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一个推广的 Hardy–Hilbert 型不等式及其逆式

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摘 要 本文引入单参数 λ 及应用 Beta 函数, 给一个 Hardy–Hilbert 型不等式以具有最佳常数因子的推广. 作为应用, 给出了它的逆向形式.

关键词 Hardy–Hilbert 型不等式; 权系数; Beta 函数
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On an Extension of Hardy–Hilbert's Type Inequality and a Reversion

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Abstract This paper deals with an extension of Hardy–Hilbert's type inequality with a best constant factor by introducing a parameter λ and using the Beta function. As application, a reversion is considered.

Keywords Hardy–Hilbert's type inequality; weight coefficient; Beta function
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1 引言

设 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n, b_n \geq 0$, 使得 $0<\sum_{n=1}^{\infty} a_n^p<\infty$ 及 $0<\sum_{n=1}^{\infty} b_n^q<\infty$, 则有如下不等式 (见文 [1])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}. \tag{1}$$

这里, 常数因子 $\frac{\pi}{\sin(\pi/p)}$ 为最佳值. 式 (1) 称为 Hardy–Hilbert 不等式, 它在分析学有重要的应用 (见文 [2]). 1997–1998 年, 杨与高在文 [3–4] 利用 Euler 常数, 给出式 (1) 的一个加强形式. 不等式 (1) 的较为精确的形式是

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1}<\frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=0}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}b_n^q\right\}^{\frac{1}{q}}. \tag{2}$$

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这里, 常数因子 $\frac{\pi}{\sin(\pi/p)}$ 仍为最佳值 (见文 [1]). 1998 年, 为推广 Hilbert 积分不等式, 杨 [5] 首先引入了独立参数 λ 与 Beta 函数. 1999 年, 杨 [6] 给出式 (2) 的一个加强; 杨与 Debnath [7] 给式 (2) 以如下推广: 若

$$2-\min\{p,q\}<\lambda\leq 2,\quad 0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p<\infty \ \text{ 及 }\ 0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^q<\infty,$$

则有

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_mb_n}{(m+n+1)^{\lambda}}<k_{\lambda}(p)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}. \tag{3}$$

这里, 常数因子 $k_{\lambda}(p)(=B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}))$ 为最佳值. 2003 年, 杨等 [8] 对上面的结果及权系数方法作了充分的评述; 杨 [9] 给出了如下的 Hardy–Hilbert 型不等式: 对于 $\alpha\geq e^{7/6}$, 有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\ln\alpha mn}<\frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}n^{p-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q-1}b_n^q\right\}^{\frac{1}{q}}. \tag{4}$$

这里, 常数因子 $\frac{\pi}{\sin(\pi/p)}$ 为最佳值. 2004–2005 年, Mario 和 Krnic 等 [10–11] 考虑了式 (1) 及积分类似的一些新的推广; 杨 [12–14] 则考虑了它们的逆向形式.

本文目的是建立 (4) 的引入参量 $\lambda>0$ 和 Beta 函数的最佳推广式, 它联系着如下二重级数

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{(\ln m+\ln n+\ln\alpha)^{\lambda}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\ln^{\lambda}\alpha mn}(\alpha\geq e^{7/6}). \tag{5}$$

作为应用, 还考虑了逆向形式.

为此, 必须估算如下权系数

$$\omega_{\lambda}(r,n,\alpha):=\sum_{m=1}^{\infty}\frac{\ln^{\lambda-1}\sqrt{\alpha}n}{m\ln^{\lambda}\alpha mn}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{2-\lambda}{r}}(r>1,\ \alpha\geq e^{7/6}) \tag{6}$$

及建立一些引理.

