關於完全度量空間中的誘導極限及點集序列

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摘要: 在本文第一部份中我們首先對完全度量空間裏的點序列定義高次誘導極限點，然後論述關於此類極限點在連續變換下的一個極限定理. 在第二部份中，我們對於空間中的點集序列引入廣義均勻收斂的觀念，並從而導出兩個有關雙重點集序列極限方面的定理.

CONCERNING DERIVED LIMITS AND SET SEQUENCES IN COMPLETE METRIC SPACES

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1. *Derived Limit of Higher Order.* Let $\mathfrak{X}$ be a complete metric space in which $\varrho(x,y)$ is the distance function defined for every pair of points x, y, and every Cauchy sequence converges to a limit. Let $X=\{x_n\}$ be an arbitrary bounded sequence of points in $\mathfrak{X}$. If x_0 appears infinitely many times in $\{x_n\}$ and if $\{x_n\}$ contains only a finite number of points other than x_0, then $\{x_n\}$ is said to have the limit x_0 of zero order. Denote by X' the derived set of X, and by X^0 the set of all those points each of which appears infinitely many times in $\{x_n\}$. Then every point in the union $X^0 \cup X'$ is called a limiting point of $\{x_n\}$, where X^0 may be named the derived set of zero order.

We shall now consider the case where $X=\{x_n\}$ has only a finite number of successive derived sets (derivatives), say $X', X'', \cdots, X^{(k)}, X^{(k+1)}=0$ (empty), where k is the order of X and each point of $X^{(k)}$ is called a k-th limiting point. It is easily seen that $X' \supseteq X'' \supseteq \cdots \supseteq X^{(k)}$, and all the derivatives are closed sets. If $X^{(k)}$ contains a single point, say x_0, then we denote

$$\lim_{n\to\infty}{}^{(k)} x_n = x_0, \tag{1}$$

and call x_0 *the k-th derived limit or the limit of* $\{x_n\}$ *of order* k, where the case of zero order (i.e. $k=0$) is included. In particular, for $k=0$ or 1 we may use the usual notation $\lim x_n$ instead of $\lim^{(k)} x_n$.

2. *A Limit Theorem Concerning Mapping Sequences.* Let $\mathfrak{X}_1, \cdots, \mathfrak{X}_\nu$ be ν complete metric spaces with distance functions $\varrho_1(x,y), \cdots, \varrho_\nu(x,y)$ respectively. Clearly the product space $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_\nu$ can again be made as a complete metric space by introducing a certain distance function into the space, e.g. $\varrho(x,y) = \sqrt{\left(\sum_{i=1}^{\nu} \left(\varrho_i(x_i, y_i)\right)^2\right)}$, where $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$ are any two points of the product space and $x_i, y_i \varepsilon \mathfrak{X}_i$ $(i=1, \cdots, \nu)$. Generally, $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_\nu$ is said to be *properly metrized*, if the distance function $\varrho(x,y) = F(\varrho_1(x,y), \cdots, \varrho_\nu(x_\nu, y_\nu))$ satisfies the two postulates of Lindenbaum together with the condition that $\varrho \to 0$ implies $\varrho_i \to 0$ and vice versa. Thus it is clear that the properly metrized product space is always a complete metric space.

We shall call $f(x)$ a *continuous mapping* of x of $\mathfrak{X}$ into a complete metric space $\mathfrak{X}^*$ (with distance function ϱ^*) if (i) for all $x \varepsilon \mathfrak{X}$ we have $f(x) \varepsilon \mathfrak{X}^*$, where the totality of $f(x)$ forms a bounded set in $\mathfrak{X}^*$; (ii) for each fixed $x_0 \varepsilon \mathfrak{X}$ and an arbitrary $\varepsilon > 0$ we can find a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$\varrho^*[f(x), f(x_0)] < \varepsilon \quad \text{whenever} \quad \varrho(x, x_0) < \delta, \quad x \varepsilon \mathfrak{X}. \tag{2}$$

