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流固耦合介质轴对称动力问题解法的改进 [1)]

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摘要　用直接求解常微分方程组解文 [1] 所得的控制方程，减少了传递矩阵计算工作量，避免了子阵求逆，使问题的求解得到了简化.

关键词　Laplace-Hankel 变换，轴对称动力问题，多层地基

在文 [1] 中，曾用 Laplace-Hankel 变换导出了流固耦合介质轴对称动力问题的一组常微分方程组. 为求解单层流固两相介质动力响应的初参数解答，需引入两相介质单位面积流体的准流量. 在确定面值未知各物理量时需进行子阵求逆和提供流量边界条件. 为此，本文采用直接求解常微分方程组，利用所得矩阵的特殊结构重新求解这一问题.

1　单层流固两相介质动力问题初参数解答的改进

控制方程为[1]

$$\frac{\mathrm{d}^2\tilde{u}_r}{\mathrm{d}z^2} - \xi^2\tilde{u}_r - \frac{\lambda+G}{G}\xi\tilde{e} + \frac{1}{G}\xi\tilde{\sigma}_f = \frac{\rho}{G}s^2\tilde{u}_r \tag{1a}$$

$$\frac{\mathrm{d}^2\tilde{u}_z}{\mathrm{d}z^2} - \xi^2\tilde{u}_z + \frac{\lambda+G}{G}\frac{\mathrm{d}\tilde{e}}{\mathrm{d}z} - \frac{1}{G}\frac{\mathrm{d}\tilde{\sigma}_f}{\mathrm{d}z} = \frac{\rho}{G}s^2\tilde{u}_z \tag{1b}$$

$$\frac{\mathrm{d}^2\tilde{\sigma}_f}{\mathrm{d}z^2} - \xi^2\tilde{\sigma}_f = \frac{s}{k'_d}\tilde{e} - \rho_f s^2\tilde{e} \tag{1c}$$

$$\frac{\mathrm{d}^2\tilde{e}}{\mathrm{d}z^2} - \xi^2\tilde{e} = \frac{s}{k'_d M}\tilde{e} + \frac{\rho-\rho_f}{M}s^2\tilde{e} \tag{1d}$$

需要说明的是，原文公式 (14b) 有误，应该是 (1b) 的形式.

现利用直接求解常微分方程方法求解 (1d) ～ (1a)，令

$$\beta^2 = \frac{s}{k'_d M} + \frac{\rho-\rho_f}{M}s^2,\quad \psi^2 = \xi^2+\beta^2,\quad l = 1-\frac{\rho s^2}{M\beta^2}$$

$$b = 1-l-\frac{G}{M},\quad \alpha^2 = \xi^2+\frac{\rho s^2}{G}$$

由 (1d) 可得

$$\tilde{e}(\xi, z, s) = e^{-\psi z}\tilde{e}(\xi, 0, s) \tag{2}$$

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由本构关系 $\sigma_z' = \lambda e + 2G\dfrac{\partial u_z}{\partial z}$ 可得

$$\tilde{\sigma}_z + \tilde{\sigma}_f = \lambda\tilde{e} + 2G\frac{\partial \tilde{u}_z}{\partial z} \text{ 且 } \tilde{e} = \xi\tilde{u}_r + \frac{\partial \tilde{u}_z}{\partial z}$$

则

$$\tilde{e}(\xi, z, s) = \frac{1}{M}[\tilde{\sigma}_z(\xi, z, s) + \tilde{\sigma}_f(\xi, z, s) + 2G\xi\tilde{u}_r(\xi, z, s)] \tag{3}$$

于是

$$\tilde{e}(\xi, z, s) = \frac{1}{M}[\tilde{\sigma}_z(\xi, o, s) + \tilde{\sigma}_f(\xi, o, s) + 2G\xi\tilde{u}_r(\xi, o, s)]e^{-\psi z} \tag{4}$$

利用(4)代入(1c)求出 $\tilde{\sigma}_f(\xi, z, s)$ 后,同理由(1a),(1b)即可求出 $\tilde{u}_r(\xi, z, s)$,$\tilde{u}_z(\xi, z, s)$,进而由式(3)得到 $\tilde{\sigma}_z(\xi, z, s)$ 的解.

