# Housing in a Neoclassical Growth Model

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Preliminary Draft, May 2007

**Abstract**

A simple social-planning two-sector general equilibium model with residential housing entering utility function is analysed with emphasis on local dynamics. Conditons for saddle path stability are listed and their plausibility is likely to pass testing based on calibration results.

Key words: Two-sector models; Dynamic stability; Housing

## 1 Introduction

There is increasing recognition on the importance of the interplay between housing markets and the macroeconomy. However, full analyses on housing from an aggregate perspective by mainstream macroeconomics are surprisingly scarce, while conventional housing economics tends to be limited to urban scope. This paper tries to join in the recently small yet growing macro-housing literature which borders on both macroeconomy and real estate research. Leung (2004) provides a selective survey in this field. Existing research efforts mainly focus on the relationship between housing and taxation (Turnovsky and Okuyama 1994, Berkovec and Fullerton 1992), between housing and business cycles (Davis and Heathcote 2005, Greenwood and Hercowitz 1991, Jin and Zeng 2004, Matsuyama 1990), and between policy and real estate (Jin and Zeng 2007).

This paper cuts in from a different perspective: what's the qualitative effect that economic fundamentals have on housing market, particularly, on housing price? Will housing price change indeterminately like a bubble? We present a socially planning two-sector general equilibium model with residential housing entering utility function, which is related to Turnovsky and Okuyama (1994). The subsequent analysis has connections with two-sector transitional dynamics literature (Bond et al. 1996, Mulligan and Sala-I-Martin 1993, Eicher and Turnovsky 2001, Herrendorf and Valentinyi 2006), since local dynamic stability is essential for providing an answer to the above questions. We find that housing

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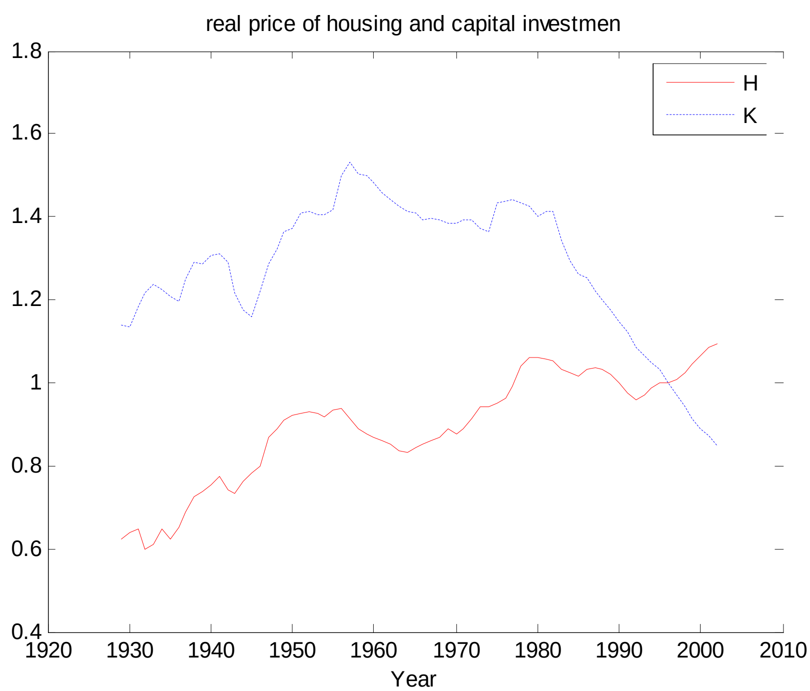
price in balanced growth path reponds to economic fundamentals in the way catering to common sense and is saddle path stable under certain conditions, whose plausibility is tested using calibration results in the latter part of this article.

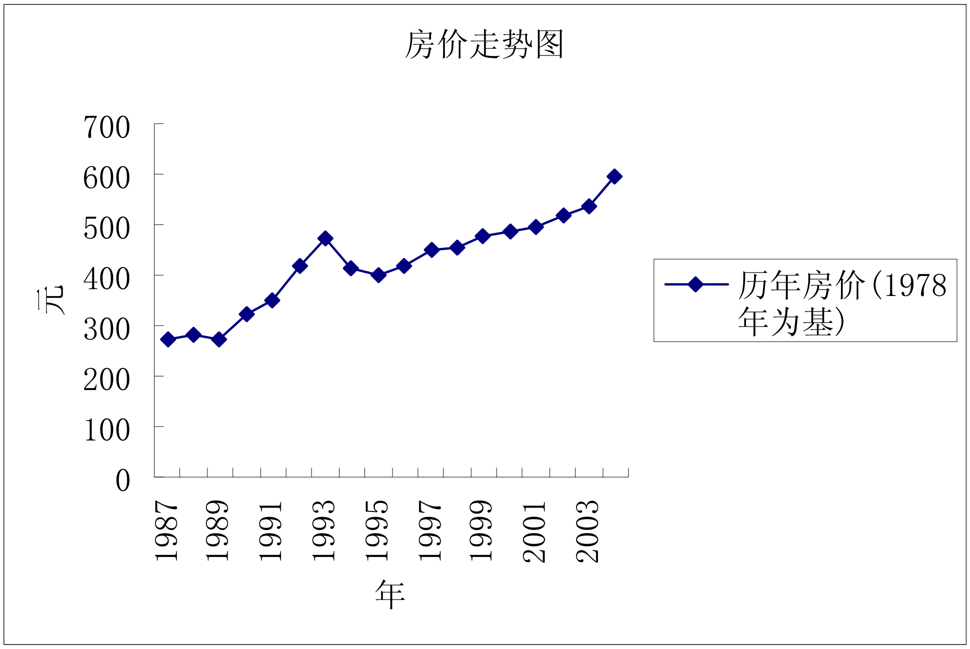
The rest of the paper is organized as follows. The next section give some empirical facts which exhibits the importance of housing sector in macroeconomy. Section 3 presents a simple social-planning two-sector model with some distinguishing features compared with literatures in both fields, and lists its balanced growht path in explicit form. The fourth section transform the model into a reduced form so as to analyse its local stability more tractably. Quantative assesements exploiting calibration results to have a look at the prediction power of the model are provided in section 5. The final section concludes the paper.