Now we may state the following

THEOREM 1. *Let* $\{x_n\}, \{y_n\}, \{z_n\}, \cdots (x_n \varepsilon \mathfrak{X}_1, y_n \varepsilon \mathfrak{X}_2, z_n \varepsilon \mathfrak{X}_3, \cdots)$ *be* ν *bounded sequences having limits of orders* $a, b, c, \cdots (\geqq 1)$ *respectively, and let* $f(x, y, z, \cdots)$ *be a continuous mapping of* $(x, y, z, \cdots)$ *of the properly metrized space* $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_\nu$ *into a complete metric space such that* $\{f(x_n, y_n, z_n, \cdots)\}$ *is of order* $S = (a+b+c+\cdots) - \nu + 1$. *Then*

$$\lim_{n\to\infty}{}^{(s)} f(x_n, y_n, z_n, \cdots) = f(\lim_{n\to\infty}{}^{(a)} x_n, \lim_{n\to\infty}{}^{(b)} y_n, \lim_{n\to\infty}{}^{(c)} z_n, \cdots). \tag{3}$$

Sketch of The Proof. We shall only sketch a proof of the theorem by stating the most essential steps as follows:

(i) *If* $\{v_n\}$ *is a bounded sequence having k-th derived limit* λ *in a complete metric space* $\mathfrak{X}$, *and if* $f(v)$ *is a continuous mapping of* v *of* $\mathfrak{X}$ *into a complete metric space* $\mathfrak{X}^*$, *then* $\{f(v_n)\}$ *is of order* $\leqq k$, *and only the limiting point* $f(\lambda)$ *can possibly have the greatest order* k.

(ii) *If the* ν *sequences* $\{x_n\}$, $\{y_n\}, \cdots$ *have limits* $\alpha, \beta, \cdots$ *of orders* $a, b, \cdots$ *in* $\mathfrak{X}_1$, $\mathfrak{X}_2, \cdots$ *respectively, and if* $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_\nu$ *is properly metrized, then the order* s *of* $\{(x_n, y_n, \cdots)\}$ *in the product space does not exceed the number* $(a+b+\cdots)-\nu+1$, *and only the limiting point* $(\alpha, \beta, \cdots)$ *can possibly have the greatest order* $(a+b+\cdots)-\nu+1$.

The statement (i) can easily be established by induction argument. For $k=1$, we see that $f(v_n) \to f(\lambda)=l$ (as $n \to \infty$) so that $f(\lambda)$ is a limiting point in $\mathfrak{X}^*$. Now the order of $\{f(v_n)\}$ must be $\leqq 1$. For, otherwise there would exist different limiting points l', l'' such that $f(\nu_{\mu_n}) \to l'$, $f(\nu_{\nu_n}) \to l''$. But since $\{\nu_{\mu_n}\}$, $\{\nu_{\nu_n}\}$ are sub-sequences we have $\nu_{\mu_n} \to \lambda$, $\nu_{\nu_n} \to \lambda$, and by continuity of f, we must have $l'=f(\lambda)=l''$, which is a contradiction. Similarly, if $k=2$ and if $\{f(v_n)\}$ is of order 2, then the only possible second order limiting point of $\{f(\nu_n)\}$ must be $f(\lambda)$, because from the case $k=1$ we have seen that any first order limiting point of $\{\nu_n\}$ can only be mapped into a limiting point of order $\leqq 1$ by the continuous f, and moreover, λ is known as the only second order limiting point of $\{\nu_n\}$. Hence in general the order of $\{f(\nu_n)\}$ must be $\leqq 2$. Generalizing the above argument and using induction we easily prove the general case $k \geqq 2$.

To prove (ii) we need only make repeated use of induction. For $\nu=1$, the statement is trivial. For $\nu=2$, we may first prove the simple case $a \geqq 1$, $b=1$ by induction on a. Then, by generalizing the argument and using induction on b we may easily establish the case $a \geqq 1$, $b \geqq 1$. The procedure and reasoning can easily be extended to the general case $\nu \geqq 2$.

