right) \tag{4}
\end{equation*}
$$

where $\tau$ is defined in Proposition 3.1.
Corollary 3.1. Let $\mathbf{X} \in \mathfrak{L}_{m, n}^{+\beta}(q)$, and $\mathbf{S}=\mathbf{X}^{*} \mathbf{X}=\mathbf{W}_{1} \mathbf{\Lambda} \mathbf{W}_{1}^{*} \in \mathfrak{S}_{m}^{+\beta}(q)$. Then

$$
\begin{equation*}
(d \mathbf{X})=2^{-q}|\boldsymbol{\Lambda}|^{\beta(n-m+1) / 2-1}(d \mathbf{S}) \wedge\left(\mathbf{V}_{1}^{*} d \mathbf{V}_{1}\right) \tag{5}
\end{equation*}
$$

with $\mathbf{V}_{1} \in \mathcal{V}_{n, q}^{\beta}$.
Proof. The proof is obtained immediately from Propositions 3.1 and 3.2 .
Let $\mathbf{A} \in \mathfrak{L}_{n, m}^{+\beta}(q)$ then $\mathbf{A}^{+} \in \mathfrak{L}_{n, m}^{+\beta}(q)$ denotes its Moore-Penrose inverse.
Theorem 3.1 (Moore-Penrose inverse $\mathbf{S}=\mathbf{S}^{*}$ ). Let $\mathbf{S} \in \mathfrak{S}_{m}^{+\beta}(q)$ as in Proposition 3.2. Then ignoring the sign, if $\mathbf{V}=\mathbf{S}^{+} \in \mathfrak{S}_{m}^{+\beta}(q)$

$$
\begin{equation*}
(d \mathbf{V})=\prod_{i=1}^{q} \lambda_{i}^{\beta(-2 m+q+1)-2}(d \mathbf{S}) \tag{6}
\end{equation*}
$$

Proof. The proof follows by observing that if $\mathbf{S}=\mathbf{W}_{1} \boldsymbol{\Lambda} \mathbf{W}_{1}^{*}$ is the nonsingular part of the spectral decomposition of $\mathbf{S}$, then $\mathbf{V}=\mathbf{S}^{+}=\mathbf{W}_{1} \boldsymbol{\Lambda}^{-1} \mathbf{W}_{1}^{*}$. Therefore, from (4)

$$
\begin{equation*}
(d \mathbf{S})=2^{-q} \pi^{\tau} \prod_{i=1}^{q} \lambda_{i}^{\beta(m-q)} \prod_{i<j}^{q}\left(\lambda_{i}-\lambda_{j}\right)^{\beta}(d \mathbf{\Lambda}) \wedge\left(\mathbf{W}_{1}^{*} d \mathbf{W}_{1}\right) \tag{7}
\end{equation*}
$$

And

$$
(d \mathbf{V})=2^{-q} \pi^{\tau} \prod_{i=1}^{q} \lambda_{i}^{-\beta(m-q)-2} \prod_{i<j}\left(\lambda_{i}^{-1}-\lambda_{j}^{-1}\right)^{\beta}\left(\mathbf{W}_{1}^{*} d \mathbf{W}_{1}\right) \wedge(d \mathbf{\Lambda})
$$

taking into account that (ignoring the sign),

$$
\left(d \boldsymbol{\Lambda}^{-1}\right)=\bigwedge_{i=1}^{q} d \lambda_{i}^{-1}=\bigwedge_{i=1}^{q}(-1) \frac{d \lambda_{i}}{\lambda_{i}^{2}}=\prod_{i=1}^{q} \lambda_{i}^{-2} \bigwedge_{i=1}^{q} d \lambda_{i}=\prod_{i=1}^{q} \lambda_{i}^{-2}(d \boldsymbol{\Lambda}) .
$$

