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ON THE SNAPPING OF A THIN SPHERICAL CAP

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Abstract

In this paper, the snapping of a thin spherical cap under edge moment is considered. The snapping of the same cap under line load distributed along a circle as shown in Fig. 1 has been disscussed by Chien Wei-zang[4, 5] in two unpublished papers. His results are briefly summarized in this paper.

Consider a thin spherical cap of thickness h, span $2a$ and radius of curvature $2f/a^2$ under edge moment M per unit length uniformly distributed along the edge of the cap as shown in Fig. 4. The deflection w and the membrane radial stress N_r at a distance r from the axis of the cap satisfy the equations

$$\frac{D}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}-\frac{1}{r}\frac{d}{dr}\left[rN_r\left(\frac{2f}{a^2}r+\frac{dw}{dr}\right)\right]=0, \tag{21a}$$

$$\frac{1}{hE}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r)+\frac{2f}{a^2}r\frac{dw}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2=0, \tag{21b}$$

with the boundary conditions

$$w=0,\quad D\left(\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}\right)=-M_r,\quad N_r=0\quad at\quad r=a. \tag{22}$$

By integrating Eq. (21a), we get

$$Dr\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}-rN_r\left(\frac{2f}{a^2}r+\frac{dw}{dr}\right)=0. \tag{23}$$

In order to simplify the following calculations, let us introduce the following dimensionless variables:

$$\left.\begin{aligned} &\rho = \frac{r}{a}, \quad y = \sqrt{12(1-\mu^2)}\,\frac{w}{h}, \quad \theta = -\frac{dy}{d\rho}, \\ &S_r = \frac{12(1-\mu^2)\,a^2}{Eh^3} N_r, \quad S = \rho\, s_r, \\ &k^2 = 2\sqrt{12(1-\mu^2)}\,\frac{f}{h}, \quad m = \frac{12(1-\mu^2)\sqrt{12(1-\mu^2)}a^2}{Eh^4} M. \end{aligned}\right\} \tag{24}$$

By regarding θ and S as functions of ρ, Eqs. (23), (21b) and boundary conditions (22) reduce to

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\,\theta) + k^2 S = -\frac{\theta S}{\rho}, \tag{25a}$$

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\, S) - k^2\theta = -\frac{\theta^2}{2\rho}, \tag{25b}$$

$$\frac{d\theta}{d\rho} + \mu\frac{\theta}{\rho} = m, \quad S = 0 \quad at \quad \rho = 1. \tag{25c}$$

This is a system of two non-linear differential equations. Exact solution of these equations is very difficult. An approximate solution is given in this paper. Here we may note that two simple exact solutions can be found when m takes two special values. The first corresponds to the case when the cap is bent into a flat plate. In this case

$$\theta = k^2\rho, \quad S = -\frac{k^4}{16}\rho(1-\rho^2), \quad m = (1+\mu)\,k^2. \tag{26}$$

The second corresponds to the case when the cap is turned over into an inverted spherical cap. In this case

$$\theta = 2k^2\rho, \quad S = 0, \quad m = 2(1+\mu)\,k^2. \tag{27}$$

In order to investigate the behavior of the cap near the form of a flat plate, it is convenient to make the following substitution:

$$\theta = k^2\rho + \theta', \quad S = -\frac{k^4}{16}\rho(1-\rho^2) + S', \quad m = (1+\mu)k^2 + m'. \tag{28}$$

Substituting expressions (28) into Eqs. (25), we get

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\,\theta') + \frac{k^4}{16}(1-\rho^2)\,\theta' = -\frac{\theta' S'}{\rho}, \tag{29a}$$

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\, S') = -\frac{\theta'^2}{2\rho}, \tag{29b}$$

$$\frac{d\theta'}{d\rho} + \mu \frac{\theta'}{\rho} = m', \quad S' = 0 \quad at \quad \rho = 1. \tag{29c}$$

Equations (25) and (29) are fundamental equations in this paper.