Finally we may deduce Theorem 1 from (i) and (ii). For, it is obvious that we may consider $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_\nu$ as the complete metric space $\mathfrak{X}$ in (i) so that $\nu_n=(x_n, y_n, \cdots)$, $k=(a+b+\cdots)-\nu+1=s$, $\lambda=(\alpha, \beta, \cdots)$, and our theorem follows.

3. *Limiting Elements of Set Sequences in $\mathfrak{X}$.* In what follows we use $X, Y, Z, \cdots$ to denote point sets (containing points x's, y's, z's, $\cdots$ respectively) in the complete metric space $\mathfrak{X}$. By generalizing the ordinary notion of limit for a sequence of points in a metric space we now define the limiting element for an arbitrary sequence of point sets as follows:

Definition 3.1. To every $\{X_n\}$ in $\mathfrak{X}$ there corresponds a set X_∞, called limiting element, which is defined to be the totality of all limiting points of convergent sub-sequence $\{x_{r_n}\}$'s with $x_{r_n} \varepsilon X_{r_n}$ $(n=1, 2, 3, \cdots)$. In notation,

$$X_\infty = \operatorname*{Lim}_{n\to\infty} X_n. \tag{4}$$

Here we use Lim (not lim) to indicate that the limit operation is applied to the sequence of sets. Obviously our definition is quite general and comprehensive in its meaning. For example, in Euclidean 2-space *if* $\{X_n\}$ is a sequence of plane sets tending to a circular region (circle) as limit, then X_∞ simply represents the circle (including its circumference) on the plane. Moreover, *if* $\{X_{2n+1}\}$ and $\{X_{2n}\}$ tend to a circular region and a triangular region respectively, then X_∞ represents the union of these two closed plane regions. In particular, if each set X_n consists of only one point x_n (*i. e.* $X_n=(x_n)$), then $\operatorname*{Lim}_{n\to\infty} X_n$ represents the totality of all the limiting points of $\{x_n\}$, *i. e.* $X_\infty = X^\circ \cup X'$, where $\{x_n\}=X$.

Denote $\{X_n \cup Y_n\} = \{\Sigma_n\}$ and $\{X_n \cap Y_n\} = \{\Pi_n\}$, where $\cap$ is known as the set-theoretic intersection relation. Then it is easily observed that

$$\sum\nolimits_\infty = X_\infty \cup Y_\infty, \qquad \prod\nolimits_\infty \subseteq X_\infty \cap Y_\infty. \tag{5}$$

4. *Differences of Sets, Generalized Uniform Convergence.* Let $\{X_{nm}\}$ $(n, m=1, 2, 3, \cdots)$ be a double sequence of bounded point sets in $\mathfrak{X}$. Using the definition of limiting elements we may express

$$
\begin{array}{l}
X_{11},\ X_{12},\ X_{18},\ \cdots,\ X_{1\infty},\\
X_{21},\ X_{22},\ X_{23},\ \cdots,\ X_{2\infty},\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
X_{\infty 1},\ X_{\infty 2},\ X_{\infty 3},\ \cdots,\ X_{\infty\infty}\ (?)
\end{array}
$$

Now since $\operatorname*{Lim}_{n\to\infty} X_{n\infty}$ and $\operatorname*{Lim}_{n\to\infty} X_{\infty n}$ are not necessarily the same, we cannot of course assume that the double array has the limit $X_{\infty\infty}$ at the corner. This naturally leads us to the generalization of the notion of uniform convergence. Let us now introduce the following*

Definition 4.1. *Given two point sets X, Y in $\mathfrak{X}$, every point x of X has a ϱ-distance from Y, and every y of Y has a ϱ-distance from X, then the maximum or the lexast upper bound of all these distances is defined to be the difference of X, Y, and it is denoted by $||X-Y||$.*