2 一些引理

首先, 我们需要 Beta 函数的如下公式 (见文 [11]):

$$B(u,v):=\int_0^{\infty}\frac{1}{(1+t)^{u+v}}t^{u-1}dt=B(v,u)\ \ (u,v>0). \tag{7}$$

引理 2.1 设 $r>1$, $n\in N$, $\alpha\geq e^{7/6}$ 及 $2-r<\lambda\leq 2$. 定义函数 $R_{\lambda}(r,n,\alpha)$ 为

$$\begin{aligned}R_{\lambda}(r,n,\alpha):=&\ln^{1-\lambda}\sqrt{\alpha}n\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{\frac{2-\lambda}{r}}\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}\frac{u^{(\lambda-2)/r}}{(1+u)^{\lambda}}du\\&-\left(\frac{7}{12}+\frac{2-\lambda}{12r\ln\sqrt{\alpha}}\right)\frac{1}{\ln^{\lambda}\sqrt{\alpha}n}-\frac{\lambda}{12\ln^{\lambda+1}\alpha n},\end{aligned} \tag{8}$$

则有 $R_{\lambda}(r,n,\alpha)>0$.

证明 分部积分, 有

$$I := \int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}} \frac{u^{(\lambda-2)/r}}{(1+u)^\lambda}du = \frac{r}{r+\lambda-2}\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}} \frac{1}{(1+u)^\lambda}du^{1+\frac{\lambda-2}{r}}$$

$$= \frac{r\ln^\lambda\sqrt{\alpha}n}{(r+\lambda-2)\ln^\lambda \alpha n}\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{2+\frac{\lambda-2}{r}}$$

$$+ \frac{r^2\lambda}{(r+\lambda-2)(2r+\lambda-2)}\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}} \frac{1}{(1+u)^\lambda}du^{2+\frac{\lambda-2}{r}}$$

$$> \left[\frac{r\ln\sqrt{\alpha}}{\ln^\lambda \alpha n} + \frac{r^2\lambda\ln^2\sqrt{\alpha}}{(2r+\lambda-2)\ln^{\lambda+1}\alpha n}\right]\frac{\ln^{\lambda-1}\sqrt{\alpha}n}{(r+\lambda-2)}\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{\frac{\lambda-2}{r}}.$$

由 (8) 式, 有

$$R_\lambda(r,n,\alpha) > \left(\frac{r\ln\sqrt{\alpha}}{r+\lambda-2} - \frac{7}{12} - \frac{2-\lambda}{12r\ln\sqrt{\alpha}}\right)\frac{1}{\ln^\lambda \alpha n} + \left[\frac{r^2\ln^2\sqrt{\alpha}}{(r+\lambda-2)(2r+\lambda-2)} - \frac{1}{12}\right]\frac{\lambda}{\ln^{\lambda+1}\alpha n}. \tag{9}$$

由 $r > 1$, $\alpha \geq e^{7/6}$ 及 $2 - r < \lambda \leq 2$, 有

$$\frac{r\ln\sqrt{\alpha}}{r+\lambda-2} - \frac{7}{12} - \frac{2-\lambda}{12r\ln\sqrt{\alpha}} = \frac{(12\ln\sqrt{\alpha}-7)r^2\ln\sqrt{\alpha} + (7\ln\sqrt{\alpha}-1)r(2-\lambda) + (2-\lambda)^2}{12r(r+\lambda-2)\ln\sqrt{\alpha}} \geq 0;$$

$$\frac{r^2\ln^2\sqrt{\alpha}}{(r+\lambda-2)(2r+\lambda-2)} - \frac{1}{12} \geq \frac{49r^2}{144(r+\lambda-2)(2r+\lambda-2)} - \frac{1}{12}$$
$$= \frac{25r^2 + 36r(2-\lambda) - 12(2-\lambda)^2}{144(r+\lambda-2)(2r+\lambda-2)} > \frac{25r^2 - 12r^2}{144(r+\lambda-2)(2r+\lambda-2)} > 0.$$

因而由 (9) 式, 有 $R_\lambda(r,n,\alpha) > 0$. 证毕.