Then,

$$
\begin{equation*}
\left(\mathbf{W}_{1}^{*} d \mathbf{W}_{1}\right) \wedge(d \boldsymbol{\Lambda})=2^{q} \pi^{-\tau}\left[\prod_{i=1}^{q} \lambda_{i}^{-\beta(m-q)-2} \prod_{i<j}\left(\lambda_{i}^{-1}-\lambda_{j}^{-1}\right)^{\beta}\right]^{-1}(d \mathbf{V}) \tag{8}
\end{equation*}
$$

Finally (ignoring the sign), observe that,

$$
\begin{equation*}
\prod_{i<j} \frac{\left(\lambda_{i}^{-1}-\lambda_{j}^{-1}\right)^{\beta}}{\left(\lambda_{i}-\lambda_{j}\right)^{\beta}}=\prod_{i<j} \frac{1}{\left(\lambda_{i} \lambda_{j}\right)^{\beta}}=\prod_{i=1}^{q} \lambda_{i}^{-\beta(q-1)} \tag{9}
\end{equation*}
$$

The desired result is obtained by substituting (8) and (9) in (7).

Theorem 3.2 (Moore-Penrose inverse). Let $\mathbf{X} \in \mathfrak{L}_{n, m}^{+\beta}(q)$ as in Proposition [3.1. Then ignoring the sign, if $\mathbf{Y}=\mathbf{X}^{+}$

$$
\begin{equation*}
(d \mathbf{Y})=\prod_{i=1}^{q} d_{i}^{-2 \beta(m+n-q)}(d \mathbf{X}) \tag{10}
\end{equation*}
$$

Proof. The proof is analogous to that given for Theorem 3.1, using (3).
Theorems 3.1 and 3.2 were obtained by Díaz-García et al. (1997) and Li and Xue (2010) for the real and quaternion cases, respectively.

The next result was proposed by Uhlig (1994) in the real case as a conjecture. Subsequently, Díaz-García and Gutiérrez (1997) proposed an indirect proof of this conjecture. Later Díaz-García and Gutiérrez-Jáimez (2009b) provided an alternative proof based on the exterior product, also in the real case. In 2010, Li and Xue (2010) extended this result to the quaternion case, generalising the indirect proof stated in Díaz-García and Gutiérrez (1997). We now propose two alternative statements of these results for real normed division algebras.
Theorem 3.3 (First Uhlig's conjecture via $S V D$ ). Let $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_{m}^{+\beta}(n)$ such that $\mathbf{X}=$ $\mathbf{B}^{*} \mathbf{Y B}$, where $\mathbf{B} \in \mathfrak{L}_{m, m}^{\beta}(m)$. In addition, consider the nonsingular part of the $S D, \mathbf{X}=$ $\mathbf{G}_{1} \boldsymbol{\Delta} \mathbf{G}_{1}^{*}$ and $\mathbf{Y}=\mathbf{H}_{1} \boldsymbol{\Lambda} \mathbf{H}_{1}^{*}$, where $\boldsymbol{\Delta}=\operatorname{diag}\left(\delta_{1}, \ldots, \delta_{n}\right), \boldsymbol{\Lambda}=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{n}\right) \in \mathcal{D}_{n}^{1}$ and $\mathbf{G}_{1}, \mathbf{H}_{1} \in \mathcal{V}_{n, m}^{\beta}$. Then

$$
\begin{align*}
(d \mathbf{X}) & =|\mathbf{B}|^{\beta n}\left|\mathbf{G}_{1}^{*} \mathbf{B}^{*} \mathbf{H}_{1}\right|^{\beta(m-n-1)+2}(d \mathbf{Y})  \tag{11}\\
& =|\mathbf{B}|^{\beta n}\left|\mathbf{H}_{1}^{*} \mathbf{B G}_{1}\right|^{\beta(m-n-1)+2}(d \mathbf{Y})  \tag{12}\\
& =|\mathbf{B}|^{\beta n}|\boldsymbol{\Delta}|^{\beta(m-n-1) / 2+1}|\boldsymbol{\Lambda}|^{-\beta(m-n-1) / 2-1}(d \mathbf{Y}), \tag{13}
\end{align*}
$$

with

$$
(d \mathbf{Y})=2^{-n} \pi^{\tau} \prod_{i=1}^{n} \lambda_{i}^{\beta(m-n)} \prod_{i<j}^{n}\left(\lambda_{i}-\lambda_{j}\right)^{\beta}(d \boldsymbol{\Lambda}) \wedge\left(\mathbf{H}_{1}^{*} d \mathbf{H}_{1}\right)
$$