Clearly in accordance with the definition we may express

$$(6)\qquad ||X-Y|| = \overline{\underset{x\varepsilon X,\, y\varepsilon Y}{\text{bound}}}(\varrho(x,Y), \varrho(y,X)),$$

where

$$\varrho(x,Y) = \underset{y\varepsilon Y}{\underline{\text{bound}}}\,\varrho(x,y);\qquad \varrho(y,X) = \underset{x\varepsilon X}{\underline{\text{bound}}}\,\varrho(x,y).$$

It can be easily observed that

$$
\begin{array}{ll}
\text{i)} & ||X-Y|| = ||Y-X|| > 0,\\
\text{ii)} & ||X-Y|| = 0 \text{ if and only if } \overline{X} = \overline{Y},\\
\text{iii)} & ||X-Y|| + ||Y-Z|| \geqq ||X-Z||.
\end{array}
$$

Let us now give a formal proof of iii). Without loss of generality we may assume

$$\text{bound}\,(\varrho(x,Z), \varrho(z,X)) = \varrho(x_0,Z),\quad x_0\,\varepsilon\,\overline{X}.$$

Clearly we have

*Cf. F. Hausdorff, Mengenlehre, I, pp. 236, 293.

$$\overline{\text{bound}}\,(\varrho(x, Y), \varrho(y, X)) \geqq \overline{\text{bound}}\,\varrho(x, Y) \geqq \varrho(x_0, Y) = \varrho(x_0, y_0), \quad y_0 \varepsilon \overline{Y},$$

$$\overline{\text{bound}}\,(\varrho(y, Z), \varrho(z, Y)) \geqq \overline{\text{bound}}\,\varrho(y, Z) \geqq \varrho(y_0, Z) = \varrho(y_0, z_0), \quad z_0 \varepsilon \overline{Z}.$$

Adding these inequalities together and using Definition 4.1, we get

$$||X - Y|| + ||Y - Z|| \geqq \varrho(x_0, y_0) + \varrho(y_0, z_0) \geqq \varrho(x_0, z_0) \geqq \varrho(x_0, Z) = ||X - Z||,$$

which is what we want to prove. Hence we have the following simple conclusion:

THEOREM 2. *All the bounded open sets (or closed sets) contained n the metric space $\mathfrak{X}$, when considered as elements, also form a metric space with respect to the distance function (difference) $||X - Y||$ which is defined for every pair of sets X, Y in $\mathfrak{X}$.*

Having defined the difference for any two sets, we may now introduce the notion of uniform convergence for set sequences.

Definition 4.2. *The sequence $X_{n1}, X_{n2}, X_{n3}, \cdots$ is said to converge to the limiting element $X_{n\infty}$ uniformly with respect to n, if for every given $\varepsilon > 0$, there exists a large number N (depending on ε only) such that for all n,*

$$||X_{nj} - X_{n\infty}|| < \varepsilon \tag{7}$$

whenever $j \geqq N = N(\varepsilon)$.

We are now going to establish the following

THEOREM 3. *If the sequence $X_{n1}, X_{n2}, X_{n3}, \cdots$ of bounded point sets in $\mathfrak{X}$ converges to $X_{n\infty}$ uniformly with respect to n, then we have*

$$\operatorname*{Lim}_{n\to\infty}\left(\operatorname*{Lim}_{m\to\infty} X_{nm}\right) = \operatorname*{Lim}_{m\to\infty}\left(\operatorname*{Lim}_{n\to\infty} X_{nm}\right). \tag{8}$$