若 $f \in C^4[1,\infty)$, $(-1)^n f^{(n)}(x) > 0$, $f^{(n)}(\infty) = 0$ $(n = 0, 1, \ldots, 4)$ 及 $\int_1^\infty f(x)dx < \infty$, 则有 (见文 [12])

$$\sum_{k=1}^\infty f(k) = \int_1^\infty f(x)dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1)\varepsilon \quad (0 < \varepsilon < 1). \tag{10}$$

引理 2.2 若 $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $n \in N$, $\alpha \geq e^{7/6}$ 及 $2 - \min\{r,s\} < \lambda \leq 2$, $\omega_\lambda(r,n,\alpha)$ 由 (6) 式所定义, 则有

$$\omega_\lambda(r,n,\alpha) < k_\lambda(r) := B\left(\frac{r+\lambda-2}{r}, \frac{s+\lambda-2}{s}\right). \tag{11}$$

证明 设 $f_n(x)$ 为

$$f_n(x) := \frac{\ln^{\lambda-1}\sqrt{\alpha}n}{x\ln^\lambda \alpha nx}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}x}\right)^{\frac{2-\lambda}{r}} \ (x > 0),$$

则 $f_n(x)$ 具有应用 (10) 式的条件. 我们有

$$f_n'(1) = -\left(\frac{r\ln\sqrt{\alpha}+2-\lambda}{r\ln\sqrt{\alpha}\ln^\lambda \alpha n} + \frac{\lambda}{\ln^{\lambda+1}\alpha n}\right)\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}}\right)^{\frac{2-\lambda}{r}}\ln^{\lambda-1}\sqrt{\alpha}n.$$

对下面的积分作变换 $u = \frac{\ln\sqrt{\alpha}x}{\ln\sqrt{\alpha}n}$, 有

$$\int_1^\infty f_n(x)dx = \int_1^\infty \frac{\ln^{\lambda-1}\sqrt{\alpha}n}{x(\ln\sqrt{\alpha}n + \ln\sqrt{\alpha}x)^\lambda}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}x}\right)^{\frac{2-\lambda}{r}}dx = k_\lambda(r) - \int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}\frac{u^{(\lambda-2)/r}}{(1+u)^\lambda}du.$$

由 (10) 式及上面的结果, 有

$$\begin{aligned}\omega_\lambda(r,n,\alpha) &= \sum_{k=1}^{\infty} f_n(k) < \int_1^\infty f_n(x)dx + \frac{1}{2}f_n(1) - \frac{1}{12}f_n'(1)\\ &= k_\lambda(r) - \int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}} \frac{u^{(\lambda-2)/r}}{(1+u)^\lambda}du + \frac{\ln^{\lambda-1}\sqrt{\alpha}n}{2\ln^\lambda \alpha n}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}}\right)^{\frac{2-\lambda}{r}}\\ &\quad + \frac{1}{12}\left(\frac{r\ln\sqrt{\alpha}+2-\lambda}{r\ln\sqrt{\alpha}\ln^\lambda \alpha n} + \frac{\lambda}{\ln^{\lambda+1}\alpha n}\right)\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}}\right)^{\frac{2-\lambda}{r}}\ln^{\lambda-1}\sqrt{\alpha}n\\ &= k_\lambda(r) - R_\lambda(r,n,\alpha)\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}}\right)^{\frac{2-\lambda}{r}}\ln^{\lambda-1}\sqrt{\alpha}n,\end{aligned}$$

故由引理 2.1, (11) 式成立. 证毕.

注 1 对于 $\lambda = 2$, 由 (11) 式, 有

$$\omega(n,\alpha) := \sum_{m=1}^{\infty} \frac{\ln\sqrt{\alpha}n}{m\ln^2 \alpha mn} = \omega_2(r,n,\alpha) < 1. \tag{12}$$

引理 2.3 若 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha \ge e^{7/6}$, $2-\min\{p,q\}<\lambda\le 2$, 则对 $0<\varepsilon<p+\lambda-2$, 有

$$\begin{aligned}J &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{mn\ln^\lambda \alpha mn}\left(\frac{1}{\ln\sqrt{\alpha}m}\right)^{\frac{2-\lambda+\varepsilon}{p}}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{\frac{2-\lambda+\varepsilon}{q}}\\ &> \frac{1}{\varepsilon\ln^\varepsilon \alpha}B\left(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p}, \frac{q+\lambda-2}{q}+\frac{\varepsilon}{p}\right) - O(1).\end{aligned} \tag{13}$$