再利用 τ_{rz} 的本构方程 $\tau_{rz} = G\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right)$ 有

$$\tilde{\tau}_{rz}(\xi, z, s) = G\left(\frac{\partial \tilde{u}_r}{\partial z} - \xi\tilde{u}_z\right) \tag{5}$$

由上面所求得的 $\tilde{u}_r(\xi, z, s)$ 和 $\tilde{u}_z(\xi, z, s)$ 极易求出 $\tilde{\tau}_{rz}(\xi, z, s)$. 为引入初参数 $\tilde{\tau}_{rz}(\xi, 0, s)$,可令 $z = 0$,则据 $\tilde{\tau}_{rz}(\xi, z, s)$ 的表达式得到 $\tilde{\tau}_{rz}(\xi, 0, s)$ 与其它初参数的关系,再利用 $\tilde{\tau}_{rz}(\xi, 0, s)$ 消掉初参数 $\tilde{u}_z(\xi, 0, s)$ 从而确定了 $\tau_{rz}(\xi, z, s)$.

将(1a)～(1c),(3),(5)所得的解整理成矩阵形式得

$$\{X\} = \{\Phi\}_{5\times 5}\{X_1\} \tag{6}$$

式中

$$\{X\} = [\tilde{u}_r(\xi, z, s)\ \ \tilde{u}_z(\xi, z, s)\ \ \tilde{\sigma}_f(\xi, z, s)\ \ \tilde{\tau}_{rz}(\xi, z, s)\ \ \tilde{\sigma}_z(\xi, z, s)]^{\mathrm{T}}$$

$$\{X_1\} = [\tilde{u}_r(\xi, 0, s)\ \ \tilde{u}_z(\xi, 0, s)\ \ \tilde{\sigma}_f(\xi, 0, s)\ \ \tilde{\tau}_{rz}(\xi, 0, s)\ \ \tilde{\sigma}_z(\xi, 0, s)]^{\mathrm{T}}$$

$[\Phi]$ 各元素如下

$$\Phi_{11} = e^{-\alpha z} + 2\xi^2\left(b\frac{e^{-\psi z} - e^{-\alpha z}}{\psi^2 - \alpha^2} + l\frac{e^{-\xi z} - e^{-\alpha z}}{\xi^2 - \alpha^2}\right)$$

$$\Phi_{13} = \frac{\xi}{G}\left[b\frac{e^{-\psi z} - e^{-\alpha z}}{\psi^2 - \alpha^2} + (l - 1)\frac{e^{-\xi z} - e^{-\alpha z}}{\xi^2 - \alpha^2}\right]$$

$$\Phi_{15} = \frac{\xi}{G}\left(b\frac{e^{-\psi z} - e^{-\alpha z}}{\psi^2 - \alpha^2} + l\frac{e^{-\xi z} - e^{-\alpha z}}{\xi^2 - \alpha^2}\right)$$

$$\Phi_{21} = 2\xi\left(\psi b\frac{e^{-\psi z} - e^{-\alpha z}}{\psi^2 - \alpha^2} + \xi l\frac{e^{-\xi z} - e^{-\alpha z}}{\xi^2 - \alpha^2}\right)$$

$$\Phi_{22} = e^{-\alpha z}$$

$$\Phi_{23} = \frac{1}{G}\left[\psi b\frac{e^{-\psi z} - e^{-\alpha z}}{\psi^2 - \alpha^2} + (l - 1)\frac{e^{-\xi z} - e^{-\alpha z}}{\xi^2 - \alpha^2}\right]$$

$$\Phi_{25} = \frac{1}{G}\left(\psi b\frac{e^{-\psi z} - e^{-\alpha z}}{\psi^2 - \alpha^2} + \xi l\frac{e^{-\xi z} - e^{-\alpha z}}{\xi^2 - \alpha^2}\right)$$

$$\Phi_{31} = 2G\xi l(\mathrm{e}^{-\psi z} - \mathrm{e}^{-\xi z})$$

$$\Phi_{33} = \mathrm{e}^{-\xi z} + l(\mathrm{e}^{-\psi z} - \mathrm{e}^{-\xi z})$$

$$\Phi_{35} = l(\mathrm{e}^{-\psi z} - \mathrm{e}^{-\xi z})$$

$$\Phi_{41} = -4G\xi^2\left(\psi b\frac{\mathrm{e}^{-\psi z} - \mathrm{e}^{-\alpha z}}{\psi^2 - \alpha^2} + \xi l\frac{\mathrm{e}^{-\xi z} - \mathrm{e}^{-\alpha z}}{\xi^2 - \alpha^2}\right)$$

$$\Phi_{43} = -2\xi\left[\psi b\frac{\mathrm{e}^{-\psi z} - \mathrm{e}^{-\alpha z}}{\psi^2 - \alpha^2} + \xi(l-1)\frac{\mathrm{e}^{-\xi z} - \mathrm{e}^{-\alpha z}}{\xi^2 - \alpha^2}\right]$$