Proof. Considering Propositions 3.1, 3.2 and Díaz-García and Gutiérrez-Jáimez (2012, Proposition 1) the proof is analogous to that given in Díaz-García and Gutiérrez-Jáimez (2009b) for the real case.

Proposition 3.3 ( $Q R$ decomposition, $Q R D$ ). Let $\mathbf{X} \in \mathfrak{L}_{n, m}^{+\beta}(q)$, then there exists a matrix $\mathbf{H}_{1} \in \mathcal{V}_{q, n}^{\beta}$ and an upper quasi-triangular matrix $\mathbf{T} \in \mathfrak{T}_{U_{m, q}^{+\beta}}^{+\beta}$ such that $\mathbf{X}=\mathbf{H}_{1} \mathbf{T}$ is the nonsingular part of the $Q R$ decomposition and

$$
\begin{equation*}
(d \mathbf{X})=\prod_{i=1}^{q} t_{i i}^{\beta(n-i+1)-1}\left(\mathbf{H}_{1}^{*} d \mathbf{H}_{1}\right) \wedge(d \mathbf{T}) \tag{14}
\end{equation*}
$$

Proposition 3.4 (Cholesky's decomposition, $C H D$ ). Let $\mathbf{S} \in \mathfrak{S}_{m}^{+\beta}(q)$. Then $\mathbf{S}=\mathbf{T}^{*} \mathbf{T}$, where $\mathbf{T} \in \mathfrak{T}_{U_{m, q}^{+\beta}}^{+\beta}$ such that

$$
\mathbf{T}=\left(\mathbf{T}_{1} \mathbf{T}_{2}\right)
$$

with $\mathbf{T}_{1} \in \mathfrak{T}_{U_{q, q}}^{+\beta}, q \times q$ upper triangular matrix. Also, let $\mathbf{X} \in \mathfrak{L}_{m, n}$, with $\mathbf{X}=\mathbf{H}_{1} \mathbf{T}(Q R$ Decomposition) and $\mathbf{S}=\mathbf{X}^{*} \mathbf{X}=\mathbf{T}^{*} \mathbf{T}$ (Cholesky decomposition) such that

$$
\mathbf{S}=\left(\begin{array}{cc}
\mathbf{T}_{1}^{*} \mathbf{T}_{1} & \mathbf{T}_{1}^{*} \mathbf{T}_{2} \\
\mathbf{T}_{2}^{*} \mathbf{T}_{1} & \mathbf{T}_{2}^{*} \mathbf{T}_{2}
\end{array}\right)=\left(\begin{array}{cc}
\underset{q \times q}{\mathbf{S}_{11}} & \underset{q \times m-q}{\mathbf{S}_{12}} \\
\underset{m-q \times q}{ } & \underset{m-q \times m-q}{\mathbf{S}_{22}}
\end{array}\right) \quad \text { with } \quad \mathbf{S}_{11} \in \mathfrak{P}_{q}^{\beta}
$$

Then
1.

$$
\begin{equation*}
(d \mathbf{S})=2^{q} \prod_{i=1}^{q} t_{i i}^{\beta(m-i)+1}(d \mathbf{T}) \tag{15}
\end{equation*}
$$

2. Also,

$$
\begin{align*}
(d \mathbf{X}) & =2^{-q}\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|^{\beta(n-m+1) / 2-1}(d \mathbf{S}) \wedge\left(\mathbf{H}_{1}^{*} d \mathbf{H}_{1}\right),  \tag{16}\\
& =2^{-q}\left|\mathbf{S}_{11}\right|^{\beta(n-m+1) / 2-1}(d \mathbf{S}) \wedge\left(\mathbf{H}_{1}^{*} d \mathbf{H}_{1}\right. \tag{17}
\end{align*}
$$

Propositions 3.3 and 3.4 were studied by Díaz-García and González-Farías (2005b) for the real singular case and by Li and Xue (2009) for the quaternion nonsingular case.