## 2 Empirical facts

The increasing house price has been not only a hot economic issue but also an important political factor. Recently, the Premier in Korea has to resign due to failing in prohibiting the rise of house price. The importance of real estate has quickly attracted public attention in recent years in many countries, both in developing countries such as China and in OEDC ones, e.g., Canada. As a starting point, we shall get knowledge about what is the secular trend of house price? For this purpose, we draw the following figure to have a look at price change in USA.

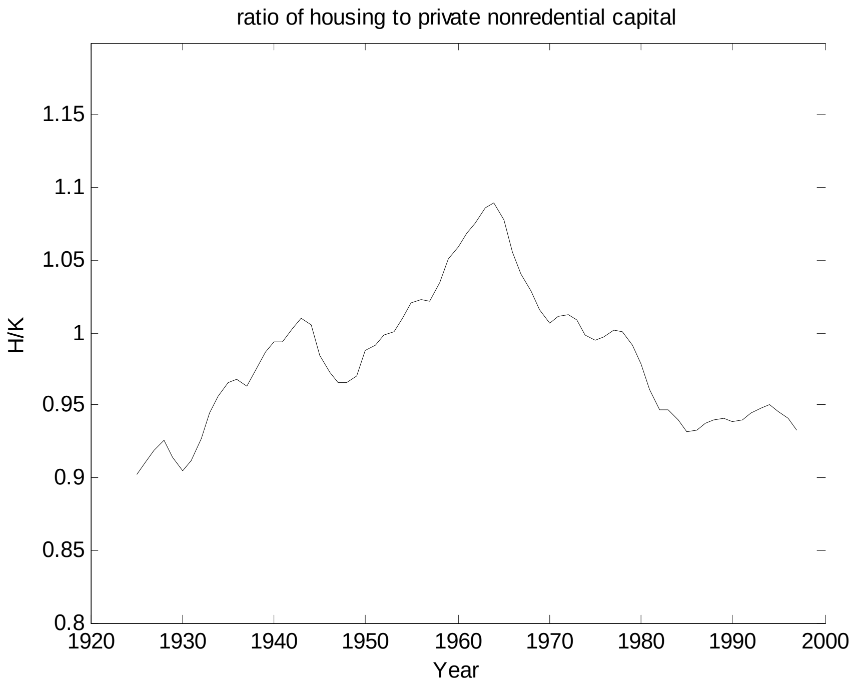



As shown in the above figure, dotted blue line represents price of private fixed capital other than housing, and real red line that of house price. In general, they all have a secular increasing trend but differ somewhat along the period of 73 years. Especially since 1970, house price increase more firmly than other two.

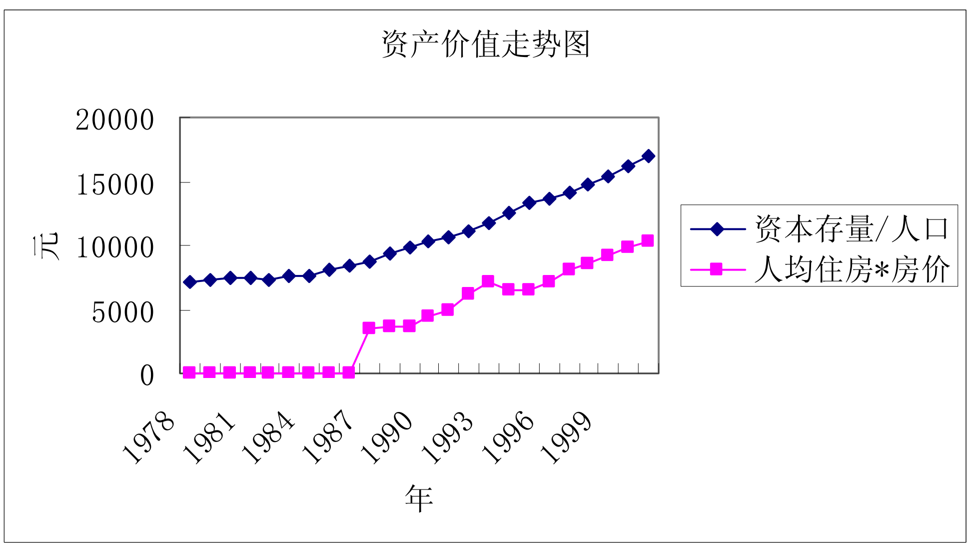



The above figure is the path of real house price in China. Early time may not be suitable for description of price change since price does not reflect market signal in planned economy. But from 1995 on when market reform keeps proceeding, the trend of house price is upward without any surprice.

The importance of housing market may also reflect in the fact that value of housing stock is very large. We further have a look at the ratio in real fixed private capital stock. Data from NIPA table are processed to get the below figure.




Along a long period of about 70 years, the average ratio of housing stock value to private fixed productive capital is near 1, which indicates that housing stock has a much greater influence on the macroeconomy than what we perceived previously.




