Randomly Modulated Periodic Signals

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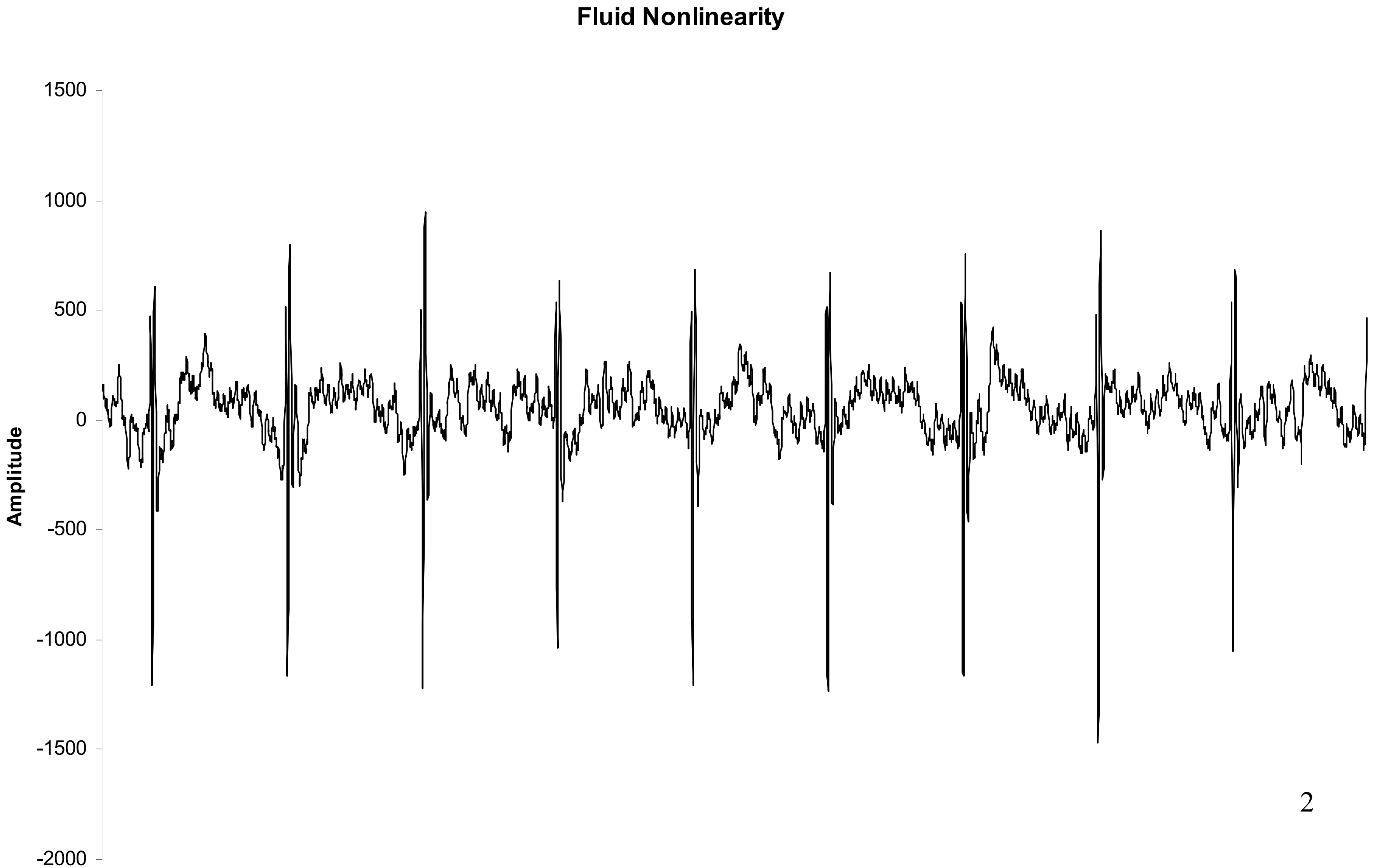