证明 对于固定的 $y\ge 1$, 对下面的积分作变换 $u=(\ln\sqrt{\alpha}x)/(\ln\sqrt{\alpha}y)$, 有

$$\begin{aligned}&\int_1^\infty \frac{(\ln\sqrt{\alpha}y)^{\lambda-1}}{x\ln^\lambda \alpha xy}\left(\frac{\ln\sqrt{\alpha}y}{\ln\sqrt{\alpha}x}\right)^{\frac{2-\lambda+\varepsilon}{p}}dx = \int_{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}y}}^\infty \frac{u^{(\lambda-2-\varepsilon)/p}}{(1+u)^\lambda}du\\ &= \int_0^\infty \frac{u^{(\lambda-2-\varepsilon)/p}}{(1+u)^\lambda}du - \int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}y}} \frac{u^{(\lambda-2-\varepsilon)/p}}{(1+u)^\lambda}du\\ &> \int_0^\infty \frac{u^{(\lambda-2-\varepsilon)/p}}{(1+u)^\lambda}du - \int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}y}} u^{(\lambda-2-\varepsilon)/p}du\\ &= \int_0^\infty \frac{u^{(\lambda-2-\varepsilon)/p}}{(1+u)^\lambda}du - \frac{p}{p+\lambda-2-\varepsilon}\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}y}\right)^{\frac{\lambda-2-\varepsilon}{p}+1},\end{aligned}$$

则可算得

$$\begin{aligned}J &> \int_1^\infty\int_1^\infty \frac{1}{xy\ln^\lambda \alpha xy}\left(\frac{1}{\ln\sqrt{\alpha}x}\right)^{\frac{2-\lambda+\varepsilon}{p}}\left(\frac{1}{\ln\sqrt{\alpha}y}\right)^{\frac{2-\lambda+\varepsilon}{q}}dxdy\\ &= \int_1^\infty \frac{1}{y}\left(\frac{1}{\ln\sqrt{\alpha}y}\right)^{1+\varepsilon}\left[\int_1^\infty \frac{(\ln\sqrt{\alpha}y)^{\lambda-1}}{x\ln^\lambda \alpha xy}\left(\frac{\ln\sqrt{\alpha}y}{\ln\sqrt{\alpha}x}\right)^{\frac{2-\lambda+\varepsilon}{p}}dx\right]dy\\ &> \int_1^\infty \frac{1}{y}\left(\frac{1}{\ln\sqrt{\alpha}y}\right)^{1+\varepsilon}dy\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{p+\lambda-2-\varepsilon}{p}-1}du\\ &\quad - \frac{p}{p+\lambda-2-\varepsilon}\left(\ln\sqrt{\alpha}\right)^{\frac{\lambda-2-\varepsilon}{p}+1}\int_1^\infty \frac{1}{y}\left(\frac{1}{\ln\sqrt{\alpha}y}\right)^{\frac{\lambda-2-\varepsilon}{p}+2+\varepsilon}dy\\ &= \frac{1}{\varepsilon\ln^\varepsilon \alpha}B\left(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p}, \frac{q+\lambda-2}{q}+\frac{\varepsilon}{p}\right) - O(1).\end{aligned}$$

因而有 (13) 式. 证毕.

3 主要结果

定理 3.1 若 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\geq e^{7/6}$, $2-\min\{p,q\}<\lambda\leq 2$, $a_n,b_n\geq 0$, 使得 $0<\sum_{n=1}^{\infty}\frac{n^{p-1}a_n^p}{\ln^{\lambda-1}\sqrt{\alpha}n}<\infty$ 及 $0<\sum_{n=1}^{\infty}\frac{n^{q-1}b_n^q}{\ln^{\lambda-1}\sqrt{\alpha}n}<\infty$, 则有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\ln^{\lambda}\alpha mn}<k_\lambda(p)\left\{\sum_{n=1}^{\infty}\frac{n^{p-1}a_n^p}{\ln^{\lambda-1}\sqrt{\alpha}n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}b_n^q}{\ln^{\lambda-1}\sqrt{\alpha}n}\right\}^{\frac{1}{q}},\tag{14}$$