The above figure is about China's housing stock per capita. Evidently, value of housing stock is even more than half of total capital stock. Thus, research in such field has important significance both in theory and in reality.

# 3 A two-sector economy with housing

Corresponding to the decentralized model in Turnovsky and Okuyama (1994), a socialy planning two-sector model is esblished to observe what effects economic fundamentals have on housing price in an artificial frictionless world. There are two production factors, capital $K$ and labor $L$, allocated in two sectors, general goods sector and housing sector. Each sector is featured with exogenously growing technologies $A$ or $B$ whose growth rates, $x_1$ or $x_2$, is not necessarily equal[1]. Labor size is exogenously growing in a rate of $n$. Throughout, the general good will be taken as numeraire commodity and can be used in both consumption and investment. Depreciations to capital $K$ and housing $H$ reflects in parameters $\delta_1$ and $\delta_2$. For simplication, all housing is assumed to be residential and owner-occupied. The model is as follows[2]:

$$(P1)\begin{cases} \max \int_0^\infty U(C,H)e^{-\rho t}dt \\ \text{s.t.} \begin{cases} \dot{K} = A(sK)^\alpha (vL)^{1-\alpha} - C - \delta_1 K \\ \dot{H} = B((1-s)K)^\beta ((1-v)L)^{1-\beta} - \delta_2 H \\ \dfrac{\dot{L}}{L} = n,\ \dfrac{\dot{A}}{A} = x_1,\ \dfrac{\dot{B}}{B} = x_2 \end{cases} \end{cases}$$

Initial conditions $L_0$, $K_0$, $H_0$, $A_0$ and $B_0$ are given. $s$ and $v$ represent the proportions of factors $K$ and $L$ allocated to general goods sector respectively. For convenience, we difine $Y_1 \equiv A(sK)^\alpha (vL)^{1-\alpha}$ and $Y_2 \equiv B((1-s)K)^\beta ((1-v)L)^{1-\beta}$ and specifies:[3]

$$U(C,H) = \ln C + \gamma \ln H$$

This model differs from other two-sector growth models in two aspects. One is that there is a special kind of stock, i.e. housing $H$, entering untility function; the other, motion equation for $H$. is specified separately. The former implies that residential services and general consumption might not be perfect substitute to each other, and there is a proportional relationship between housing stock and residential service flows.

Factor $L$ is assigned to grow exogenously while allocation variables are endogenously determined. Such specification marks the model as two-sector neoclassical class. It's convenient to see the dynamics of the ratio of co-state variables to two motion equations, which is in fact the relative price of products between two sectors, i.e. housing price.

[1] Such specification has reflection in structural change growth literature originated from Baumol (1967) and carried forward in recent years such as Ngai and Pissarides (2005). It accommodates partial imbalances in a balanced whole. This feature differs our model from those in macro-housing and dynamics fields, and adds complexity to local stability analysis.

[2] See appendix 1 for a detailed knowledge about the connection between model (P1) and literature.

[3] If utility function is specified as $U(C,H) = \frac{(C^\eta H^{1-\eta})^{1-\sigma}-1}{1-\sigma}$, the qualitative results won't change except rates of balanced growth is alterred quantitatively. The setup in the text is a special case of the above form, in which $\sigma = 1$ and $\gamma = \frac{1-\eta}{\eta}$.

### 3.1 Balanced growth equilibrium

This subsection analyses the above model's balanced growth equilibrium along which all variables grow at constant, though possibly different, rates. The Hamilton function of the above system can be written as:

$$\tilde{\mathcal{H}} = \ln C + \gamma \ln H + \lambda[Y_1 - C - \delta_1 K] + \theta[Y_2 - \delta_2 H]$$

where $\lambda$ and $\theta$ are shadow prices[4] of $K$ and $H$ respectively. The price of housing at time t can be presented by $P \equiv \frac{\theta}{\lambda}$[5]. FOCs to $C$, $s$ and $v$ and Euler equations for $K$ and $H$ are as follows correspondingly:

$$\frac{1}{C} = \lambda \tag{1a}$$

$$\alpha\frac{Y_1}{s} = \beta P\frac{Y_2}{1-s} \tag{1b}$$

$$(1-\alpha)\lambda\frac{Y_1}{v} = (1-\beta)\theta\frac{Y_2}{1-v} \tag{1c}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta_1 - \frac{\alpha Y_1}{K} - \beta p\frac{Y_2}{K} \tag{1d}$$

$$\frac{\dot{\theta}}{\theta} = \rho + \delta_2 - \frac{\gamma}{\theta H} \tag{1e}$$

$$TVCs: \lim_{t\to\infty} e^{-\rho t}\lambda_t K_t = \lim_{t\to\infty} e^{-\rho t}\theta_t H_t = 0$$

From (1b) and (1c):

$$v = \frac{1}{1 + \frac{\alpha(1-\beta)}{(1-\alpha)\beta}(\frac{1}{s} - 1)} \tag{2}$$

From (1b) and (1d):

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta_1 - \frac{\alpha Y_1}{sK} \tag{3}$$

Note that $\frac{\alpha Y_1}{sK} - \delta_1$ is acturally the real interest rate in the economy. From (1a), (1e) and the definition $P \equiv \frac{\theta}{\lambda}$:

$$\frac{\dot{\theta}}{\theta} = \rho + \delta_2 - \gamma\frac{C}{PH} \tag{4}$$

Then

$$\frac{\dot{P}}{P} = \frac{\dot{\theta}}{\theta} - \frac{\dot{\lambda}}{\lambda} = \frac{\alpha Y_1}{sK} - \gamma\frac{C}{pH} + \delta_2 - \delta_1 \tag{5}$$

[4] i.e., co-state variables to motion equation of $\dot{K}$ and $\dot{H}$.