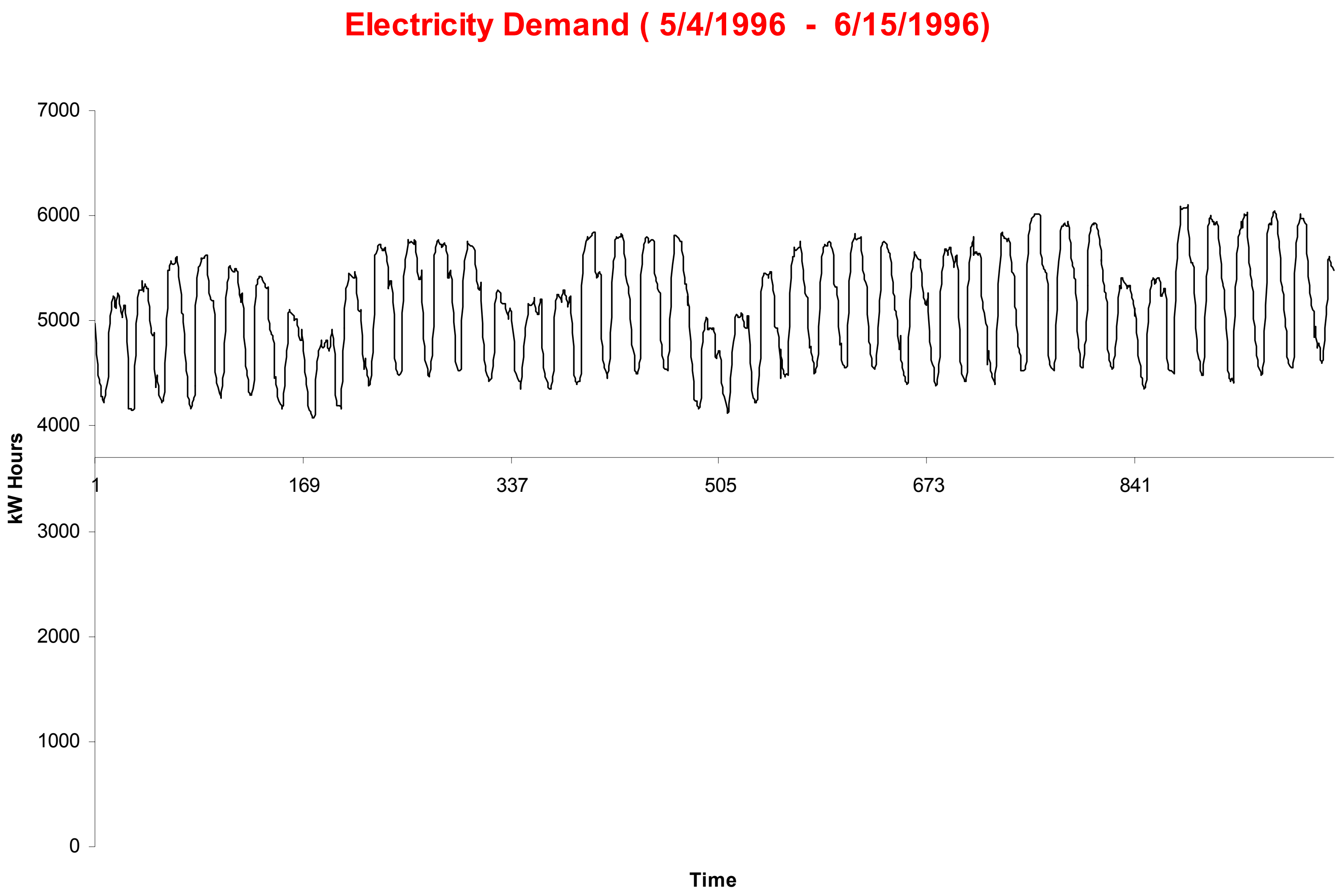
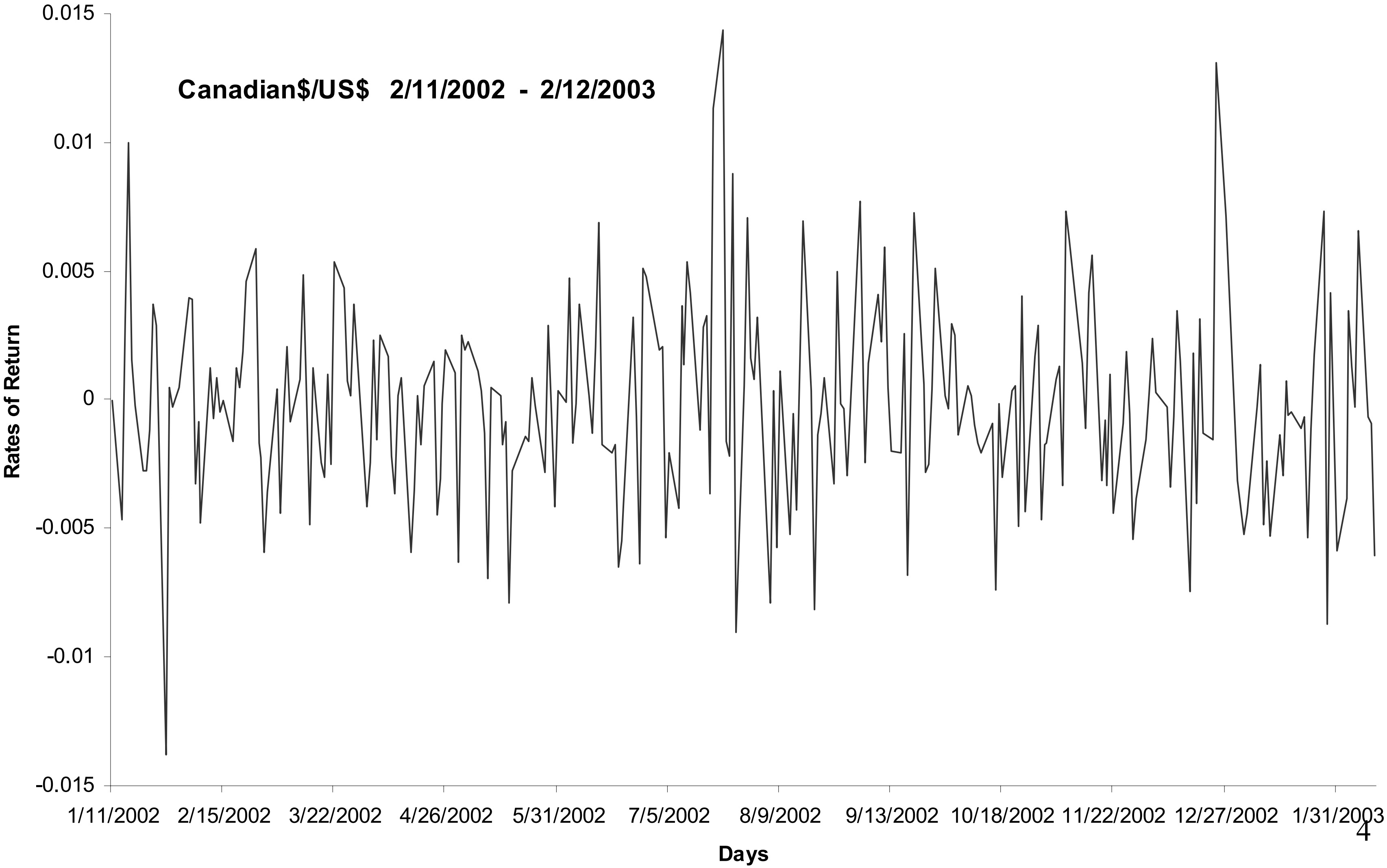
Rotating Cylinder Data