这里, 常数因子 $k_\lambda(p)(=B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}))$ 是最佳值. 特别地, 对于 $\lambda=2$, 有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\ln^{2}\alpha mn}<\left\{\sum_{n=1}^{\infty}\frac{n^{p-1}a_n^p}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}b_n^q}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{q}}.\tag{15}$$

证明 配方, 由带权的 Hölder 不等式及 (6) 式 (见文 [16]), 有

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\ln^{\lambda}\alpha mn}&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\ln^{\lambda}\alpha mn}\\&\quad\times\left[\left(\frac{\ln\sqrt{\alpha}m}{\ln\sqrt{\alpha}n}\right)^{\frac{2-\lambda}{pq}}\left(\frac{m^{\frac{1}{q}}}{n^{\frac{1}{p}}}\right)a_m\right]\left[\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{2-\lambda}{pq}}\left(\frac{n^{\frac{1}{p}}}{m^{\frac{1}{q}}}\right)b_n\right]\\&\leq\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\ln^{\lambda}\alpha mn}\left(\frac{\ln\sqrt{\alpha}m}{\ln\sqrt{\alpha}n}\right)^{\frac{2-\lambda}{q}}\left(\frac{m^{p-1}}{n}\right)a_m^p\right\}^{\frac{1}{p}}\\&\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\ln^{\lambda}\alpha mn}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{2-\lambda}{q}}\left(\frac{n^{q-1}}{m}\right)b_n^q\right\}^{\frac{1}{q}}\\&=\left\{\sum_{m=1}^{\infty}\omega_\lambda(q,m,\alpha)\frac{m^{p-1}a_m^p}{\ln^{\lambda-1}\sqrt{\alpha}m}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega_\lambda(p,n,\alpha)\frac{n^{q-1}b_n^q}{\ln^{\lambda-1}\sqrt{\alpha}n}\right\}^{\frac{1}{q}},\end{aligned}\tag{16}$$

则由 (11) 式, 因 $k_\lambda(p)=k_\lambda(q)$, 有 (14) 式.

对于 $0<\varepsilon<p+\lambda-2$, 置 $\widetilde{a}_n$, $\widetilde{b}_n$ 为

$$\widetilde{a}_n=\frac{1}{n}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{\frac{2-\lambda+\varepsilon}{p}},\quad \widetilde{b}_n=\frac{1}{n}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{\frac{2-\lambda+\varepsilon}{q}}\quad(n\in N),$$

则有

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}\frac{n^{p-1}\widetilde{a}_n^p}{(\ln\sqrt{\alpha}n)^{\lambda-1}}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}\widetilde{b}_n^q}{(\ln\sqrt{\alpha}n)^{\lambda-1}}\right\}^{\frac{1}{q}}\\&\quad=\sum_{n=1}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}=\sum_{n=1}^{2}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}+\sum_{n=3}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\\&\quad<\sum_{n=1}^{2}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}+\int_{\frac{e}{\sqrt{\alpha}}}^{\infty}\frac{1}{x(\ln\sqrt{\alpha}x)^{1+\varepsilon}}dx\\&\quad=\sum_{n=1}^{2}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}+\frac{1}{\varepsilon}=\frac{1}{\varepsilon}(1+o(1))\quad(\varepsilon\to 0^+).\end{aligned}\tag{17}$$

若 (14) 式的常数因子 $k_\lambda(p)$ 不为最佳值, 则有正常数 k $(k < k_\lambda(p))$, 使当常数因子 $k_\lambda(p)$ 换成 k 时, (14) 式仍然保持成立. 特别地, 由 (13) 与 (17) 式, 有

$$\begin{aligned}&\frac{1}{\ln^{\varepsilon}\alpha}B\left(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p},\frac{q+\lambda-2}{q}+\frac{\varepsilon}{p}\right)-\varepsilon O(1)<\varepsilon J\\&<\varepsilon k\left\{\sum_{n=1}^{\infty}\frac{n^{p-1}\widetilde{a}_n^p}{\ln^{\lambda-1}\sqrt{\alpha}n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}\widetilde{b}_n^q}{\ln^{\lambda-1}\sqrt{\alpha}n}\right\}^{\frac{1}{q}}<k\ (1+o(1)),\end{aligned}$$

则有 $k_\lambda(p)\le k$ $(\varepsilon\to 0^+)$. 这矛盾于 $k<k_\lambda(p)$. 因而 (14) 式的常数因子 $k_\lambda(p)$ 是最佳值. 证毕.