[5] Such disposal method is used frequently in two-sector growth literature.

From (1a) and (3):

$$\frac{\dot{C}}{C} = \frac{\alpha Y_1}{sK} - \rho - \delta_1 \tag{6}$$

Since substituting (2) into (1b) would make $s$ and $\nu$ the functions of other economy variables, then combining (5), (6) and rearranging two motion equations for $K$ and $H$, we get the dynamic system of $P$, $C$, $K$ and $H$, which can fully describe the transition process of this economy. For convenience, the system is listed below:

$$\begin{cases} \frac{\dot{P}}{P} = \frac{\alpha Y_1}{sK} - \frac{\gamma C}{PH} + \delta_2 - \delta_1 \\ \frac{\dot{C}}{C} = \frac{\alpha Y_1}{sK} - \rho - \delta_1 \\ \frac{\dot{K}}{K} = \frac{Y_1}{K} - \frac{C}{K} - \delta_1 \\ \frac{\dot{H}}{H} = \frac{Y_2}{H} - \delta_2 \end{cases} \tag{7}$$

The first step of analysis to (7) is to see whether balanced growth path exists. Through guessing and verifing method, it's easy to compute the growth rates $g_i, i \in \{C, K, H, P\}$ of balanced growth path along which real viables evolve.

$$g_C = g_K = n + \frac{x_1}{1-\alpha} \tag{8a}$$

$$g_H = x_2 + \frac{\beta}{1-\alpha} x_1 + n \tag{8b}$$

$$g_p = \frac{1-\beta}{1-\alpha} x_1 - x_2 \tag{8c}$$

The equation (8c) indicates that the stationary growth rate of housing price has no connection with population increase speed. This may be due to opposite effects offseting each other in this specific model. On the demand side, housing demand will increase with population growth and thus push housing price high; while on the supply side, more quantity of factors are put into production of housing which tends to lower housing price. In equilibrium in this model, only technologies growth rate affect the trend of housing price in BGP. The model provides an empirical way to test the influence of economic fundamentals on housing price since parameters in the model could be calibrated from reality data.

# 4 Dynamic stability

Although balanced growth path does exist in system (7), we can't directly judge whether $P$, $C$, $K$ and $H$ would converge to this path. As they are not stationary variables, transformation to (7) is necessary. Many alternatives may be available for such purpose, for example, $\frac{AL^{1-\alpha}}{K}$, $\frac{C}{K}$, $\frac{BK^{\beta}L^{1-\beta}}{H}$ and $\frac{C}{PH}$. However, they may be too complicated to be analysed. Here presents an easily tractable technique.

## 4.1 Equivalent reduced form

Let's take some transformations to (P1). Suppose $M^{1-\alpha} \equiv AL^{1-\alpha}$ and $Q^{1-\beta} \equiv BL^{1-\beta}$. Then straightforwardly, we have $\frac{\dot{M}}{M} = \frac{1}{1-\alpha}x_1 + n \equiv m_1$ and $\frac{\dot{Q}}{Q} = \frac{1}{1-\beta}x_2 + n \equiv m_2$. If $m_1 > m_2$, the motion equations can be transformed into:

$$\dot{K} = s^{\alpha}v^{1-\alpha}K^{\alpha}M^{1-\alpha} - C - \delta_1 K \tag{9a}$$

$$\dot{H} = (1-s)^{\beta}(1-v)^{1-\beta}K^{\beta}M^{\phi} - \delta_2 H \tag{9b}$$

where $\phi \equiv \frac{m_2}{m_1}(1-\beta) \in (0, 1-\beta)$[6]. Let's see the first case in (9a) and (9b) and make transformation of variables as $k \equiv \frac{K}{M}$, $c \equiv \frac{C}{M}$ and $h \equiv \frac{H}{M^{\beta+\phi}}$.

The utility function can then be changed to

$$\begin{aligned} U(C,H) &= \ln C + \gamma \ln H \\ &= \ln c + \gamma \ln h + [1 + \gamma(\beta+\phi)] \ln M \end{aligned}$$

Since $M_t = M_0 e^{q_1 t}$, where $M_0 = A_0 L_0^{\alpha}$, then: $\ln M_t = \ln M_0 + q_1 t$

$$\Longrightarrow \qquad U(C,H) = \ln c + \gamma \ln h + [1+\gamma(\beta+\phi)](\ln M_0 + m_1 t)$$

then

$$\int_0^{\infty} U(C,H)e^{-\rho t}dt = \int_0^{\infty} [\ln c + \gamma \ln h]e^{-\rho t}dt + \int_0^{\infty} [1+\gamma(\beta+\phi)](\ln M_0 + m_1 t)e^{-\rho t}dt \tag{11}$$

It's easy to verify that the second term on the righthand side of equation (11) is bounded. Therefore, the original model will be equivalent to the following transformed model:

$$(P2)\begin{cases} \max \int_0^{\infty} (\ln c + \gamma \ln h)e^{-\theta t}dt \\ s.t. \begin{cases} \dot{k} = \bar{A}s^{\alpha}v^{1-\alpha}k^{\alpha} - c - \delta_3 k \\ \dot{h} = \bar{B}(1-s)^{\beta}(1-v)^{1-\beta}k^{\beta} - \delta_4 h \end{cases} \end{cases}$$

