International Conference on Estuaries and Coasts
November 9-11, 2003, Hangzhou, China

A MODIFIED GDRBEM MODEL FOR WAVE SCATTERING

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Abstract: In this paper, the GDRBEM (general dual reciprocity boundary element method) wave model (Zhu et al. 2000 and Liu 2001) for solving the mild-slope equation is extended to solve the modified mild-slope equation (Chamberlain and Porter 1995), in which both the bottom curvature term related to $\nabla^2 h$ and the slope-squared term related to $(\nabla h)^2$ neglected in the conventional mild-slope equation are included to account for relatively steep and rapidly undulating bathymetry. Using the modified GDRBEM model, wave scattering by a submerged parabolic shoal is calculated and compared with experimental solutions (Suh et al. 2001) and the GDRBEM solutions (Liu 2001). It is shown that the modified GDRBEM model is much more accurate than the GDRBEM model for wave propagation in a region where the bottom slope is not moderate and the mild-slope assumption is violated.

Key words: Wave scattering, The modified mild-slope equation, The GDRBEM model, A submerged parabolic shoal

1. INTRODUCTION

Since the mild-slope equation (MSE) was derived by Berkhoff (1972), it has proved to be a very useful governing equation for a wide range of water wave problems as both refractive and diffractive effects have been included in this single equation for a wide wave spectrum from short water waves to long water waves. However, as its name indicates, the MSE is only valid for the geometry with "mild" bottom slope. This limitation restricts the application of the MSE to seabed geometry of first order in bottom slope ∇h.

Some attention has been then paid to break this limitation. On one hand, some early work has been done to test the assumption of "mild" bottom slope numerically. For example, using numerical solution based on the hybrid finite element method, Tsay and Liu (1983) declared that the MSE can produce accurate results even for bottom slope as large as one in one. However, Booij's (1983) further analysis revealed that Tsay and Liu's discovery is correct only for waves propagating parallel to the contours of the sloping bed, for waves propagating normal to the contours, the MSE produces accurate results provided that the slope is less than one in three. On the other hand, a great deal of effort has been made to improve the conventional mild-slope equation. Responding to the failure of the MSE to approximate adequately wave scattering by singly and doubly periodic ripple beds, Kirby (1986) derived an extended MSE (EMSE) which still includes the first order term related to ∇h only but differs from the conventional MSE. Then, by keeping all the terms to second order, including both the bottom curvature term related to $\nabla^2 h$ and the slope-squared term related to $(\nabla h)^2$, Chamberlain and Porter (1995) derived a modified MSE (MMSE). In addition, Chandrasekera and Cheung (1997) rederived Chamberlain and Porter's (1995) MMSE by using Berkhoff's

(1972, 1976) approach and Suh et al. (1997) derived a time-dependent equation which reduced to Chamberlain and Porter's (1995) MMSE for a monochromatic wave.

There has been some further theoretical work to support these MMSEs, see Porter and Staziker (1995), Miles and Chamberlain (1998), Agnon (1999) and Agnon and Pelinovsky (2001). Based on these MMSEs, some new numerical models have been also subsequently proposed. Using a standard error-checking Runge-Kutta method, Chamberlain and Porter (1999) proposed a numerical model to solve the MMSE for wave scattering by both surface-piercing islands and submerged shoals. However, Chamberlain and Porter's (1999) model is restricted to axi-symmetrical topography. In addition, Chandrasekera and Cheung (1997) gave a numerical solution of Chamberlain and Porter's (1995) MMSE by using the hybrid finite element and Suh et al. (2001) applied the finite difference method to solve their time-dependant MMSE. It can be seen from these numerical solutions that the effect of the curvature and slope-squared terms are significant.

Since the GDRBEM only requires a discretization on the boundary of a computational domain plus some extra inner interpolation points, it is much more computing efficient than other numerical models to solve the MSE, as pointed out by Zhu et al. (2000) and Liu (2001). In this paper, we plan to extend the GDRBEM model to a modified GDRBEM model to solve the MMSE.

2. THE MODIFIED MILD-SLOPE EQUATION

Under a Cartesian coordinate system in which x and y denote a pair of orthogonal horizontal coordinates and z denotes the vertical coordinate measured positively upwards from the undisturbed free surface, the incident wave potential of a train of monochromatic waves propagating along the positive x-axis over a seabed of variable water depth $h(x, y)$ can be express

$$\phi^I(x, y) = -\frac{ig}{\omega} A e^{ik_o x} \tag{1}$$

with g being the gravitational acceleration, A the incident wave amplitude, ω the angular frequency and k_o the wave number with respect to the constant water depth.