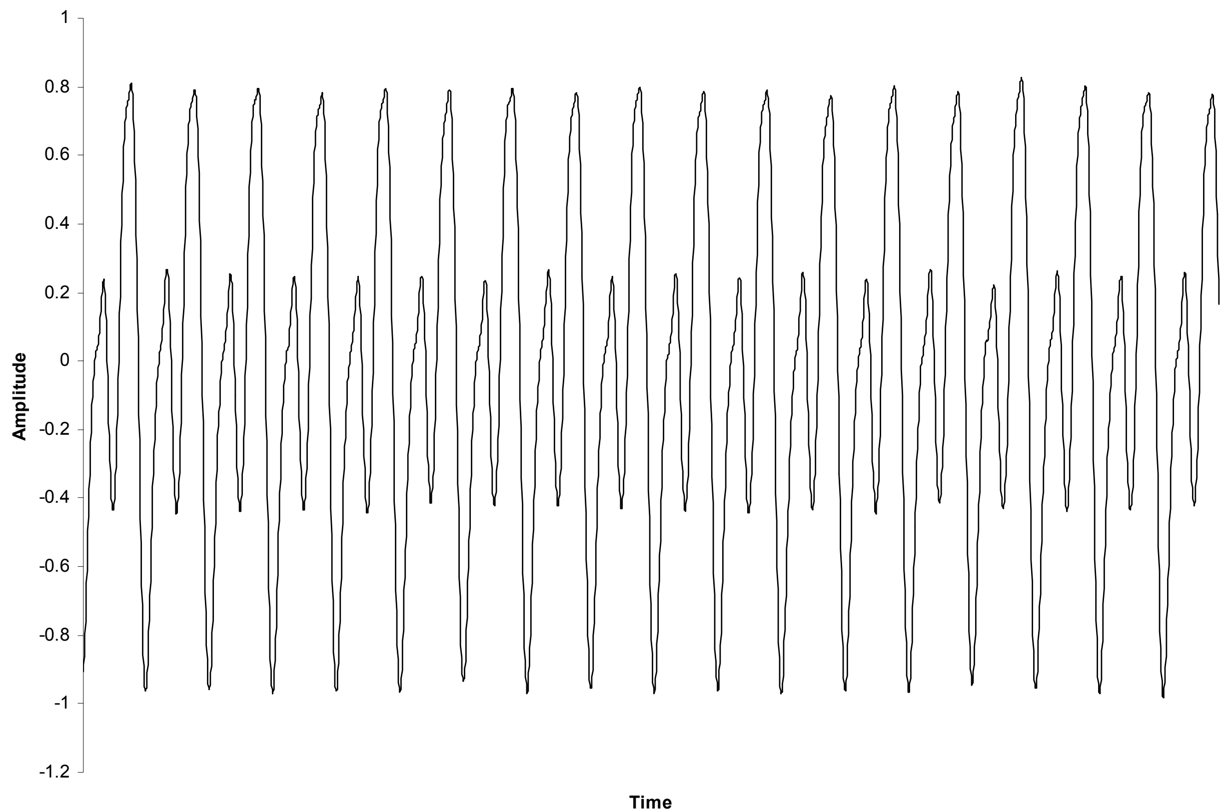
Hourly Alberta Electricity Demand



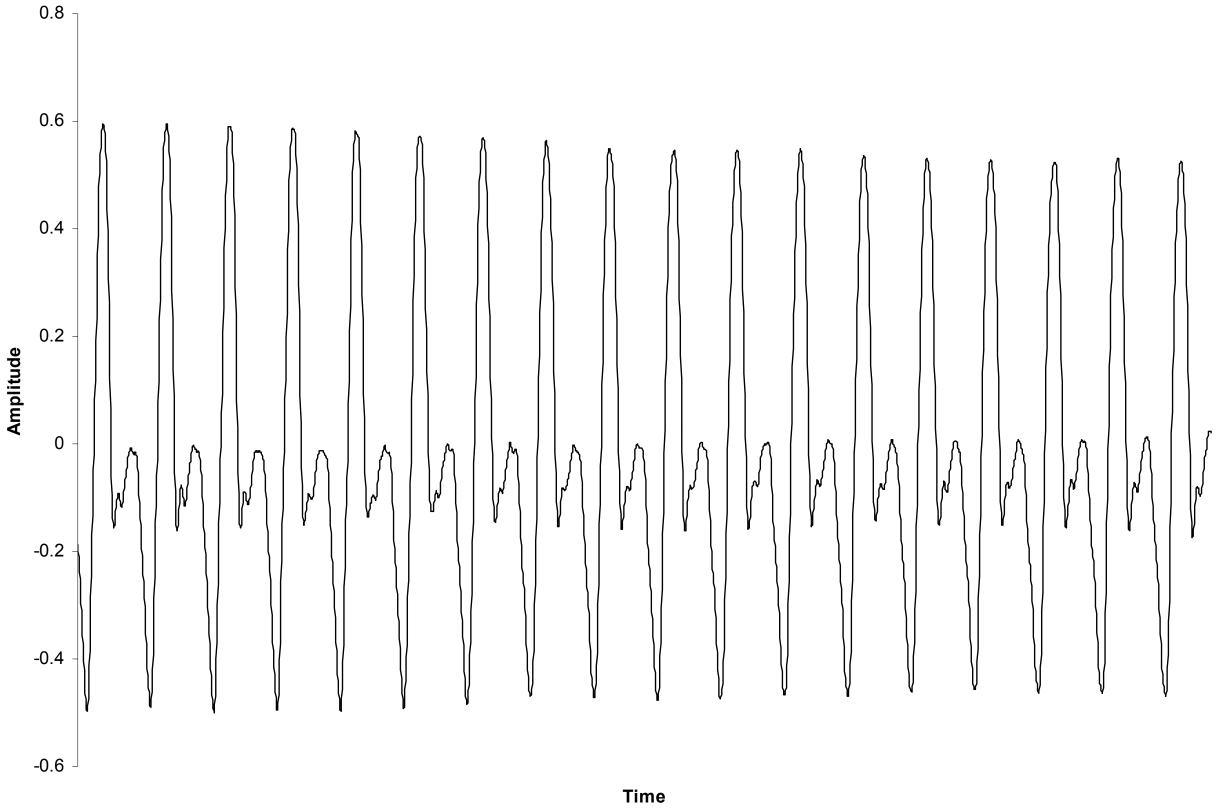
Canadian $/US $ Rates of Return



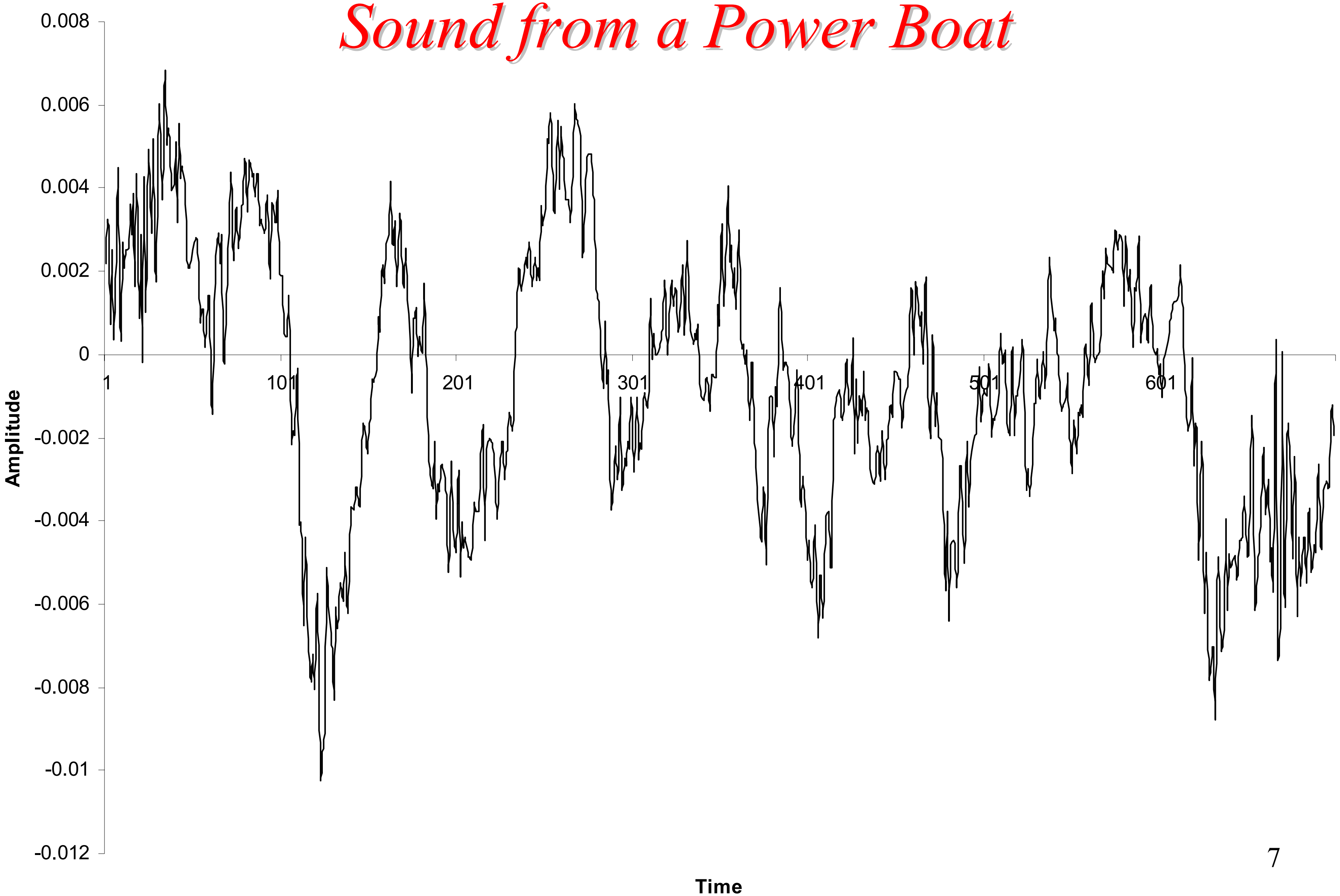
Bassoon Note



Flute Note



Sound from a Power Boat



Definition of a RMP

A signal is called a *randomly modulated periodicity* with period $T=N\delta$ if it is of the form

$$x(t_n) = \mu_0 + N^{-1} \sum_{k=1}^{K} \begin{bmatrix} \left(s_{1k} + u_{1k}(t_n)\right)\cos\left(2\pi f_k t_n\right) + \\ \left(s_{2k} + u_{2k}(t_n)\right)\sin\left(2\pi f_k t_n\right) \end{bmatrix}$$

$$f_k = \frac{k}{T} \quad t_n = n\delta \quad Eu_{1k}(t_n) = Eu_{2k}(t_n) = 0$$

$$\text{for each } k = 1, \cdots, K \quad \text{where } K \le \frac{T}{2\delta}$$