[6] Correspondingly, if $m_1 \leq m_2$, the motion equations become:

$$\dot{K} = s^{\alpha}v^{1-\alpha}K^{\alpha}Q^{\frac{m_1}{m_2}(1-\alpha)} - C - \delta_1 K \tag{10-a}$$

$$\dot{H} = (1-s)^{\beta}(1-v)^{1-\beta}K^{\beta}Q^{1-\beta} - \delta_2 H \tag{10-b}$$

Just take one of them, and we will infer similar conclusions to the other case where $\bar{A} = \bar{B} = 1$, $\delta_3 \equiv \delta_1 + m_1$ and $\delta_4 \equiv \delta_2 + \beta m_1 + (1-\beta)m_2$. Objective function is equivalent due to the property that monotonic increasing transformation does not affect utility[7], as long as the parameter $\rho$ is above zero.

Compared with (P1), (P2) is almost identical to it. Denote $\hat{\lambda}$ and $\hat{\theta}$ as the shadow price of $k$ and $h$ respectively. Then $p \equiv \hat{\theta}/\hat{\lambda}$ will represent the relative spot price of adjusted products in sector $h$ to those in sector $k$. Analogous to the computation in previous baseline model, it can be verified that $k$, $h$, $c$ and $p$ are stationary variables and their dynamic system has the form very much similar to the system (7) Therefore, analysis on stability property of our baseline model (P1) can be made in the same way as that on (P2).

### 4.2 Dynamic system

The significance of stability analysis lies in that if the system is not stable, price trend of housing would diverge to infinity or zero. If its dynamic system has indeterminate property, price bubble is possible. Whether it will diverge or increase sharply like a bubble under the influence of economy fundamentals is the problem we are concerned with. Define $y_1 \equiv \bar{A}s^{\alpha}v^{1-\alpha}k^{\alpha}$ and $y_2 \equiv \bar{B}(1-s)^{\beta}(1-v)^{1-\beta}k^{\beta_1}$. Hamiltonian for problem (P2)

$$\mathcal{J} = \ln c + \gamma \ln h + \hat{\lambda}[y_1 - c - \delta_3 k] + \hat{\theta}[y_2 - \delta_4 h]$$

FOCs to $c$, $s$ and $v$: and Euler equations to $k$ and $h$ are almost identical to the solution process of (P1). So, we only list key steps.

$$z \equiv \frac{s}{v}k = \frac{1}{q_1}p^{\frac{1}{\alpha-\beta}} \tag{12a}$$

$$s = s(p,k) = \frac{\alpha(1-\beta)}{\alpha-\beta} - \frac{(1-\alpha)\beta}{\alpha-\beta}\frac{1}{q_1 k}p^{\frac{1}{\alpha-\beta}} \tag{12b}$$

$$v = v(p,k) = \frac{\alpha(1-\beta)}{\alpha-\beta}q_1 p^{\frac{-1}{\alpha-\beta}}k - \frac{(1-\alpha)\beta}{\alpha-\beta} \tag{12c}$$

where $q_1 \equiv (\frac{\alpha}{\beta})^{\frac{\beta}{\alpha-\beta}}(\frac{1-\alpha}{1-\beta})^{\frac{1-\beta}{\alpha-\beta}}(\frac{\bar{A}}{\bar{B}})^{\frac{1}{\alpha-\beta}}$. Note $z$, a transformation of real interest rate, is only a function of $p$. If $\alpha > \beta$, then $s > v$, which means that general goods sector is capital K intensive and housing sector is compound factor M intensive. It's easy to verify that when $\alpha > \beta$, $s\frac{d\nu}{ds} - \nu > 0$, $\frac{\partial s}{\partial p} < 0$ and $\frac{\partial s}{\partial k} > 0$. It is not difficult to get the dynamic

[7] Changing objective function to other forms like $\int_0^{\infty}[\ln C + \gamma \ln H]Le^{-\rho t}dt$, where $C$ and $H$ denote comsumption per capita and housing stock per capita, would not alter conclusion qualitatively. What will be changed is the subjective discount parameter from $\rho$ to $\theta$, which is a complicated function of model parameters. More generally, there could be a corresponding between $\rho$ and $\theta$, depending on the specific form of utility function.

system of $p$, $c$, $k$ and $h$, which can fully describe the transition process of this economy.

$$\begin{aligned}
\dot{p} &= q_3 p^{1+\frac{\alpha-1}{\alpha-\beta}} - \frac{\gamma c}{h} + (\delta_4 - \delta_3)p && \text{(13a)} \\
\dot{c} &= q_3 p^{\frac{\alpha-1}{\alpha-\beta}} c - (\rho + \delta_3)c && \text{(13b)} \\
\dot{k} &= q_4 v(p,k) p^{\frac{\alpha}{\alpha-\beta}} - c - \delta_3 k && \text{(13c)} \\
\dot{h} &= q_2(1 - \nu(p,k)) p^{\frac{\beta}{\alpha-\beta}} - \delta_4 h && \text{(13d)}
\end{aligned}$$

where $q_i$ are all constants. $q_2 \equiv B(\frac{\beta(1-\alpha)}{\alpha(1-\beta)})^\beta q_1^{-\beta}$, $q_3 \equiv \alpha \bar{A} q_1^{1-\alpha}$ and $q_4 \equiv \bar{A} q_1^{-\alpha}$.