$$\Phi_{44} = \mathrm{e}^{-\alpha z}$$

$$\Phi_{45} = -2\xi\left(\psi b\frac{\mathrm{e}^{-\psi z} - \mathrm{e}^{-\alpha z}}{\psi^2 - \alpha^2} + \xi l\frac{\mathrm{e}^{-\xi z} - \mathrm{e}^{-\alpha z}}{\xi^2 - \alpha^2}\right)$$

$$\Phi_{51} = 2G\xi\left[\mathrm{e}^{-\psi z} - \mathrm{e}^{-\alpha z} + l(\mathrm{e}^{-\xi z} - \mathrm{e}^{-\psi z}) - 2\xi^2\left(b\frac{\mathrm{e}^{-\psi z} - \mathrm{e}^{-\alpha z}}{\psi^2 - \alpha^2} + l\frac{\mathrm{e}^{-\xi z} - \mathrm{e}^{-\alpha z}}{\xi^2 - \alpha^2}\right)\right]$$

$$\Phi_{53} = (1-l)(\mathrm{e}^{-\psi z} - \mathrm{e}^{-\xi z}) - 2\xi^2\left[b\frac{\mathrm{e}^{-\psi z} - \mathrm{e}^{-\alpha z}}{\psi^2 - \alpha^2} + (l-1)\frac{\mathrm{e}^{-\xi z} - \mathrm{e}^{-\alpha z}}{\xi^2 - \alpha^2}\right]$$

$$\Phi_{55} = \mathrm{e}^{-\psi z} + l(\mathrm{e}^{-\xi z} - \mathrm{e}^{-\psi z}) - 2\xi^2\left(b\frac{\mathrm{e}^{-\psi z} - \mathrm{e}^{-\alpha z}}{\psi^2 - \alpha^2} + l\frac{\mathrm{e}^{-\xi z} - \mathrm{e}^{-\alpha z}}{\xi^2 - \alpha^2}\right)$$

$$\Phi_{12} = \Phi_{14} = \Phi_{24} = \Phi_{32} = 0$$

$$\Phi_{34} = \Phi_{42} = \Phi_{52} = \Phi_{54} = 0$$

从式(6)不难看出,采用本文方法在不需要引入两相介质单位面积流体的准流量即可求得问题的解,且所得矩阵为 5×5 阶,其中有 8 个元素为零,零元素的分布形成了矩阵的特殊结构,为进一步简化成层地基的动力问题解答提供了基础.

2 面值未知物理量求解的改进

对于 N 层流固两相介质(见原文图 1),将上述均质解答近似应用到每一层中由层间完全接触条件,从底层向上递推得

$$\{X_{n+1}\} = [A]\{X_1\} \tag{7}$$

式中 $[A] = [\Phi_n][\Phi_{n-1}]\cdots[\Phi_2][\Phi_1]$

地表面($z=0$)和位于地下足够深的固定表面($z = h_{N+1}$)的边界条件为

$$\sigma_f(r,0,t) = 0,\quad \tau_{rz}(r,0,t) = 0,\quad \sigma_z(r,0,t) = -p(r,t) \tag{8}$$

$$u_r(r,h_{N+1},t) = 0,\quad u_z(r,h_{N+1},t) = 0 \tag{9}$$

由矩阵 $\{\Phi\}_{5\times5}$ 的特殊结构易得:$A_{12} = A_{14} = A_{24} = 0$,结合边界条件(8),(9)的变换式代入(7)得

$$\left.\begin{aligned}\tilde{u}_r(\xi,h_{N+1},s) &= A_{11}\tilde{u}_r(\xi,0,s) + A_{15}\tilde{\sigma}_z(\xi,0,s) = 0\\ \tilde{u}_z(\xi,h_{N+1},s) &= A_{21}\tilde{u}_z(\xi,0,s) + A_{22}\tilde{u}_z(\xi,0,s) + A_{25}\tilde{\sigma}_z(\xi,0,s) = 0\end{aligned}\right\}$$

解(10)**可得面值未知物理量** $\tilde{u}_r(\boldsymbol{\xi},0,s)$，$\tilde{u}_z(\boldsymbol{\xi},0,s)$ **与已知面力** $\tilde{\sigma}_z(\boldsymbol{\xi},0,s)$ **的显式关系**

$$\left.\begin{aligned}\tilde{u}_r(\boldsymbol{\xi},0,s) &= -\frac{A_{15}}{A_{11}}\tilde{\sigma}_z(\boldsymbol{\xi},0,s)\\ \tilde{u}_z(\boldsymbol{\xi},0,s) &= \frac{A_{15}A_{21}-A_{11}A_{25}}{A_{11}A_{12}}\tilde{\sigma}_z(\boldsymbol{\xi},0,s)\end{aligned}\right\}$$