Proof. Let x^* be an arbitrary limiting point contained in $\operatorname*{Lim}_{n\to\infty} X_{n\infty}$. Then there is a sequence $\{x_{\nu_n}\}$ converging to x^* with $x_{\nu_n} \varepsilon X_{\nu_n\infty}$. Given an arbitrary $\varepsilon > 0$, we can determine a large number $M = M(\varepsilon, x^*)$ depending on ε and x^* only such that

$$(9) \qquad \varrho(x_{\nu_n}, x^*) < \frac{1}{4}\varepsilon, \qquad \| X_{kj} - X_{k\infty} \| < \frac{1}{4}\varepsilon,$$

whenever $\nu_n \geqq M, j \geqq M, (k=1,2,3,\cdots)$. This is always possible, since, by uniform convergence, we have $N=N(\varepsilon)$ such that the second inequality of (9) holds for $j \geqq N$, and we may take $M > N$. Now the second inequality implies that there are points x'_{kj} of X_{kj} $(j=M, M+1, \cdots)$ satisfying

$$(10) \qquad \varrho(x'_{\nu_n j}, x_{\nu_n}) < \frac{1}{4}\varepsilon,$$

for all $\nu_n \geqq M$. Comparing (9) with (10), we get

$$(11) \qquad \varrho(x'_{\nu_n j}, x^*) < \frac{1}{4}(\varepsilon + \varepsilon) = \frac{1}{2}\varepsilon,$$

for all $\nu_n, j \geqq M$. Since $\{x'_{\nu_n j}\}$ is bounded, there exists at least a convergent sub-sequence $\{x'_{\mu_n j}\}$ of it such that $\lim_{n\to\infty} x'_{\mu_n j} = x'_j$, Clearly $x'_j \varepsilon X_{\infty j}$ $(j=M, M+1, \cdots)$ and from (11) we have

$$(12) \qquad \varrho(x'_j, x^*) \leqq \frac{1}{2}\varepsilon < \frac{2}{3}\varepsilon.$$

Thus there is a convergent sub-sequence of $\{x'_j\}$, say $\{x'_{j_n}\}$, so that $\lim_{n\to\infty} x'_{j_n} = x''$ belongs to $\operatorname{Lim}_{j\to\infty} X_{\infty j}$ and obviously we have (by (12))

$$(13) \qquad \varrho(x'', x^*) \leqq \frac{2}{3}\varepsilon < \varepsilon.$$

This shows that x^* has a distance $< \varepsilon$ from $\operatorname{Lim}_{j\to\infty} X_{\infty j}$. Note that ε is arbitrary and $\operatorname{Lim}_{j\to\infty} X_{\infty j}$ is closed, we may therefore infer that x^* is a point of $\operatorname{Lim}_{j\to\infty} X_{\infty j}$. Hence we have

$$(14) \qquad \operatorname{Lim}_{n\to\infty} X_{n\infty} \subseteqq \operatorname{Lim}_{j\to\infty} X_{\infty j}.$$

It remains to show that "$\subseteqq$" can be replaced by "$\supseteqq$". The following is just a reverse of the foregoing procedure with suitable modifications. Let x^* be any point of $\operatorname*{Lim}_{j\to\infty} X_{\infty j}$ and ε an arbitrary number > 0. By uniform convergence there exists a large number N (depending on ε only) such that

$$(15) \qquad \| X_{nj} - X_{n\infty} \| < \frac{1}{4}\varepsilon,$$

for all $j \geqq N$ and $n = 1, 2, 3, \cdots$. By our supposition there exsits an $M \geqq N$ such that

$$(16) \qquad \varrho(x_M, x^*) < \frac{1}{4}\varepsilon,$$

with $x_M \varepsilon X_{\infty M}$. Clearly there is a convergent (column) sub-sequence $\{x_{\nu_n}\}$ such that $\lim x_{\nu_n} = x_M$ and $x_{\nu_n} \varepsilon X_{\nu_n M} (n = 1, 2, 3, \cdots)$. Hence there is a $K \geqq M$ such that

$$(17) \qquad \varrho(x_{\nu_n}, x_M) < \frac{1}{4}\varepsilon,$$

whenever $\nu_n \geqq K$. Comparing (16) with (17) we get

$$(18) \qquad \varrho(x_{\nu_n}, x^*) < \frac{1}{2}\varepsilon,$$

for all $\nu_n \geqq K$. Note that $x_{\nu_n} \varepsilon X_{\nu_n M}$ and (15) implies