According to Chamberlain and Porter (1995), the total wave potential, $\phi(x, y)$, by which the refraction of these waves due to the topographic change of seabed and the existence of the shoal are described, may satisfy the so-called MMSE

$$\nabla \cdot (u_0 \nabla \phi) + [k^2 u_0 + u_1 \nabla^2 h + u_2 (\nabla h)^2]\phi = 0, \tag{2}$$

in which $\nabla = (\partial / \partial x, \partial / \partial y)$ and

$$u_0(x, y) = \frac{1}{2k} \tanh kh \left(1 + \frac{K}{\sinh K}\right), \tag{3}$$

$$u_1(x, y) = \frac{1 - \tanh^2 kh}{4(K + \sinh K)} (\sinh K - K \cosh K), \tag{4}$$

$$u_2(x, y) = \frac{k(1 - \tanh^2 kh)}{12(K + \sinh K)^3} [K^4 + 4K^3 \sinh K - 9 \sinh K \sinh 2K$$

$$3K(K + 2\sinh K)(\cosh^2 K - 2\cosh K + 3)], \tag{5}$$

where $K = 2kh$ and the wave number k is determined by the dispersion relation

$$\omega^2 = gk \tanh kh \, . \tag{6}$$

It can be seen that the difference between the conventional MSE and the MMSE is that there are two extra terms in the MMSE (2) which are related to the bottom curvature $\nabla^2 h$ and the slope-square $(\nabla h)^2$, respectively.

3. INTEGRAL EQUATIONS

In the GDRBEM (Zhu et al. 2000 and Liu 2001), to reduce the interpolation error to a minimum, it is preferable that the right-hand side of the governing equation be kept as simple as possible. At the same time, the simplicity of the main differential operator should be taken into account in order to obtain the corresponding particular solution analytically. In addition, for the problem of wave refraction and diffraction, in order to convert the original MSE defined in the whole infinite region into an equivalent integral equation in the inner region Ω_i with variable water depth, we generally need utilize Sommerfeld's (1964) far-field radiation condition to eliminate the integral along the infinite circle Γ_∞ (Liu 2001, p.33), therefore it is better to choose the Helmholtz operator as the main differential operator. For the same reason here, we deliberately rewrite the governing equation (2) as

$$\nabla^2(u_0\phi) + k_o^2(u_0\phi) = R, \tag{7}$$

where

$$R = (k_o^2 - k^2)u_0\phi + \nabla u_0 \cdot \nabla\phi + \phi\nabla^2 u_0 - u_1\nabla^2 h\phi - u_2(\nabla h)^2\phi\ . \tag{8}$$

Let

$$\phi^*(X,\xi) = \frac{i}{4}H_0^{(1)}(k_o\rho)$$

be the Hankel function of the first kind of zero order with $\rho = \|X - \xi\|$ being the distance between a source point ξ and a field point $X = (x, y)$. It is well-known that $\phi^*(X,\xi)$ is the fundamental solution of the Helmholtz equation:

$$\nabla^2\phi + k_o^2\phi = -\delta(X-\xi)\ . \tag{9}$$

In the inner region Ω_i, for any fixed source point ξ, multiplying both sides of Eq.(7) by $\phi^*(X,\xi)$ and using the Green's second identity, we can rewrite Eq.(7) as

$$c_\xi^{(i)}u_0(\xi)\phi(\xi) - \int_\Gamma \frac{\partial u_0}{\partial \vec{n}}\phi\phi^*\,d\Gamma + u_0(\xi)|_\Gamma \int_\Gamma\left(\phi\frac{\partial\phi^*}{\partial\vec{n}} - \phi^*\frac{\partial\phi}{\partial\vec{n}}\right)d\Gamma = \int_{\Omega_i} R\phi^*\,d\Omega\ , \tag{10}$$

where $\vec{n}$ is the outward normal unit vector for the inner domain Ω_i, and $c_\xi^{(i)}$ is a geometric parameter: $c_\xi^{(i)} = \alpha(\xi)/(2\pi)$ for $\xi \in \Gamma$, $c_\xi^{(i)} = 1$ for $\xi \in \Omega_i$ and $c_\xi^{(i)} = 0$ for $\xi \in \Omega_o$, respectively, with $\alpha(\xi)$ being the internal angle of the boundary at point ξ.