Random Modulations

The vector of the K modulations

$$\mathbf{u}(t_n) = \{u_{1k}(t_n), u_{2k}(t_n) : k = 1, \cdots, K\}$$

are jointly dependent random processes

for all $0 < t_1 < \cdots < t_m < T$

Finite Dependence

Condition needed to ensure that averaging over frames yields asymptotically gaussian estimates

$$\{\mathbf{u}(t_1),\cdots,\mathbf{u}(t_m)\} \quad \& \quad \{\mathbf{u}(t_1^{'}),\cdots,\mathbf{u}(t_r^{'})\}$$

are independently distributed if

$t_m + D < t_1^{'}$ for some D & and all

$$t_1 < \cdots < t_m \quad \& \quad t_1^{'} < \cdots < t_n^{'}$$

Fourier Series for Components

Thus $x(t_n) = s(t_n) + u(t_n)$ *where*

$$s(t_n) = s_0 + N^{-1}\sum_{k=1}^{K}\left[s_{1k}\cos\left(2\pi f_k t_n\right) + s_{2k}\sin\left(2\pi f_k t_n\right)\right]$$

$$u(t_n) = N^{-1}\sum_{k=1}^{K}\left[u_{1k}\cos\left(2\pi f_k t_n\right) + u_{2k}\sin\left(2\pi f_k t_n\right)\right]$$

Signal Plus Noise

$s(t_n)$ *is the mean of* $x(t)$

$\{u(t_n)\}$ *has a periodic joint distribution*

The modulation is part of the signal

It is not measurement noise

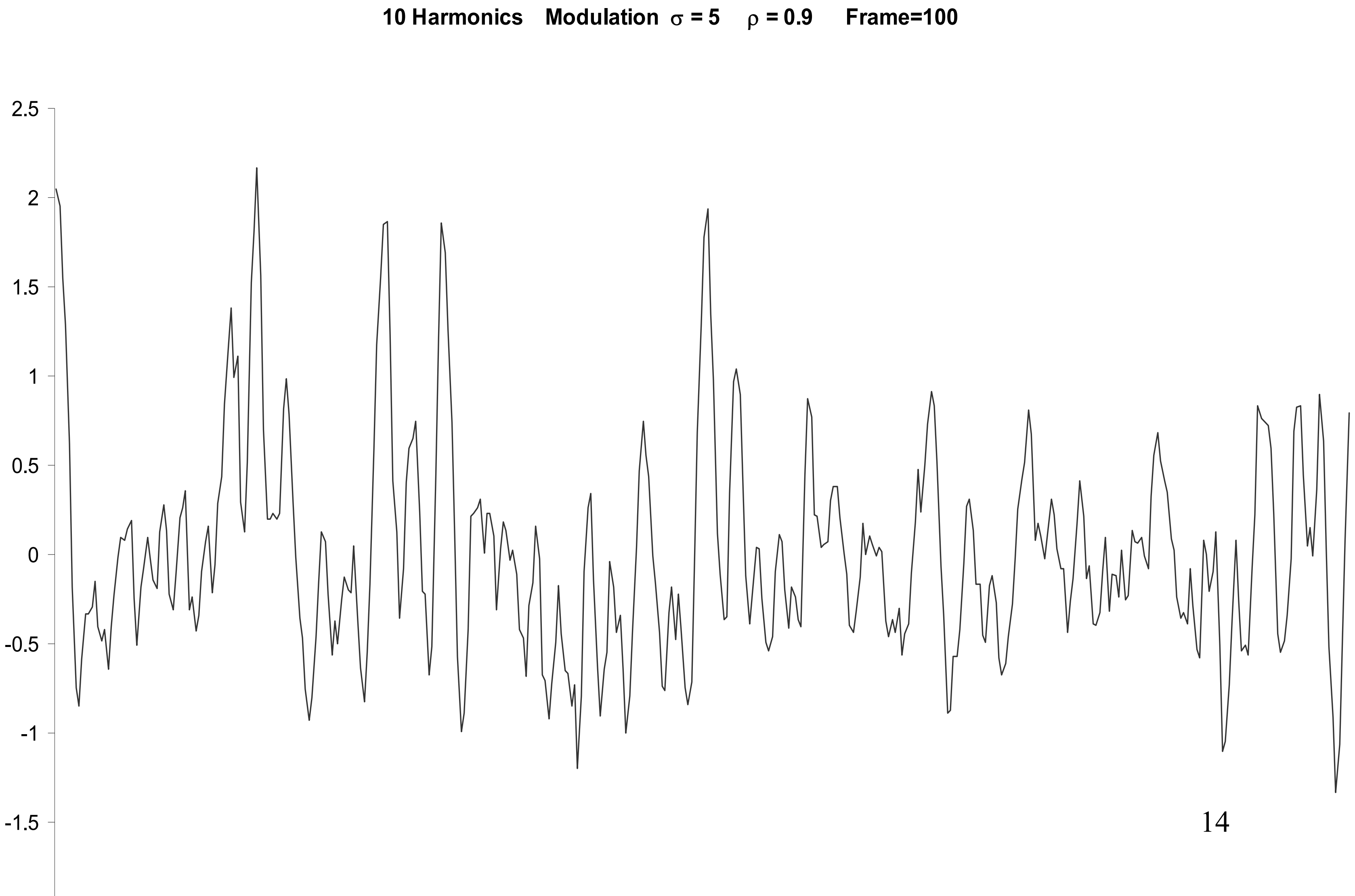
Artificial Data Examples

$$x(t_n) = s_0 + N^{-1}\sum_{k=1}^{K}\begin{bmatrix}\left(1+\sigma u_{1k}\left(t_n\right)\right)\cos\left(2\pi f_k t_n\right) + \\ \left(1+\sigma u_{2k}\left(t_n\right)\right)\sin\left(2\pi f_k t_n\right)\end{bmatrix}$$

