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压电材料平面应力状态的直线裂纹问题一般解

侯密山
(石油大学机械系, 东营 257062)

摘要　研究了含直线裂纹系的压电材料平面应力问题. 单个裂纹和双裂纹问题的封闭解答表明,在裂纹尖端,应力、电场强度和电位移有 1/2 阶的奇异性. 并与前人结果比较了产生电场奇异性的物理因素.

关键词　压电介质,线裂纹,平面应力问题,强度因子,封闭解

引　　言

近年来,Sosa[1], Suo[2], Pak[3]和文[4,5]分别研究了含缺陷的压电材料平面和反平面问题. 在这些研究中,均忽略了缺陷内空气介质对电场的传导作用,简化了问题的求解. 对孔洞问题,因压电材料的介电常数远远大于空气介质的介电常数,材料中的电位移在孔边的法向分量相当小,如此忽略是恰当的. 但裂纹不同于孔洞,其内空气介质如同薄膜一样,电场在裂纹面处应是连续的. 对此,本文将裂纹面处的电场边界条件取为既能体现空气介质作用,又可避免解双区域问题的连续性电场边界条件,恰当描述了电场在裂纹面处的物质表现. 用复变函数方法,精确分析了压电材料平面应力状态的直线裂纹问题.

1　基本方程与问题的求解

取 xoy 平面为横观各向同性压电材料的各向异性面,则平面应力问题的本构方程为

$$\varepsilon_x = S_{11}\sigma_x + S_{13}\sigma_y + d_{31}E_y, \quad \varepsilon_y = S_{13}\sigma_x + S_{33}\sigma_y + d_{33}E_y, \quad 2\varepsilon_{xy} = S_{44}\sigma_{xy} + d_{15}E_x \qquad (1)$$

$$D_x = d_{15}\sigma_{xy} + \varepsilon_{11}E_x, \quad D_y = d_{31}\sigma_x + d_{33}\sigma_y + \varepsilon_{33}E_y \qquad (2)$$

式中 $\sigma_x = F''_{yy}(x,y)$, $\sigma_y = F''_{xx}(x,y)$, $\sigma_{xy} = -F''_{xy}(x,y)$, $E_x = \phi'_x(x,y)$, $E_y = -\phi'_y(x,y)$. 记累计求和 $\sum_m = \sum_{k=1,2,\cdots,m}$, 则应力函数 $F(x,y)$ 和电势函数 $\phi(x,y)$ 可分别表示为[1]

$$F(x,y) = 2\,\mathrm{Re}\sum_3 F_k(z_k), \qquad \phi(x,y) = 2\mathrm{Re}\sum_3 \lambda_k F'_k(z_k) \qquad (3)$$

式中 μ_k, $k=1,2,3$ 是复参数,为特性方程(4)的三个不相等的复数根

$$[S_{11}\mu^4 + (2S_{13}+S_{44})\mu^2 + S_{33}](\varepsilon_{11}+\varepsilon_{33}\mu^2) - \mu^2[d_{31}\mu^2 + (d_{33}-d_{15})]^2 = 0 \qquad (4)$$

$$z_k = x + \mu_k y, \quad \lambda_k = [d_{31}\mu_k^3 + (d_{33}-d_{15})\mu_k]/(\varepsilon_{11}+\varepsilon_{33}\mu_k^2) \qquad (5)$$

设 S 为横观各向同性压电介质所占的无限大平面区域. 沿实轴的 $L = L_1 + L_2 + \cdots + L_n$, $L_i = a_ib_i$ 为自由共线裂纹,则应力边界条件为

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$$\sigma_y^+(t) = \sigma_y(t) = 0\,,\qquad \tau_{xy}^+(t) = \tau_{xy}(t) = 0\,,\ t\in L \tag{6}$$

考虑裂纹缺陷内空气介质对电场的作用,按 Maxwell **电磁场理论,在裂纹面与空气介质的交界处,两介质中的电位移的法向分量** D_n **连续,电场强度的切向分量** E_τ **连续. 又裂纹两表面之间的间隙远远小于裂纹尺寸,其内空气介质如同薄膜一样,则沿裂纹法向方向,空气介质中的** D_n, E_τ **是不变量. 由此有裂纹面处的连续性电场边界条件**

$$D_y^+(t) = D_y^-(t)\,,\qquad E_x^+(t) = E_x^-(t)\,,\qquad t\in L \tag{7}$$

连续性条件(7)**式表达了压电材料中的电场在裂纹面处的连续性,体现了裂纹缺陷内空气介质对电场的作用,从理论上完善了电场在裂纹面处的物质表现,可求得问题的精确解.**

记 $\phi_k(z_k) = F''_k(z_k)$ **和** $\Omega_k(z_k) = \overline{\phi}_k(z_k)$ **为域** S **上的全纯函数**(L **和端点除外**),**对自由裂纹**

$$\phi_k(z_k) = A_k + O(1/z_k^2)\,,\qquad \Omega_k(z_k) = \overline{A}_k + O(1/z_k^2)\,,\qquad k = 1,2,3 \tag{8}$$