In the outer region Ω_o with constant water depth, note that $u_0(x,y) \equiv \bar{u}_0 = const$, Eq.(7) is equivalent to the Helmholtz equation

$$\nabla^2\phi_s + k_o^2\phi_s = 0\ . \tag{11}$$

Multiplying both sides of Eq.(11) by $\phi^*(X,\xi)$, one can rewrite Eq.(11) as

$$c_\xi^{(o)}\phi_s(\xi) + \int_\Gamma\left(\phi_s\frac{\partial\phi^*}{\partial\vec{n}'} - \phi^*\frac{\partial\phi_s}{\partial\vec{n}'}\right)d\Gamma = 0\ , \tag{12}$$

where $\vec{n}'$ is the outward normal unit vector of the outer domain Ω_o and $c_\xi^{(o)}$ is also a geometric parameter: $c_\xi^{(o)} = \alpha(\xi)/(2\pi)$ for $\xi \in \Gamma$, $c_\xi^{(o)} = 0$ for $\xi \in \Omega_i$ and $c_\xi^{(o)} = 1$ for $\xi \in \Omega_o$, respectively.

The continuity of the wave potential and flux across the common boundary Γ shared by Ω_i and Ω_o demands

$$\phi_s = \phi - \phi^I, \qquad \frac{\partial\phi_s}{\partial\vec{n}'} = \frac{\partial\phi^I}{\partial\vec{n}} - \frac{\partial\phi}{\partial\vec{n}}\ . \tag{13}$$

Substituting Eq.(13) into Eq.(12), we obtain

$$\int_\Gamma\left(\phi\frac{\partial\phi^*}{\partial\vec{n}} - \phi^*\frac{\partial\phi}{\partial\vec{n}}\right)d\Gamma = c_\xi^{(o)}[\phi(\xi) - \phi^I(\xi)] + \int_\Gamma\left(\phi^I\frac{\partial\phi^*}{\partial\vec{n}} - \phi^*\frac{\partial\phi^I}{\partial\vec{n}}\right)d\Gamma\ . \tag{14}$$

Finally, the two integral equations (10) and (14) can be merged into

$$c_\xi u_0(\xi)\phi(\xi) - \int_\Gamma \frac{\partial u_0}{\partial \vec{n}}\phi\phi^* d\Gamma = c_\xi^{(o)}\bar{u}_0\phi^I(\xi) - \bar{u}_0\int_\Gamma (\phi^I \frac{\partial \phi^*}{\partial \vec{n}} - \phi^* \frac{\partial \phi^I}{\partial \vec{n}}) d\Gamma - \int_{\Omega_i} R\phi^* d\Omega \tag{15}$$

with $c_\xi = \alpha(\xi)/(2\pi)$ for $\xi \in \Gamma$, $c_\xi = 1$ for $\xi \in \Omega_i$ and $\xi \in \Omega_o$, respectively.

It is clear that, in the right hand side of the Eq.(15), there is still one domain integral which involves the unknown function $\phi(X)$ and its gradient $\nabla\phi$. To eliminate it, the DRBEM will be employed in the following section.

4. THE MODIFIED GDRBEM

Considering that the main differential operators in governing equation is the Helmholtz operator, we shall follow Zhu et al. (2000) to chose the interpolation functions f_j to be the following RBFs:

$$f_j(X) = 1 + \|X - X_j\|^2 + \|X - X_j\|^3 . \tag{16}$$

Thus, the right-hand side term R in Eq.(7) is expanded as

$$R(X) = \sum_{j=1}^{m+l} \alpha_j f_j(X), \tag{17}$$

where α_j are the coefficients to be determined with the collocation method by demanding the satisfaction of $m+l$ equations

$$R(X)|_{X_i} = \sum_{j=1}^{m+l} \alpha_j f_j(X_i), \quad \text{i=1,…,m+l}, \tag{18}$$

at m points on Γ and l interior collocation points within the domain Ω_i.