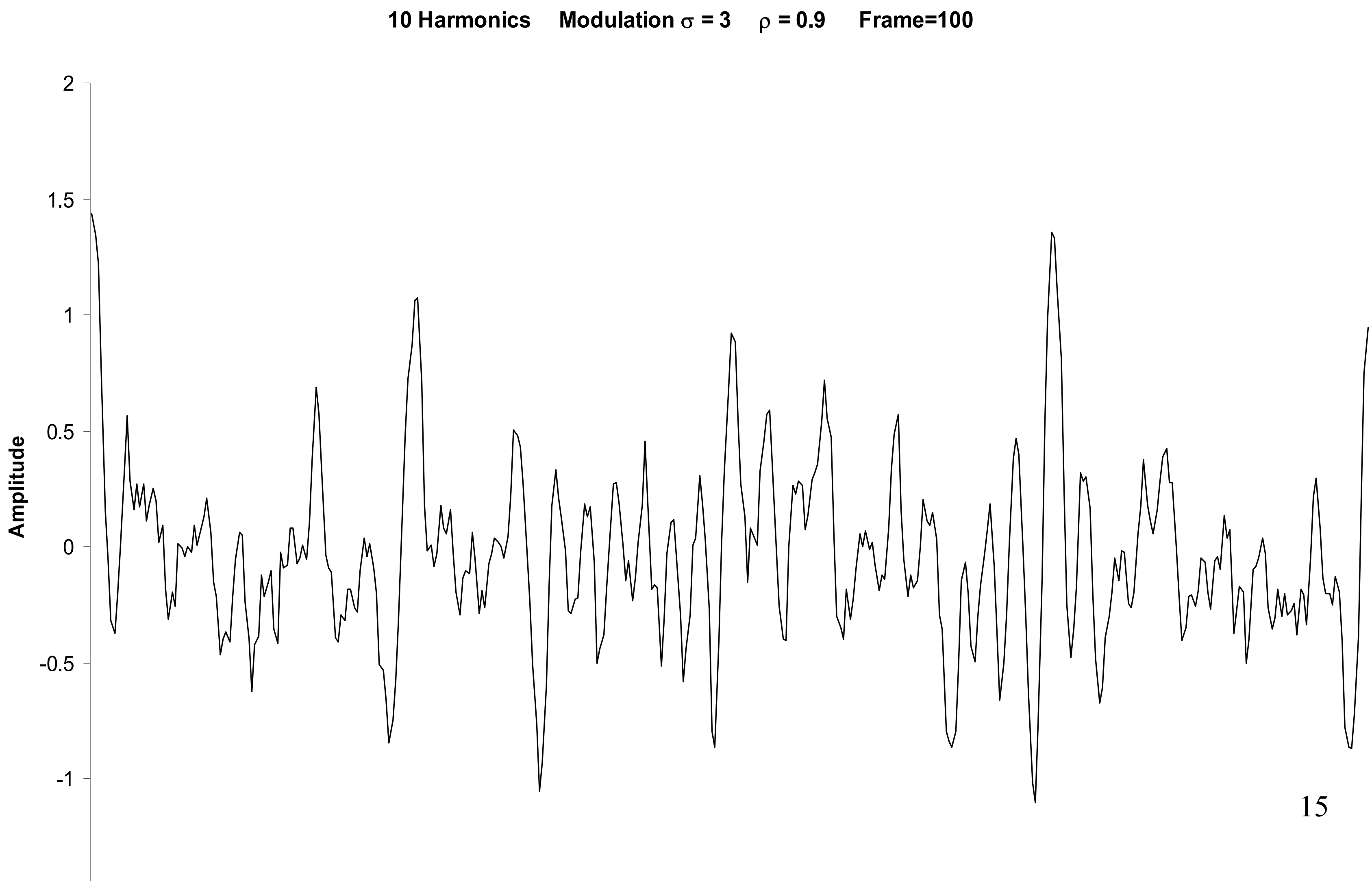
$$u_{1k}\left(t_n\right) = \rho u_{1k}\left(t_n - T\right) + e_1\left(t_n\right)$$

$$u_{2k}\left(t_n\right) = \rho u_{2k}\left(t_n - T\right) + e_2\left(t_n\right)$$