We begin with the investigation of the small bending of the cap in accordance with the classical linear theory. By neglecting second order terms in Eqs. (25), we get

$$\frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho} (\rho\theta) + k^2 S = 0, \tag{30a}$$

$$\frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho} (\rho S) - k^2 \theta = 0, \tag{30b}$$

$$\frac{d\theta}{d\rho} + \mu \frac{\theta}{\rho} = m, \quad S = 0 \quad at \quad \rho = 1. \tag{30c}$$

The solution of Eqs. (30) under the boundary conditions

$$\theta = \beta, \quad S = 0 \quad at \quad \rho = 1$$

is

$$\theta = \frac{\beta}{2} \left\{ \frac{J_1(jk\rho)}{J_1(jk)} + \frac{J_1(j^3k\rho)}{J_1(j^3k)} \right\}, \tag{33}$$

$$S = \frac{j^2\beta}{2} \left\{ \frac{J_1(jk\rho)}{J_1(jk)} - \frac{J_1(j^3k\rho)}{J_1(j^3k)} \right\}, \qquad j = \sqrt[4]{-1} = \sqrt{i}. \tag{34}$$

The relation between the edge moment m and edge slop β can be found from Eq. (30c). Thus we obtain

$$\frac{m}{\beta} = m_1 = -1 + \mu + \frac{k}{2} \left\{ \frac{jJ_0(jk)}{J_1(jk)} + \frac{j^3J_0(j^3k)}{J_1(j^3k)} \right\}. \tag{35}$$

Values of m_1 for several values of k are given in Table 1.

Integrating formula (33), we get

$$y = \frac{\beta}{2k} \left\{ \frac{J_0(jk\rho) - J_0(jk)}{jJ_1(jk)} + \frac{J_0(j^3k\rho) - J_0(j^3k)}{j^3J_1(j^3k)} \right\}. \tag{36}$$

Deflection curves for $k=0, 2, 4, 6$ are shown in Fig. 5. When k is small, deflection curves resemble that of a flat plate. But when k is large, deflection curves have several waves. The transition from n waves to $(n+1)$ waves occurs when k satisfies the following condition:

$$\left(\frac{d\theta}{d\rho}\right)_{\rho=0} = 0, \quad i.e. \quad \frac{1}{k}\,\mathrm{Im}\, j\, J_1(j\,k) = 0. \tag{40}$$

The first two roots of this equation are

$$k_1 = 3.77, \qquad k_2 = 8.28. \tag{41}$$

Let us now consider in some details the case when $k_1 < k < k_2$. When m is small, the bending moment at the center of the cap is negative and the shape of the deflection curve resembles to that for $k=4$ as shown in Fig. 5. But when m is sufficiently large, it is evident that the bending moment at the center of the cap must be positive and the corresponding deflection curve resembles that for $k=0$ as shown in Fig. 5. The transition from a waved deflection curve to an unwaved deflection curve cannot occur in a continuous proccess. The transition must be carried out by a sudden jamp, i.e. by snapping or return back proccess. Therefore $k=3.77$ may be regarded as the upper bound of k for the occurrence of snapping. A more precise value given below is $k=3.54$, which justifies the above reasoning.

In order to determine the critical value of k for the occurrence of snapping, it is sufficient to consider the stability of the cap when it is bent to the form of a flat plate. It is quite clear that the form of a flat plate is the most unstable state of all deformations corresponding to different values of m. If this state is stable, there will be no snapping action. If this state is unstable, then snapping is possible. Therefore the critical value of k for the unstability of the flat plate form is also the critical value for the occurrence of snapping.

The critical value of k for the unstability of the flat plate form is the first eigenvalue of the following differential equation:

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\,\theta') + \frac{k^4}{16}(1-\rho^2)\,\theta' = 0\,, \tag{43a}$$

$$\frac{d\theta'}{d\rho} + \mu\,\frac{\theta'}{\rho} = 0 \quad at \quad \rho = 1\,. \tag{43b}$$

This value is determined by an approximate method. It is found that

$$3.4813 \leqslant k_1 \leqslant 3.5946. \tag{55a}$$

Therefore if we take

$$k_1 = 3.5379, \tag{55b}$$

the possible error of this value is less than 1.6%.