式中: A_k **由无限远处载荷** $\sigma^\infty = (\sigma_x^\infty, \sigma_y^\infty, \sigma_{xy}^\infty)$, $E^\infty = (E_x^\infty, E_y^\infty)$ **确定**; $O(1/z_k^2)$ **为函数记号,在域内全纯,且对充分大的**$|z|$ **有指出的阶次. 则**(7)**式和**(8)**式可分别表示为**

$$\left.\begin{aligned}&\sum_3[\phi_k^+(t) + \Omega_k^-(t)] = 0\,, \qquad \sum_3[\phi_k^-(t) + \Omega_k^+(t)] = 0\\ &\sum_3[\mu_k\phi_k^+(t) + \overline{\mu}_k\Omega_k^-(t)] = 0\,, \qquad \sum_3[\mu_k\phi_k^-(t) + \overline{\mu}_k\Omega_k^+(t)] = 0\,, \qquad t\in L\end{aligned}\right\} \tag{10}$$

由连续性电场边界条件(10)**式,解** Riemann-Hilbert **问题**[6]**得**

$$\sum_3[\lambda_k\phi_k(z) - \overline{\lambda}_k\Omega_k(z)] = B_1\,,\qquad \sum_3[g_k\lambda_k(z) - \overline{g}_k\Omega_k(z)] = B_2 \tag{11}$$

$$B_1 = \sum_3[\lambda_k A_k - \overline{\lambda}_k\overline{A}_k]\,,\qquad B_2 = \sum_3[g_k A_k - \overline{g}_k\overline{A}_k] \tag{12}$$

将(11)**式和**(12)**式代入到应力边界条件**(9)**式有**

$$U_j^+(t) - \overline{U_j^-}(t) = B_j\,,\qquad U_j^-(t) - \overline{U_j^+}(t) = B_j\,,\qquad j = 3,4\,,\qquad t\in L \tag{13}$$

式中

$$U_3(z) = \sum_2[(W + T_k)\phi_k(z) - \overline{R}_k\Omega_k(z)]\,,\qquad U_4(z) = \sum_2[(\mu_k W + \mu_3 T_k)\phi_k(z) - \mu_3\overline{R}_k\Omega_k(z)] \tag{14}$$

$$B_3 = (g_3 + \overline{g}_3)B_1 - (\lambda_3 + \overline{\lambda}_3)B_2\,,\qquad B_4 = (\overline{\mu}_3 g_3 + \mu_3\overline{g}_3)B_1 - (\overline{\mu}_3\lambda_3 + \mu_3\overline{\lambda}_3)B_2 \tag{15}$$

$$W = g_3\overline{\lambda}_3 - \lambda_3\overline{g}_3\,,\qquad T_k = \overline{g}_3\lambda_k - \overline{\lambda}_3 g_k\,,\qquad R_k = g_3\lambda_k - \lambda_3 g_k\,,\qquad k = 1,2 \tag{16}$$

再解(13)**式得**

$$\left.\begin{aligned}U_j(z) &= \overline{\omega}_j + [B_j f(z) + P_j(z)]/[2X_n(z)]\\ \overline{U_j}(z) &= \overline{\omega}_j - [B_j f(z) + P_j(z)]/[2X_n(z)]\end{aligned}\right\}$$

式中

$$P_j(z) = C_n^{(j)} z^n + C_{n-1}^{(j)} z^{n-1} + \cdots + C_1^{(j)} z + C_0^{(j)}, \quad \bar{\omega}_j = (H_j + \overline{H_j})/2, \quad j = 3,4$$

$$H_3 = \sum_2 [(W + T_k) A_k - \overline{R}_k \overline{A} A_k], \qquad H_4 = \sum_2 [(\mu_k W + \mu_3 T_k) A_k - \mu_3 \overline{R}_k \overline{A}_k] \quad (18a)$$

$$X_n(z) = \prod_{k=1}^{n} \sqrt{(z - a_k)(z - b_k)}, \qquad f(z) = \frac{1}{\pi i} \int (t - z)^{-1} X_n(t) \mathrm{d} t \quad (18b)$$

则由(14)和(11)式解出 $\phi_k(z)$, $\Omega_k(z)$ 并用 z_k 替换 z, 得所求的 $\phi_k(z_k)$, $\Omega_k(z_k)$, k = 1,2,3.