#### 4.2.1 Steady states

The first step of analysis to system (13) is to figure out its steady states. From (13b): $q_3 (p^*)^{\frac{\alpha-1}{\alpha-\beta}} = \rho + \delta_3$, hence, $p^*$ can be figured out. From (13a):

$$\frac{c^*}{h^*} = p^* \frac{\rho + \delta_4}{\gamma} \tag{14}$$

From (13d):

$$q_2 \frac{\alpha(1-\beta)}{\alpha-\beta}(1 - q_1 (p^*)^{\frac{-1}{\alpha-\beta}} k^*)(p^*)^{\frac{\beta}{\alpha-\beta}} = \delta_4 h^* \tag{15}$$

From (13c):

$$\begin{aligned}
& \frac{1-\beta}{\alpha-\beta}(\rho+\delta_3)k^* - \bar{A}\frac{\beta(1-\alpha)}{\alpha-\beta}\left(\frac{a\bar{A}}{\rho+\delta_3}\right)^{\frac{\alpha}{1-\alpha}} \\
= \ & \delta_3 k^* + \frac{\rho+\delta_4}{\gamma\delta_4} q_2 \frac{\alpha(1-\beta)}{\alpha-\beta}(p^*)^{\frac{\alpha}{\alpha-\beta}}(1 - q_1 (p^*)^{\frac{-1}{\alpha-\beta}} k^*)
\end{aligned} \tag{16}$$

From (14), (15) and (16), $k^*$, $c^*$ and $h^*$ can be computed out. However, explicit form is too messy to be necessary to list out.

#### 4.2.2 Stability

Linearize the dynamic system (13) at its steady states.

$$\dot{z} \equiv \begin{pmatrix} \dot{p} \\ \dot{c} \\ \dot{k} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} p - p^* \\ c - c^* \\ k - k^* \\ h - h^* \end{pmatrix} \equiv J_4(z - z^*) \tag{17}$$

The coefficient matrix is the patial values of system (13) with respect to ($p$, $c$, $k$, $h$) at their steady states, i.e.:

$$J_4 \equiv = \left.\frac{\partial \dot{z}}{\partial z}\right|_{z^*}$$

Detailed computation results are as follows:

$$\begin{aligned}
a_{11} &= \rho+\delta_4-\frac{1-\alpha}{\alpha-\beta}(\rho+\delta_3)\\
a_{12} &= -\frac{\gamma}{h^*}\\
a_{13} &= 0\\
a_{14} &= \frac{\gamma c^*}{(h^*)^2}=(\rho+\delta_4)\frac{p^*}{h^*}\\
a_{21} &= -\frac{(1-\alpha)(\rho+\delta_3)}{(\alpha-\beta)p^*}\\
a_{22} &= a_{23}=a_{24}=0\\
a_{31} &= -\frac{(1-\alpha)(\rho+\delta_3)}{(\alpha-\beta)^2p^*}[(1-\beta)k^*+\beta(\frac{\alpha\bar{A}}{\rho+\delta_3})^{\frac{1}{1-\alpha}}]\\
a_{32} &= -1\\
a_{33} &= \frac{1-\beta}{\alpha-\beta}\rho+\frac{(1-\alpha)\delta_3}{\alpha-\beta}\\
a_{34} &= 0\\
a_{41} &= \frac{\bar{B}[\beta(1-\alpha)]^{\beta}[\alpha(1-\beta)]^{1-\beta}}{(\alpha-\beta)^2p^*}(\frac{\alpha\bar{A}}{\rho+\delta_3})^{\frac{\beta}{1-\alpha}}\\
&\quad\times[\beta+(1-\beta)(\frac{\rho+\delta_3}{\alpha\bar{A}})^{\frac{1}{1-\alpha}}k^*]\\
a_{42} &= 0\\
a_{43} &= -\frac{\bar{B}[\beta(1-\alpha)]^{\beta}[\alpha(1-\beta)]^{1-\beta}}{\alpha-\beta}(\frac{\rho+\delta_3}{\alpha\bar{A}})^{\frac{1-\beta}{1-\alpha}}\\
a_{44} &= -\delta_4
\end{aligned}$$

To analyse the stability of (P2), we need to figure out how many eigenvalues of $J_4$ have positive and negative real parts. Suppose $\lambda_i$, $i=\{1,2,3,4\}$ are the roots of $J_4$. It's not difficult to verify that no matter whether $\alpha>\beta$ or $\alpha<\beta$, $|J_4|>0$ and $\sum_{i=1}^4\lambda_i=\sum_{i=1}^4 a_{ii}=2\rho>0$, which imply that the signs of real parts of four eigenvalues of $J_4$ are either all positive or two negative and two positive. Since there are two jump variables and two state variables in the model, for saddle-path stability purpose it needs to give the conditions for the latter case.

According to Dockner (1985), a necessary and sufficient condition for system (17) to have two real positive and two real negative eigenvalues are $\kappa\equiv a_{11}a_{33}+2\gamma\frac{c^*}{(h^*)^2}<0$ and $0<\det(J_4)\le(\frac{\kappa}{2})^2$; or to have two negative real parts and two positive real parts of complex eigenvalues is $\det(J_4)>(\frac{\kappa}{2})^2$ and $\det(J_4)>(\frac{\kappa}{2})^2+\frac{1}{2}\rho^2\kappa$.

Another approach is to exploit Descartes' rule of signs theorem. Let $f(\lambda)=|J_4-\lambda I_4|=\lambda^4-tr(J_4)\lambda^3+\pi_2\lambda^2+\pi_3\lambda+\det(J_4)$. A necessary and sufficient conditon for two positvie and two negative eigenvalues is either $\pi_2<0$ or $\pi_3>0$.