System (18) can also be expressed in matrix form:

$$\boldsymbol{R} = \boldsymbol{F}\alpha , \tag{19}$$

thus we have

$$\alpha = \boldsymbol{F}^{-1}\boldsymbol{R}. \tag{20}$$

It is noted that the existence and the recursion formulae of the particular solution $\hat{\phi}_j(X)$ to the following equation

$$\nabla^2\phi + k_o^2\phi = f_j(X) \tag{21}$$

for a given f_j have been given by Zhu (1993). The domain integral in the right-hand side of equation (15) therefore becomes

$$\int_{\Omega_i} R\phi^* d\Omega \approx \sum_{j=1}^{m+l}\alpha_j \int_{\Omega_i} f_j(X)\phi^* d\Omega = \sum_{j=1}^{m+l}\alpha_j \int_{\Omega_i} [\nabla^2\hat{\phi}_j + k_o^2\hat{\phi}_j]\phi^* d\Omega$$

$$= \sum_{j=1}^{m+l}\alpha_j [-c_\xi^{(i)}\hat{\phi}_j(\xi) - \int_\Gamma (\hat{\phi}_j \frac{\partial\phi^*}{\partial\vec{n}} - \phi^* \frac{\partial\hat{\phi}_j}{\partial\vec{n}}) d\Gamma], \tag{22}$$

which involves boundary integrals only. Substituting (22) into (15) yields

$$c_\xi u_0(\xi)\phi(\xi) - \int_\Gamma \frac{\partial u_0}{\partial \vec{n}}\phi\phi^* d\Gamma = c_\xi^{(o)}\bar{u}_0\phi^I(\xi) - \bar{u}_0\int_\Gamma (\phi^I \frac{\partial \phi^*}{\partial \vec{n}} - \phi^* \frac{\partial \phi^I}{\partial \vec{n}}) d\Gamma - \int_{\Omega_i} R\phi^* d\Omega . \tag{23}$$

It is worth indicating that $u_0(x, y)$ is a transcendental function due to the implicit wave dispersion relationship. The calculation of $\partial u_0 / \partial \vec{n}$ is not trivial and the details can be found in the Appendix B in Liu (2001). Eq.(23) involves boundary integrals only and after appropriate discretization, a linear system of algebraic equations involving the unknown function ϕ on $m+l$ points can be established.

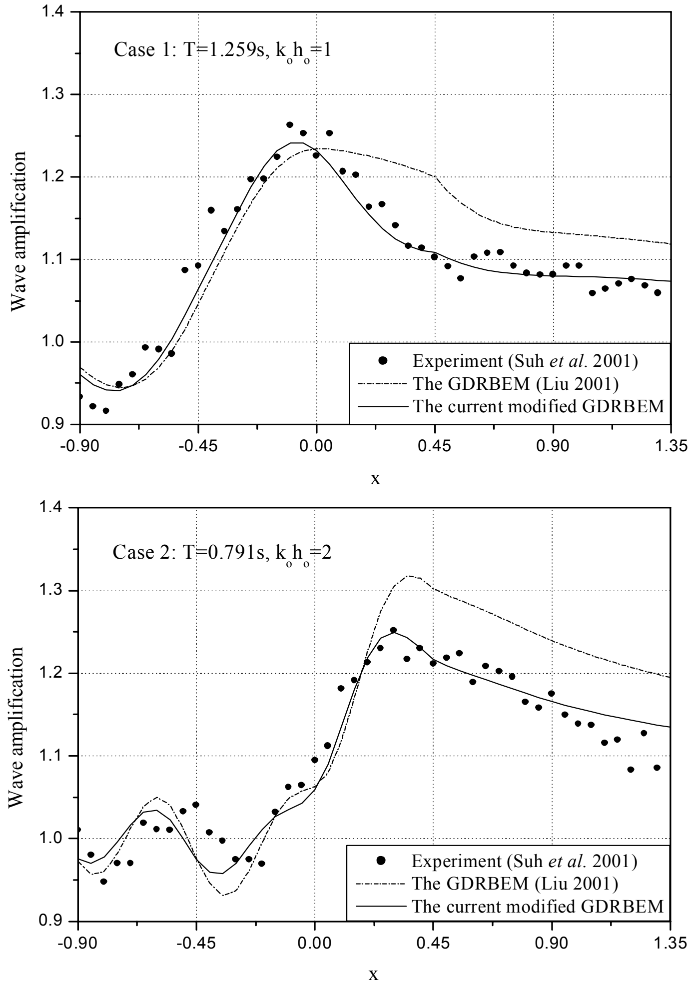
5. NUMERICAL EXAMPLES AND DISCUSSION

Suh *et al.* (2001) carried out a laboratory experiment for waves propagating in a wave tank with 23m long in the x-axis direction and 11m wide in the y-axis direction. A circular shoal is located in the middle of the wave tank. The water depth on the shoal at a distance r from the centre is given by