由(14)式知,函数 U(z) 在无限远处的性态与 $\phi(z)$ 相似,则有

$$C_n^{(j)} = H_j - \overline{H_j}, \qquad C_{n-1}^{(j)} = -(H_j - \overline{H_j}) \sum_n (a_k + b_k)/2, \qquad j = 3,4 \quad (19)$$

根据位移单值条件,记 $p_k^{(1)} = S_{11}\mu_k^2 + S_{13} - d_{31}\mu_k\lambda_k$, $p_k^{(2)} = (S_{13}\mu_k^2 + S_{33} - d_{33}\mu_k\lambda_k)/\mu_k$ 有确定其余常数 C_i, D_i, i = 0,1,2, …,n - 2 的方程组

$$\sum_3 \int_i [p_k^{(j)} (\phi_k^+(t) - \phi_k^-(t)) - \overline{P}_k^{(j)} (\Omega_k^+(t) - \Omega_k^-(t))] \mathrm{d} t = 0, \quad i = 2,3, \cdots, n, ” \quad j = 1,2$$

(20)

2　例题与讨论

例 1　单个裂纹问题的封闭解: 对单个裂纹, n = 1. 设 $a_1 = -a$, $b_1 = a$, 由前述各式有

$$U_3(z) = \bar{\omega}_3 + [B_3 + W\sigma_y^{\infty} z X_1^{-1}(z)]/2, \qquad \overline{U_3}(z) = \bar{\omega}_3 - [B_3 + W\sigma_y^{\infty} z X_1^{-1}(z)]/2 \quad (21)$$

$$U_4(z) = \bar{\omega}_4 + [B_4 - W\sigma_{xy}^{\infty} z X_1^{-1}(z)]/2, \quad \overline{U_4}(z) = \bar{\omega}_4 - [B_4 - W\sigma_{xy}^{\infty} z X_1^{-1}(z)]/2 \quad (22)$$

则得单裂纹问题的封闭解 $\phi_k(z_k)$, $\Omega_k(z_k)$, k = 1,2,3 及介质内的耦合场.

而实轴(裂纹除外)上各点的应力、电场强度和电位移分别为

$$\sigma_x = \sigma_x^{\infty} - A_x\sigma_y^{\infty} + A_x\sigma_y^{\infty} t X_1^{-1}(t), \quad \sigma_y = \sigma_y^{\infty} t X_1^{-1}(t), \quad \sigma_{xy} = \sigma_{xy}^{\infty} t X_1^{-1}(t) \quad (23a)$$

$$E_x = E_x^{\infty} - B_x\sigma_{xy}^{\infty} + B_x\sigma_{xy}^{\infty} t X_1^{-1}(t), \quad E_y = E_y^{\infty} - B_y\sigma_y^{\infty} + B_y\sigma_y^{\infty} t X_1^{-1}(t) \quad (23b)$$

$$\left.\begin{aligned} D_x &= \varepsilon_{11}(E_x^{\infty} - B_x\sigma_{xy}^{\infty}) + (d_{15} + \varepsilon_{11} B_x)\sigma_{xy}^{\infty} X_1^{-1}(t) \\ D_y &= d_{31}(\sigma_x^{\infty} - A_x\sigma_y^{\infty}) + \varepsilon_{33}(E_y^{\infty} - B_y\sigma_y^{\infty}) + (d_{31} A_x + d_{33} + \varepsilon_{33} B_y)\sigma_y^{\infty} t X_1^{-1}(t) \end{aligned}\right\}$$

(24a)

$$
\left.\begin{array}{l}
\delta_k = (-1)^k \Delta^{-1}[\widetilde{M}(W + T_{3-k}) - \overline{N} R_{3-k}], \quad \chi_k = (-1)^k \Delta^{-1}(\mu_3 - \mu_{3-k}) \\
\Delta = (N\overline{N} - M\widetilde{M}) W, \quad M = (\mu_3 - \mu_1)(W + T_2) - (\mu_3 - \mu_2)(W + T_1) \\
N = (\mu_3 - \mu_1) R_2 - (\mu_3 - \mu_2) R_1, \quad Y = (W + T_1) R_2 - (W + T_2) R_1 \\
G = (\mu_1 W + \mu_3 T_1) R_2 - (\mu_2 W + \mu_3 T_2) R_1
\end{array}\right\}
$$

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式中 $Y(a_2) = (b^2\Lambda - a^2)/\sqrt{a^2(b^2 - a^2)}$ 和 $Y(b_2) = (1 - \Lambda)k_1^{-1}$ 是尺寸影响系数,所表现的两裂纹间距及裂纹尺寸对强度因子的影响与普通各向同性弹性材料结果一致.

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A GENERAL SOLUTION OF THE PLANE STRESS PROBLEMS IN PIEZOELECTRIC MEDIA WITH LINE CRACKS

Hou Mishan

(*University of Petroleum, Dong Ying* 257062, *China*)

Abstract Considering continuous boundary condition of electric field at the cracks, a general solution of the plane stress problems in piezoelectric media with line cracks is provided by using function of complex variable and solving Riemann - Hilbert problems. Solution in closed form and field intensity factors are given for problems of a single crack and two cracks. It is shown from the solution that the stresses and electric field intensity and electric displacement have (1/2) type of singularity at the crack tip. These results are compared with some results of previous researches on physics factors of the electric field singularity.

Key words piezoelectric media, line crack, plane stress problem, intensity factor, solution in closed form