We next investigate the behavior of the cap near the form of a flat plate. Our purpose is to find out the relation between $\beta' = (\theta')_{\rho=1}$ and m' when β' is small. By neglecting second order terms in Eqs. (29), we get

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\theta') + \frac{k^4}{16}(1-\rho^2)\theta' = 0, \tag{56a}$$

$$\frac{d\theta'}{d\rho} + \mu\frac{\theta'}{\rho} = m' \quad at \quad \rho = 1. \tag{56b}$$

This equation is solved approximately. We obtain an approximate relation between m' and β' as follows:

$$\frac{m'}{\beta'} = m_1' = \frac{(1+\mu)\left[1-\left(\frac{k}{k_1}\right)^4\right]}{1-\left(\frac{k}{k_1}\right)^4 + \frac{k^4}{192(1+\mu)}}. \tag{63}$$

Several values of m'_1, taking $k_1=3.5379$, are given in Table 2.

Up to now we have disscussed only two linear problems. We do not intend to solve the non-linear equations (25) or (29). Based upon the previous results, it is possible to construct approximately the non-linear relation between m and β.

When $k > k_1$, corresponding to one value of m there may be several deflection curves. Therefore in this case β is a multiply valued function of m. But for not too large values of k, m may be a single valued function of β. It is found that the critical value k_s of k that, when $k<k_s$, m is a single valued function of β lies in the interval

$$4.237 \leqslant k_s \leqslant 4.369. \tag{66a}$$

Therefore, when $k \leqslant 4.237$, m is certainly a single valued function of β. In this case m may be expressed by a power series of β, or, what is the same thing, m' may be expressed by a power series of β'.

According to the results obtained previously, the curve $m'=m'(\beta')$ in the (β', m') plane (Fig. 6) have the following properties:

1) The slop of the tangent at the origin O is m_1, see Eq. (35).

2) The curve passes through the point O' with coordinates $\beta = k^2$, $m=(1+\mu)k^2$, see Eq. (26).

3) The slop of the tangent at the point O' is m_1', see Eq. (63).

4) The curve is skew-symmetric with respect to coordinate axes $O'\beta'$, $O'm'$. This follows immediately from Eq. (29).

Besides these results, the area $OCO'D$ in Fig. 6 can be found without any difficulty. This area represents the work done by the external edge moment m. According to the principle of conservation of energy, it must equal to the strain energy of the cap at the state corresponding to the point O'. In this way we find

$$\int_{-k^2}^{0} m'(\beta')\, d\beta' = -\frac{1}{2}(1+\mu)\, k^4 + \frac{k^8}{768}. \tag{69b}$$

Thus for the approximate representation of the relation between m' and β', we assume a curve of 7-th degree

$$\frac{m'}{k^2} = m_1' \frac{\beta'}{k^2} + m_3'\left(\frac{\beta'}{k^2}\right)^3 + m_5'\left(\frac{\beta'}{k^2}\right)^5 + m_7'\left(\frac{\beta'}{k^2}\right)^7, \tag{70}$$

where m'_1, m'_3, m'_5, m'_7, are determined by the properties cited above. In this way we obtain

$$\left.\begin{aligned} m_3' &= \frac{11}{2}(1+\mu) + \frac{m_1}{2} - 6\, m_1' - \frac{k^4}{32}, \\ m_5' &= -\frac{15}{2}(1+\mu) - \frac{3}{2} m_1 + 9 m_1' + \frac{k^4}{16}, \\ m_7' &= 3\,(1+\mu) + m_1 - 4 m_1' - \frac{k^4}{32}. \end{aligned}\right\} \tag{71}$$

Several values of m'_3, m'_5, m'_1 are given in Table 3.

Equation (70) in conjunction with coefficients (71) defines approximately the relation between the edge moment m and the edge slop β. In Fig. 7 is shown m-β curves for k=3.8, 3.9, 4.0, 4.1.

Let us now examine in some detail the behavior of the $m=m(\beta)$ curve in connection with the stability of deformation. For $k<k_1$, m is a steadily increasing function of β. For $k>k_1$, the function $m=m(\beta)$ has relative maximum and minimum. Apart from a slight difference near the origin O, a typical curve for $k>k_1$ is shown in Fig. 8, where the curve at the right hand side of the m'-axis has been added. It may be proved that the maximum value of m is the snapping moment m_s in the process of loading and the minimum value of m is the return back moment m_b in the process of unloading. Values of m_s and m_b found from Eq. (70) are plotted in Fig. 9 in dependence with k.

Since m–β curves are skew-symmetric with respect to axes $O'\beta'$, $O'm'$, it is easily seen that

This is an exact relation theoretically.