We now study an alternative approach to Theorem 3.3 with respect to an alternative factorisation measure based on QR and the Cholesky decomposition.
Theorem 3.4 (First Uhlig's conjecture via $Q R D$ ). Let $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_{m}^{+\beta}(n)$, such that $\mathbf{X}=$ $\mathbf{B}^{*} \mathbf{Y B}$, with $\mathbf{B} \in \mathfrak{L}_{m, m}^{+\beta}(m)$ fixed. Additionally, let $\mathbf{X}=\mathbf{T}^{*} \mathbf{T}$ and $\mathbf{Y}=\mathbf{L}^{*} \mathbf{L}$ with $\mathbf{T}, \mathbf{L} \in$ $\mathfrak{T}_{U_{m, n}}^{+\beta}$, such that $\mathbf{T}=\left(\mathbf{T}_{1} \mathbf{T}_{2}\right)$ and $\mathbf{L}=\left(\mathbf{L}_{1} \mathbf{L}_{2}\right)$, where $\mathbf{T}_{1}$ and $\mathbf{L}_{1}$ are $n \times n$ upper triangular matrices. Then

$$
\begin{equation*}
(d \mathbf{X})=\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|^{\beta(m-n-1) / 2+1}\left|\mathbf{L}_{1}^{*} \mathbf{L}_{1}\right|^{-\beta(m-n-1) / 2-1}|\mathbf{B}|^{\beta n}(d \mathbf{Y}) \tag{18}
\end{equation*}
$$

with

$$
(d \mathbf{Y})=2^{n} \prod_{i=1}^{n} l_{i i}^{\beta(m-i)+1}(d \mathbf{L})
$$

Proof. Let $\mathbf{Z} \in \mathfrak{L}_{n, m}^{+\beta}(n)$, such that $\mathbf{Y}=\mathbf{Z}^{*} \mathbf{Z}$. Then

$$
\begin{equation*}
\mathbf{X}=\mathbf{B}^{*} \mathbf{Y} \mathbf{B}=\mathbf{B}^{*} \mathbf{Z}^{*} \mathbf{Z B}=\mathbf{\Phi}^{*} \boldsymbol{\Phi}, \quad \text { with } \quad \boldsymbol{\Phi}=\mathbf{Z B} \tag{19}
\end{equation*}
$$

from Proposition [3.4, equation 17

$$
\begin{equation*}
(d \boldsymbol{\Phi})=2^{-n}\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|^{\beta(n-m+1) / 2-1}\left(\mathbf{Q}^{*} d \mathbf{Q}\right)(d \mathbf{X}) \tag{20}
\end{equation*}
$$

In which $\boldsymbol{\Phi}=\mathbf{Q T}$ with $\mathbf{T} \in \mathfrak{T}_{\mathrm{U}}^{\mathrm{m}, n}+\mathrm{Q}, \mathbf{Q} \in \mathfrak{U}^{\beta}(n)$, and $\mathbf{X}=\boldsymbol{\Phi}^{*} \boldsymbol{\Phi}=\mathbf{T}^{*} \mathbf{T}$. Then

$$
\begin{equation*}
(d \mathbf{X})=2^{n}\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|^{-\beta(n-m+1) / 2-1}\left(\mathbf{Q}^{*} d \mathbf{Q}\right)^{-1}(d \boldsymbol{\Phi}) \tag{21}
\end{equation*}
$$

Note that $d \boldsymbol{\Phi}=d \mathbf{Z B}$, and then by Díaz-García and Gutiérrez-Jáimez (2012, Proposition 1) we have that $(d \mathbf{\Phi})=|\mathbf{B}|^{\beta n}(d \mathbf{Z})$, from which, substituting in (21), we obtain

$$
\begin{equation*}
(d \mathbf{X})=2^{n}\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|^{-\beta(n-m+1) / 2-1}\left(\mathbf{Q}^{*} d \mathbf{Q}\right)^{-1}|\mathbf{B}|^{\beta n}(d \mathbf{Z}) \tag{22}
\end{equation*}
$$