The above two conditions are not directly observable since steady states values of variables are implicitly included in expressions. We infer from the second one to give a stronger sufficient conditon in the case of $\alpha > \beta$ as follows:

$$[\rho + \delta_4 - \frac{1-\alpha}{\alpha-\beta}(\rho+\delta_3)][\frac{1-\beta}{\alpha-\beta}\rho + \frac{(1-\alpha)\delta_3}{\alpha-\beta}] \leq (2\rho+\delta_4)\delta_4$$

or

$$(\rho+\delta_4)\delta_4\beta \geqslant (\rho+\delta_3)(1-\alpha)(\frac{\rho+\delta_3}{\alpha\bar{A}})^{\frac{1}{1-\alpha}}$$

With above conditions, dynamic system (17) and original model (P2) and (P1) will be featured with a stable saddle path. This will exclude the possibility of indeterminacy which implies bubbles of housing price.

# 5 Quantitative assesment

The explicit specifications are a weakness from theoretical perspective, but convenient for empirical testing. Exploiting actual data and calibrating values of parameters in the model will enable us predict the balanced growth rates and judge whether conditions for saddle-path stability are met.

## 5.1 Calibration

There are nine parameters in model (P1). Due to structural similarities in models, we adopt values of factor shares and depreciation rates from Jin and Zeng (2004) and use commonly acknowledged labor growth rate in literature[8]. Subjective discounting rate $\rho$ is inferred from the empirical fact that quarterly real interest rate in USA is 1%[9]. Explicitly, these six parameters are listed as

$$\begin{aligned} n &= 1.5\%,\ \rho = 0.0392,\ \alpha = 0.32,\ \beta = 0.13 \\ \delta_{1,disc} &= 0.065,\ \delta_{2,disc} = 0.015 \end{aligned}$$

where $\delta_{1,disc}$ and $\delta_{2,disc}$ are depreciation rate in discrete-time environment. They need to be transferred to corresponding value in continuous-time model, that is $\delta_1 = 0.0672$, $\delta_2 = 0.0151$. Then, there are three parameters left to be calibrated. Detailed description about data source and estimation method is provided in the second part of appendix. Here only list our calibration results:

$$x_{1,disc} = 1.1\%,\ x_{2,disc} = 0.056\%,\ \gamma = 0.191$$

[8]Such as 1.5% in a continous-time model in Eicher and Turnovsky (2001) and 1.4% in a discrete model in King and Rebelo (1993).

[9]From $e^{-\rho} = \frac{1}{1.04}$, and linearizing $e^{-\rho}$ at 0 for two orders, it will get the value of $\rho$ as 0.076. Results in literature include 0.0545 inferred from Cooley and prescott (1995), 0.04 in Eicher and Turnovsky (2001), 0.02 in Herrendorf and Valentinyi (2006), 0.05 in Greiner (1996), and 0.0513 inferred from Greenwood, Hercowitz and Krusell (1997).

Similarly, technology growth rates need to be transferred to continous-time version, that is, $x_1 = 1.09\%$, $x_2 = 0.056\%$.

### 5.2 Balanced growth rates

With above calibration results in hand, growth rate of housing price based on model prediction and actual data are as follows:

$$g_{P,prediction} = 1.34\%, \qquad g_{P,data} = 1.67\%$$

It seems that even using such a simple model and such assesment, the prediction on housing price growing rate is surprisingly near the actual trend. The difference between prediction and actual rate may be because the model (P1) is a first best economy with no friction while there are always distortions in reality. In such sense, it would be comprehensible that predicion rate is higher than actual trend.

### 5.3 Saddle-path conditions

The next step is to see whether saddle-path conditions are met. For convenience, we specifies $\bar{A} = \bar{B} = 1$. The steady states for (P2) are as follows:

$$\begin{aligned} h &= 2.425,\ k = 3.2804,\ p = 1.149,\ c = 1.0495 \\ s &= 0.9737,\ v = 0.9216 \end{aligned}$$

Obviously, the implied constraints $s > v$, arising from $\alpha > \beta$, and $s, v \in (0,1)$ are satisfied. The matrix $J_4$ is

$$\begin{pmatrix} -0.41991 & -0.078764 & 0 & 0.034089 \\ -0.42807 & 0 & 0 & 0 \\ -7.4449 & -1 & 0.53105 & 0 \\ 6.4795 & 0 & -0.42807 & -0.032744 \end{pmatrix}$$

Then eigenvalues can be computed to be

$$\lambda_{1,2,3,4} = \begin{pmatrix} -0.6641 & 0.705 & 0.141 & -0.1035 \end{pmatrix}$$

which are exactly two positive and two negative. Let's see whether conditions from Dockner (1985) are met:

$$\begin{aligned} \kappa &= -0.1548 < 0 \\ \det(J_4) &= 0.0068 > 0 \\ \det(J_4) - (\frac{\kappa}{2})^2 &= 0.00084074 > 0 \\ \det(J_4) - (\frac{\kappa}{2})^2 - \frac{1}{2}\rho^2\kappa &= 0.00095968 > 0 \end{aligned}$$

Therefore, the necessary and sufficient conditions for two positive and two negative real parts of eigenvalues of $J_4$ are exactly met, i.e., the second Dockner condition is satisfied.