评注 3.2 当 $\lambda=1$ 时, (14) 变成 (4) 式. 这说明 (14) 是 (4) 式的带有参数 λ 的推广.

4 一个逆式

引理 4.1 若 $\theta(n)=o(1)$ $(n\to\infty)$, 则

$$\left[\sum_{n=2}^{\infty}\frac{\theta(n)}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}=o(1)\quad(\varepsilon\to 0^+).\tag{18}$$

证明 因 $\theta(n)=o(1)$ $(n\to\infty)$, 则 $|\theta(n)|\le M$ $(n\in N)$, 及对于任意的 $\widetilde{\varepsilon}>0$, 存在 $N_0>1$, 使得对任意的 $n>N_0$, $|\theta(n)|<\frac{\widetilde{\varepsilon}}{2}$. 因 $\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\to\infty$ $(\varepsilon\to 0^+)$, 存在 $\delta>0$, 使得对任意的 $0<\varepsilon<\delta$,

$$\left[\sum_{n=2}^{N_0}\frac{|\theta(n)|}{n(\ln\sqrt{\alpha}n)}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}<\frac{\widetilde{\varepsilon}}{2}.$$

则对任意的 $0<\varepsilon<\delta$, 有

$$\begin{aligned}&\left|\left[\sum_{n=2}^{\infty}\frac{\theta(n)}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}\right|\\&\le\left[\sum_{n=2}^{\infty}\frac{|\theta(n)|}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}\\&\le\left[\sum_{n=2}^{N_0}\frac{|\theta(n)|}{n(\ln\sqrt{\alpha}n)}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}\\&\quad+\left[\sum_{n=N_0+1}^{\infty}\frac{|\theta(n)|}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}\\&<\frac{\widetilde{\varepsilon}}{2}+\frac{\widetilde{\varepsilon}}{2}\left[\sum_{n=N_0+1}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}<\widetilde{\varepsilon}.\end{aligned}$$

证毕.

定理 4.2 若 $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\ge e^{7/6}$, $a_n,b_n\ge 0$, 使得 $0<\sum_{n=1}^{\infty}\frac{n^{p-1}a_n^p}{\ln\sqrt{\alpha}n}<\infty$ 及 $0<\sum_{n=1}^{\infty}\frac{n^{q-1}b_n^q}{\ln\sqrt{\alpha}n}<\infty$, 则有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln^2\alpha mn}>\left\{\sum_{n=1}^{\infty}\left(1+\frac{1}{2\ln\alpha n}\right)\frac{n^{p-1}a_n^p}{\ln\alpha n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}b_n^q}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{q}},\tag{19}$$

这里, 常数因子 1 是最佳值.

证明 由 (10) 及 (12) 式, 有

$$1>\omega(n,\alpha)=\sum_{m=1}^{\infty}\frac{\ln\sqrt{\alpha}n}{m\ln^2\alpha mn}>\int_1^{\infty}\frac{\ln\sqrt{\alpha}n}{x\ln^2\alpha nx}dx+\frac{\ln\sqrt{\alpha}n}{2\ln^2\alpha n}$$
$$=\frac{\ln\sqrt{\alpha}n}{\ln\alpha n}\left(1+\frac{1}{2\ln\alpha n}\right)=1+\theta(n)\quad(\theta(n)=o(1),\ n\to\infty).\tag{20}$$

由逆向的带权的 Hölder 不等式及 (16) 式 (当 $\lambda=2$) (见文 [13]), 有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\ln^2\alpha mn}\geq\left\{\sum_{n=1}^{\infty}\omega(n,\alpha)\frac{n^{p-1}a_n^p}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(n,\alpha)\frac{n^{q-1}b_n^q}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{q}}.\tag{21}$$

再由 (20), (21) 式, 注意到 $q<0$, 有 (18) 式.