Now $\mathbf{Y}=\mathbf{Z}^{*} \mathbf{Z}=\mathbf{L}^{*} \mathbf{L}$ with $\mathbf{L} \in \mathfrak{T}_{m, n}^{+\beta}$; from Lemma 3.4

$$
\begin{equation*}
(d \mathbf{Z})=2^{-n}\left|\mathbf{L}_{1}^{*} \mathbf{L}_{1}\right|^{\beta(n-m+1) / 2-1}(d \mathbf{Y})\left(\mathbf{H}^{*} d \mathbf{H}\right) \tag{23}
\end{equation*}
$$

where $\mathbf{Z}=\mathbf{H L}$, with $\mathbf{L} \in \mathfrak{T}_{\mathrm{U}_{m, n}{ }^{+}}$and $\mathbf{H} \in \mathfrak{U}^{\beta}(n)$. Moreover, note that, due to the uniqueness of Haar's measure, $\left(\mathbf{Q}^{*} d \mathbf{Q}\right)=\left(\mathbf{H}^{*} d \mathbf{H}\right)$, see James (1954). Thus, substituting (23) in (22), we obtain

$$
\begin{aligned}
(d \mathbf{X}) & =\left|\mathbf{L}_{1}^{*} \mathbf{L}_{1}\right|^{\beta(n-m+1) / 2-1}\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|^{-\beta(n-m+1) / 2-1}|\mathbf{B}|^{\beta n}(d \mathbf{Y}) \\
& =\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|^{\beta(m-n+1) / 2}\left|\mathbf{L}_{1}^{*} \mathbf{L}_{1}\right|^{\beta-(m-n+1) / 2}|\mathbf{B}|^{\beta n}(d \mathbf{Y})
\end{aligned}
$$

The following result combines Theorems 3.1 and 3.3, which enables us to study the multivariate F and beta distributions (also termed matrix multivariate beta type II and type I, respectively) for real normed division algebras in a class of singularity.
Theorem 3.5. Let $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_{m}^{+\beta}(n)$ such that $\mathbf{X}=\mathbf{B}^{*} \mathbf{Y}^{+} \mathbf{B}$, where $\mathbf{B} \in \mathfrak{L}_{m, m}^{\beta}(m)$. In addition, consider the nonsingular part of the $S D, \mathbf{X}=\mathbf{G}_{1} \Delta \mathbf{G}_{1}^{*}$ and $\mathbf{Y}=\mathbf{H}_{1} \boldsymbol{\Lambda} \mathbf{H}_{1}^{*}$, where $\mathbf{G}_{1}, \mathbf{H}_{1} \in \mathcal{V}_{n, m}^{\beta}$ and $\boldsymbol{\Delta}=\operatorname{diag}\left(\delta_{1}, \ldots, \delta_{n}\right), \boldsymbol{\Lambda}=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{n}\right) \in \mathcal{D}_{n}^{1}$. Then

$$
\begin{equation*}
(d \mathbf{X})=|\mathbf{B}|^{\beta n}|\boldsymbol{\Delta}|^{\beta(m-n-1) / 2+1}|\boldsymbol{\Lambda}|^{-\beta(3 m-n-1) / 2-1}(d \mathbf{Y}) \tag{24}
\end{equation*}
$$

with

$$
(d \mathbf{Y})=2^{-n} \pi^{\tau} \prod_{i=1}^{n} \lambda_{i}^{\beta(m-n)} \prod_{i<j}^{n}\left(\lambda_{i}-\lambda_{j}\right)^{\beta}(d \boldsymbol{\Lambda}) \wedge\left(\mathbf{H}_{1}^{*} d \mathbf{H}_{1}\right)
$$