$$(15)' \qquad \| X_{\nu_n M} - X_{\nu_n \infty} \| < \frac{1}{4}\varepsilon,$$

where $n = 1, 2, 3, \cdots$. Thus it follows that $X_{\nu_n \infty}$ must contain a point x'_{ν_n} such that

$$(19) \qquad \varrho(x_{\nu_n}, x'_{\nu_n}) < \frac{1}{4}\varepsilon,$$

whenever $n = 1, 2, 3, \cdots$. Comparison of (18) and (19) gives

$$\varrho(x'_{\nu_n}, x^*) < \left(\frac{1}{2} + \frac{1}{4}\right)\varepsilon = \frac{3}{4}\varepsilon. \tag{20}$$

Now obviously

$$\operatorname*{Lim}_{n\to\infty}(x'_{\nu_n}) \subseteq \operatorname*{Lim}_{n\to\infty} X_{\nu_n\infty} \subseteq \operatorname*{Lim}_{n\to\infty} X_{n\infty},$$

so there is a convergent sub-sequence $\{x''_{\nu_n}\}$ of $\{x'_{\nu_n}\}$ such that

$$\lim_{n\to\infty} x''_{\nu_n} = x'' \varepsilon \operatorname*{Lim}_{n\to\infty} X_{n\infty}.$$

Hence from (20) we obtain

$$\varrho(x'', x^*) \leqq \frac{3}{4}\varepsilon < \varepsilon. \tag{21}$$

This shows that x^* has a distance $< \varepsilon$ from the closed set $\operatorname*{Lim}_{n\to\infty} X_{n\infty}$. Thus we see that x^* must belong to $\operatorname*{Lim}_{n\to\infty} X_{n\infty}$, and the relation "$\subset$" of (14) can be reversed. Our theorem is therefore established.

Note. It is clear that, in accordance with Definition 3.1, every sequence of bounded sets $\{X_n\}$ in $\mathfrak{X}$ has always a limiting element X_∞, but the upper limit

$$\overline{\lim_{n\to\infty}} \| X_n - X_\infty \|$$

is not necessarily equal to zero. Thus it is easy to see that our proof of Theorem 3 cannot be simplified very much by merely making use of the triangular relation $\|X-Y\| + \|Y-Z\| \geqq \|X-Z\|$.

THEOREM 4. *Under the same hypothesis of Theorem 3 we have*

$$\operatorname*{Lim}_{n\to\infty}(\operatorname*{Lim}_{m\to\infty} X_{nm}) = \operatorname*{Lim}_{n\to\infty} X_{nn} = X_{\infty\infty}. \tag{23}$$

Detailed proof of this theorem will be omitted here, as it can be easily produced by using the same principle as already employed in our proof of Theorem 3. As a matter of fact, by the uniform convergence of

$X_{n1}, X_{n2}, X_{n3}, \cdots \rightarrow X_{n\infty}$ it is easy to show that for every point x^* of $\underset{n\to\infty}{\mathrm{Lim}}\, X_{n\infty}$ there can always be found a diagonal sub-sequence $\{x_{\nu_n\nu_n}\}$ $(x_{\nu_n\nu_n} \varepsilon X_{\nu_n\nu_n})$ such that $\lim_{n\to\infty} x_{\nu_n\nu_n} = x^*$, and conversely, for every point x' of $X_{\infty\infty}$ there is a (column) sub-sequence $\{x_{\mu_n}\}$ of points $(x_{\mu_n} \varepsilon X_{\mu_n\infty})$ converging to x'.

I am indebted to Dr. F. Smithies for kindly reading through my manuscript and suggesting that the definition of limiting sets of §3 can be made more abstractly in terms of "*Filters*" (See Bourbaki, *Topologie générale*, Chap. 1). It may therefore be remarked that our Theorems 3 and 4 can be further generalized, but the essential reasoning involzed in such generalizations may be quite similar to that here adopted.

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