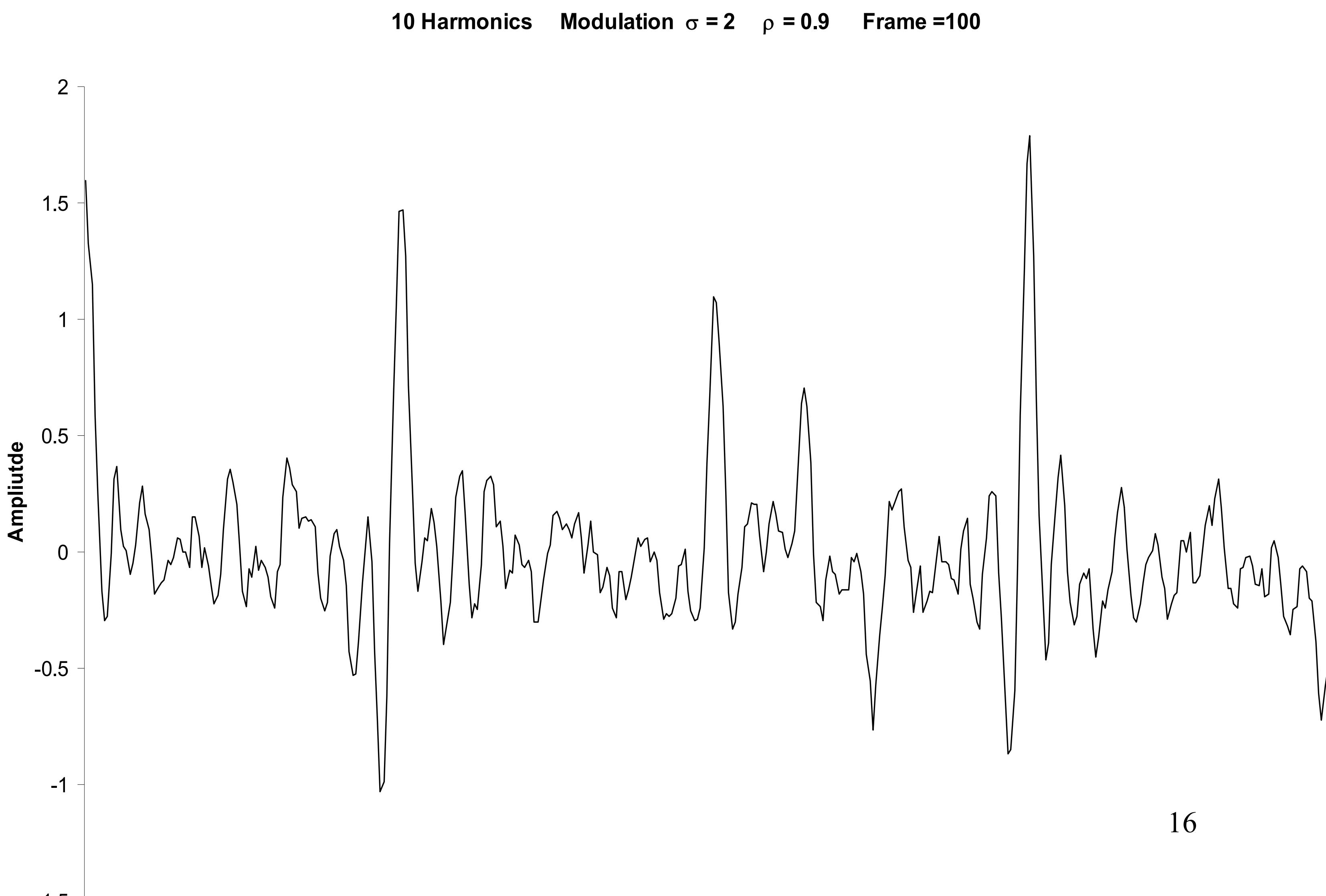
Five Standard Deviations



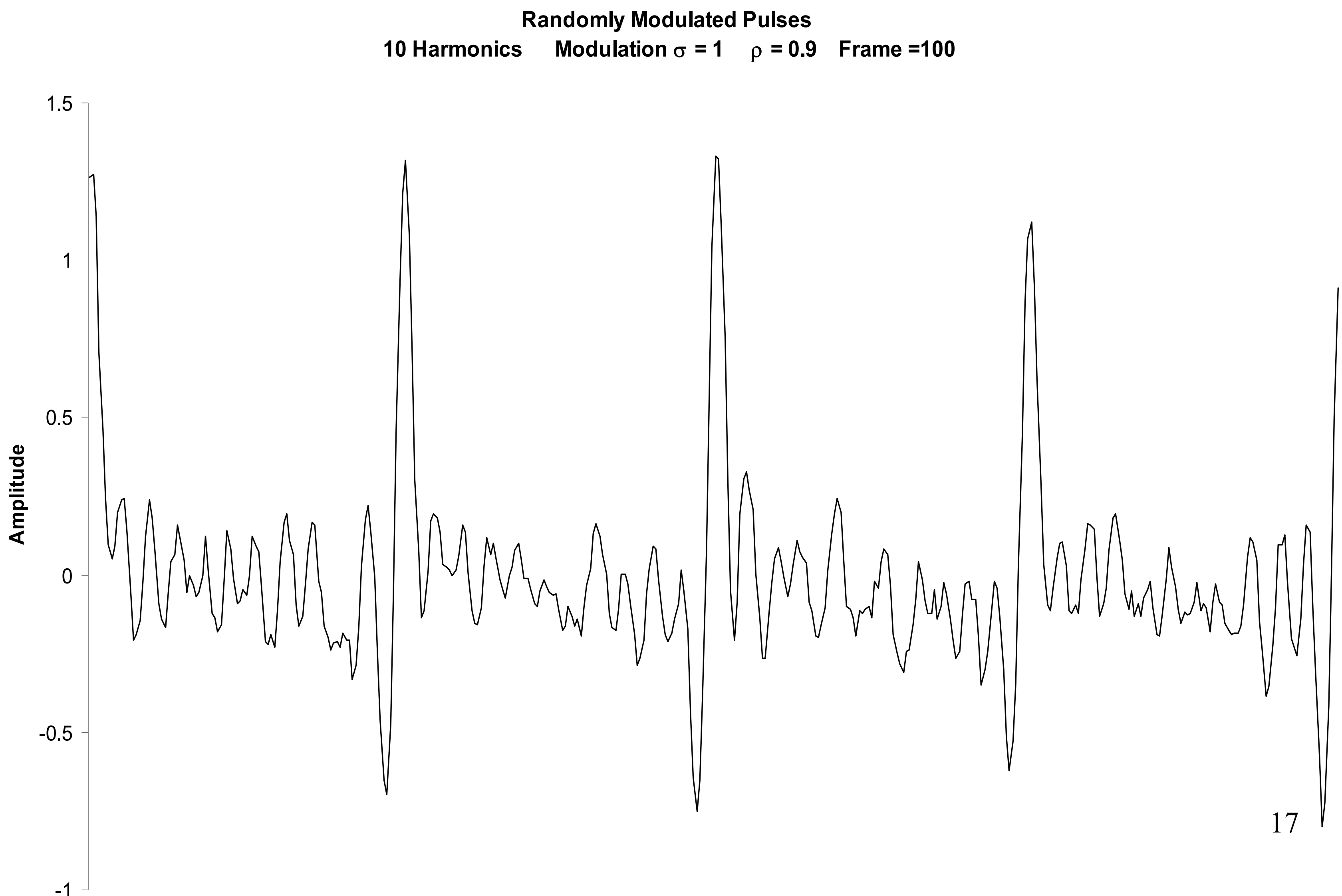
Three Standard Deviations