$$h = h_m + \frac{h_o - h_m}{R^2} r^2 , \tag{24}$$

where $h_o = 0.3$ m, $h_m = 0.12$ m is the local water depth at the centre of the shoal and $R = 0.45$ m is the radius of the shoal. In their experiment, the incident wave height was 3cm, and three different wave periods, 1.259s, 0.791s and 0.636s were used which correspond to $k_o h_o = 1.0$, 2.0 and 3.0, respectively. It is easy to see that the bottom slope of this particular shoal is no longer moderate since $\nabla^2 h = 3.5556$ and $(\nabla h)^2$ ranges from 0 at $r = 0$ to 0.64 at $r = R$. Therefore it is ideal to be used to test our modified GDRBEM model based on the MMSE.

Considering that the wave tank is very big in comparison with the circular shoal and the wave reflection from all the side walls can be neglected, we assume that the wave tank is infinite in our calculation and Sommerfeld's (1964) radiation condition at infinite is imposed. Wave amplifications along the x-axis for three different incident waves with $k_o h_o = 1.0$, 2.0 and 3.0 are respectively calculated by the GDRBEM model (Zhu et al. 2000 and Liu 2001) based on the MSE and by the current modified GDRBEM model based on the MMSE. These numerical solutions together with Suh et al.'s (2001) experimental solutions are presented in Fig. 1.



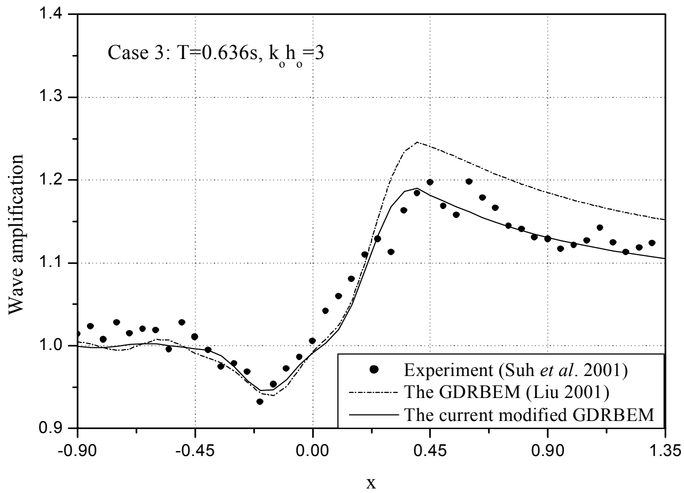


Fig 1. Comparison among the experiment data (Suh *et al.* 2001), the GDRBEM solution (Liu 2001) and the current modified GDRBEM solution for $k_o h_o = 1, 2, 3$, respectively

It can be clearly seen that the agreement between the experimental solutions and the numerical solutions from the current modified GDRBEM model is pretty good in the whole range for all three cases. However, large discrepancy between the GDRBEM solutions and the experimental solutions can be observed behind the shoal crest. This discrepancy becomes more significant when wave period increases from 0.636s ($k_o h_o = 3.0$) to 0.791s ($k_o h_o = 2.0$) and 1.259s ($k_o h_o = 1.0$), since the longer waves feel the bottom more strongly and therefore the drawback of neglecting both the bottom curvature term related to $\nabla^2 h$ and the slope-squared term related to $(\nabla h)^2$ in the MSE becomes obvious.

ACKNOWLEDGMENTS

This project is supported, in part, by the research grant from the National University of Singapore (R-264-000-121-112) and research grant from the Meteorological Services of Singapore (R-264-000-121-291). The first author likes to acknowledge the support from the Natural Science Foundation of P.R. China (NSFC10162001), the Excellent Young Teachers Program of Ministry of Education P.R. China (EYTP1784), Guangxi Shi-Bai-Qian Scholars Program (No. 2001224) and Guangxi Natural Science Foundation (GNSF0135001).

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