### 5.4 sensitivity test

Is the above assesment result robust? Since nine parameters are calibrated from actual data, we noly need to vary $\bar{A}$ and $\bar{B}$ to have a sensitivity test. We first hold $\bar{B}$ constant at one and vary $\bar{A}$. Results indicate that all conditions implied and required are met. Further increasing $\bar{A}$ would not alter conclusion but decreasing $\bar{A}$ lower to 0.2, 0.1 etc. will make complex eigenvalues emerging, which still means that saddle-path stability property unchanged. Reversely, holding $\bar{A} = 1$ and varying $\bar{B}$, the result is very similar to the above case.

Furthermore, even for those calibrated parameters, we still find that under some extensive changes, two positive and two negative eigenvalues always emerge. This may be due to the neoclassical property of the original model. The result, that saddle-path property of the model is very robust, implies that saddle-path stability is not only suitable for USA economy but also applicable to other countries although no data are processed.

## 6 Concluding remarks

The paper presents a model combining characteristics of three kinds of literature: macro-housing, two-sector dynamics and structural change growth. The objective is to see whether sharp increase in housing price is induced by economic fundamentals or not. After adopting a transforming technique, we get an equivalent reduced form to the original model, which makes local stability analysis more convenient and more tractable. Under explicit functional forms, conditions for saddle-path stability are provided. Quantitative assesments indicate that in an extensive scope of parameter values, the property of saddle path stability holds. Thus, we may get a negative answer to the above question. Moreove, the prediction power of this simple model on balanced growth rates is fairly well.

Further work may consider an extension form of the model including land, which is promising since land is an essential factor in producing houses. Capturing the core properties of housing market would help to mark macro-housing as an independent research field.

## 7 Appendix

### 7.1 relation with literature for (P1)

Turnovsky and Okuyama (1994) uses a decentralized economy model to analyze tax effect on housing price. Particularly, their model in household dimension is as follows:

$$\max \int_0^\infty U(c,H)e^{-\rho t}dt$$

$$\begin{aligned} & \dot{b} + nb + \dot{k} + nk + (\dot{\sigma h}) + n\sigma h + c + PH \\ = \quad & \omega + i(1-\tau_b)b + r(1-\tau_k)k + (r_h + \dot{\sigma}/\sigma)\sigma h(1-\tau_h) - T \end{aligned}$$

where $H$ denotes residential service and $h$ housing stock. Delete bond $b$ and various taxes, and apply the specification $H = \alpha h$ and no arbitrage condition $P = \sigma r_h$, budget constraint is then simplified as:

$$\dot{k} + nk + (\dot{\sigma h}) + n\sigma h + c + PH = \omega + rk + (r_h + \dot{\sigma}/\sigma)\sigma h$$

$$\Longrightarrow \qquad \dot{k} + nk + \sigma(\dot{h} + nh) + c = \omega + rk$$

Combining the above motion equation with specifications in production side, we can rewrite the model in a socially planning form:

$$\max \int_0^\infty U(c,h)e^{-\beta t}dt$$

$$\text{s.t.} \quad \begin{cases} \dot{k} = f(k_1) - c - nk \\ \dot{h} = g(k_2) - nh \\ k(0),\ h(0)\, given \end{cases}$$

to which our model is very analygous with some differences though. However, papers in macro-housing literature do not give any stability analysis to the model economies, which is exactly our emphasis in this paper.

## 7.2 description about parameter calibration

Data for parameter calibration come from National Income and Product Accounts (NIPA) Tables in USA Bureau of Economic Analysis[10]. In order to make data consistent with implications of variable "output" in the model, we use gross product tables on Standarlized Industry Classification (SIC) basis. More specifically, housing sector is proxyed by construction industry while general goods sector is summarized by all other private sectors. Primary data source is its GDPbyInd_VA_SIC.xls table which could be downloaded from BEA website.

[10] Web site www.bea.gov

Labor data are from NIPA table 6.9 series while fixed capital data from Net Stock Table 3 series. As real GDP data by industry are only available for the period of 1977-1997, we have to truncate labor and capital data to fit this epoch. Moreover, since capital data are year-end numbers and output and labor data are flows in an year, we use geometry average method to adjust capital data. With these data at hand and value of factor shares originated from Jin and Zeng (2004), we calculate the technology growth rates in general goods sector and housing sector respectively as $x_{1,disc} = 1.11\%$, $x_{2,disc} = -0.577\%$.

To calibrate $\gamma$, note from system (7): $\frac{\dot{P}}{P} = \frac{\alpha Y_1}{sK} - \frac{\gamma C}{PH} + \delta_2 - \delta_1$. Since $sK$ is exactly $K_1$, while data series of $Y_1$ and $K_1$ have been availble in the process of computing solow residuals in two sectors, the ratio $\frac{Y_1}{K_1}$ is then can be calculated. Exactly, $\frac{Y_1}{K_1} = 0.743$ for 1977-1997. $\frac{\dot{P}}{P}$ is also calculted out in section 4.2, that is, 1.68% in discrete-time environment. When transfered to continuous-time version, $\frac{\dot{P}}{P} = 1.67\%$. According to Davis and Heathcote (2005), aggregate output is equal to housing stock value, i.e., $\frac{PH}{Y} = 1$. Therefore, we need only to estimate the value of $\frac{C}{Y}$. From NIPA talbe 1.1.6, average real saving rate for 1929-2006 is 0.1145, which means $\frac{C}{PH} = \frac{C}{Y} = 1 - 0.1145 = 0.8855$. Finally, we can compute $\gamma$:

$$\gamma = [\,0.32 * 0.743 + 0.0151 - 0.0672 - 0.0167]\,/\,0.8855 = 0.191$$

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