对于 $0<\varepsilon<p$, 置 $\widetilde{a}_m$ 及 $\widetilde{b}_n$ 为: $\widetilde{a}_1=\widetilde{b}_1=0$;

$$\widetilde{a}_m=\frac{1}{m}\left(\frac{1}{\ln\sqrt{\alpha}m}\right)^{\frac{\varepsilon}{p}}(m\geq 2),\quad \widetilde{b}_n=\frac{1}{n}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{\frac{\varepsilon}{q}}\quad(n\geq 2),$$

则由 (18) 式, 有

$$\left\{\sum_{n=1}^{\infty}\left(1+\frac{1}{2\ln\alpha n}\right)\frac{n^{p-1}\widetilde{a}_n^p}{\ln\alpha n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}\widetilde{b}_n^q}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{q}}$$
$$=\left\{\sum_{n=1}^{\infty}(1+\theta(n))\frac{n^{p-1}\widetilde{a}_n^p}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}\widetilde{b}_n^q}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{q}}$$
$$=\left\{\sum_{n=2}^{\infty}(1+\theta(n))\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right\}^{\frac{1}{q}}$$
$$=\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\left\{1+\left[\sum_{n=2}^{\infty}\frac{\theta(n)}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]\left[\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\right]^{-1}\right\}^{\frac{1}{p}}$$
$$=\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\{1+o(1)\}^{\frac{1}{p}}\quad(\varepsilon\to 0^+).\tag{22}$$

因而有

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\ln^2\alpha mn}=\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{mn\ln^2\alpha mn}\left(\frac{1}{\ln\sqrt{\alpha}m}\right)^{\frac{\varepsilon}{p}}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{\frac{\varepsilon}{q}}$$
$$=\sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{\varepsilon}\sum_{m=2}^{\infty}\frac{1}{m\ln^2\alpha mn}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{\varepsilon}{p}}$$
$$<\sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{\varepsilon}\int_{\sqrt{\alpha^{-1}}}^{\infty}\frac{1}{x\ln^2\alpha nx}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}x}\right)^{\frac{\varepsilon}{p}}dx$$
$$=\sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{1}{\ln\sqrt{\alpha}n}\right)^{1+\varepsilon}\int_0^{\infty}\frac{u^{-\frac{\varepsilon}{p}}}{(1+u)^2}du$$
$$=B\left(1-\frac{\varepsilon}{p},1+\frac{\varepsilon}{p}\right)\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}.\tag{23}$$

若使 (19) 式的常数因子 1 不是最佳值, 则存在正常数 K ($K>1$), 使当 K 代 1 时, (19) 式

仍成立. 特别地, 由 (22) 及 (23) 式, 有

$$B\left(1-\frac{\varepsilon}{p},1+\frac{\varepsilon}{p}\right)\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}$$

$$>\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\ln^2\alpha mn}>K\left\{\sum_{n=1}^{\infty}\left(1+\frac{1}{2\ln\alpha n}\right)\frac{n^{p-1}\widetilde{a}_n^p}{\ln\alpha n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}\widetilde{b}_n^q}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{q}}$$

$$=\sum_{n=2}^{\infty}\frac{1}{n(\ln\sqrt{\alpha}n)^{1+\varepsilon}}\left\{1+o(1)\right\}^{\frac{1}{p}},$$

则有 $1\geq K$ $(\varepsilon\to 0^+)$. 这矛盾于 $K<1$. 故 (20) 式的常数因子 1 是最佳值. 证毕.

评注 4.3 (19) 式可推出

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\ln^2\alpha mn}>\left\{\sum_{n=1}^{\infty}\frac{n^{p-1}a_n^p}{\ln\alpha n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{n^{q-1}b_n^q}{\ln\sqrt{\alpha}n}\right\}^{\frac{1}{q}}. \tag{24}$$

不等式 (19) 是 (24) 式的一个加强.

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