Proof. Let $\mathbf{Z}=\mathbf{Y}^{+}$and observe that $\mathbf{Z}=\mathbf{H}_{1} \boldsymbol{\Lambda}^{-1} \mathbf{H}_{1}^{*}$. Then by Theorem 3.3,

$$
\begin{equation*}
(d \mathbf{X})=|\mathbf{B}|^{\beta n}|\boldsymbol{\Delta}|^{\beta(m-n-1) / 2+1}\left|\boldsymbol{\Lambda}^{-1}\right|^{-\beta(m-n-1) / 2-1}(d \mathbf{Z}) . \tag{25}
\end{equation*}
$$

Now by Theorem 3.1

$$
\begin{equation*}
(d \mathbf{Z})=\prod_{i=1}^{n} \lambda_{i}^{\beta(-2 m+n+1)-2}(d \mathbf{Y}) \tag{26}
\end{equation*}
$$

The result follows by substituting (26) into (25).
Observe that this result corrects an erratum in the exponent of the determinant of $\boldsymbol{\Lambda}$ in Díaz-García and Gutiérrez-Jáimez (2009b, Theorema 2.2), obtained in the real case.

## Conclusions

Note that the results presented here contain as special cases all versions of the results previously obtained in real, complex and quaternion and octonion cases, both for singular and nonsingular cases. For example from Theorem 3.4

$$
\begin{equation*}
(d \mathbf{X})=\left(\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|\left|\mathbf{L}_{1}^{*} \mathbf{L}_{1}\right|^{-1}\right)^{\beta(m-n-1) / 2+1}|\mathbf{B}|^{\beta n}(d \mathbf{Y}) \tag{27}
\end{equation*}
$$

Now, let $\mathbf{X}$ and $\mathbf{Y}$ be nonsingular matrices, that is, $m=n$, therefore $\mathbf{T}_{1}$ and $\mathbf{L}_{1}$ are square nonsingular triangular matrices and

$$
\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1}\right|=|\mathbf{X}|=\left|\mathbf{B}^{*} \mathbf{Y} \mathbf{B}\right|=\left|\mathbf{B}^{*} \mathbf{L}_{1}^{*} \mathbf{L}_{1} \mathbf{B}\right|=|\mathbf{B}|^{2}\left|\mathbf{L}_{1}^{*} \mathbf{L}_{1}\right|
$$

then,

$$
\begin{equation*}
|\mathbf{B}|^{2}=\left|\mathbf{T}_{1}^{*} \mathbf{T}_{1} \| \mathbf{L}_{1}^{*} \mathbf{L}_{1}\right|^{-1} \tag{28}
\end{equation*}
$$

The desired result

$$
(d \mathbf{X})=|\mathbf{B}|^{\beta(m-1)+2}(d \mathbf{Y})
$$

is obtained by substituting (28) into (27), see Mathai (1997) for real and complex cases and Li and Xue (2010) for the quaternion case, among others.

Finally, observe that the real dimension of real normed division algebras can be expressed as potentia of $2, \beta=2^{n}$ for $n=0,1,2,3$. Moreover, as observed by Kabe (1984), the results obtained in this work can be extended to hypercomplex cases; that is, for complex, bicomplex, biquaternion and bioctonion (or sedenionic) algebras, which of course are not division algebras (except the complex algebra), but are Jordan algebras, together with all their isomorphic algebras. Note, too, that hypercomplex algebras are obtained by replacing the real numbers with complex numbers in the construction of real normed division algebras. Thus, the results for hypercomplex algebras are obtained by simply replacing $\beta$ with $2 \beta$ in our results (we reiterate, as reported by Kabe (1984)).

## Acknowledgements

This paper was written during J. A. Díaz-García's stay as a visiting professor at the Department of Statistics and O. R. of the University of Granada, Spain.

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    Key words. Jacobians; Hausdorff measure; real, complex, quaternion and octonion singular random matrices; real normed division algebras.
    2000 Mathematical Subject Classification. 15A23; 15B33; 15A09; 15B52; 60E05