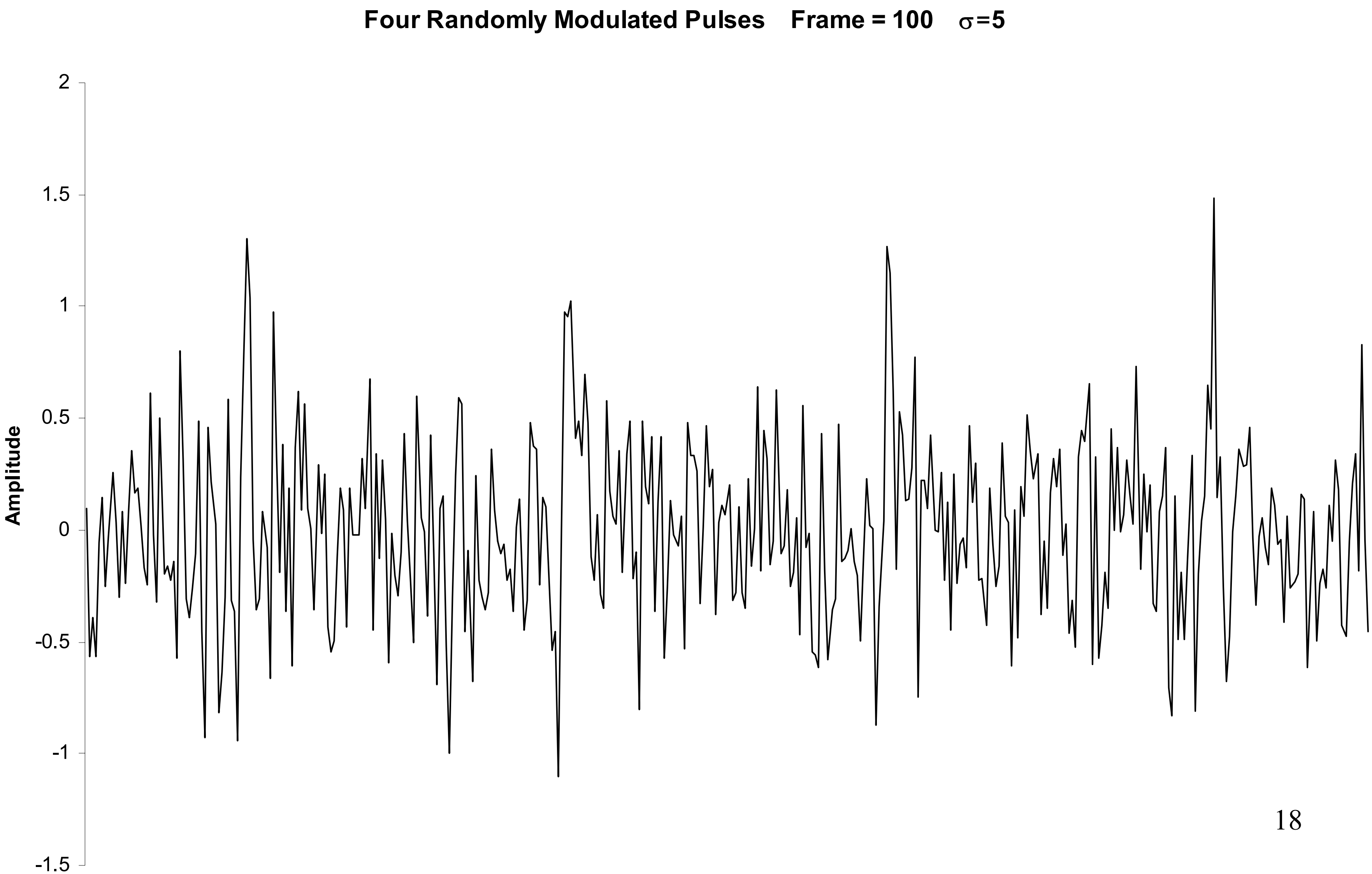
Two Standard Deviations



One Standard Deviation



No Correlation in the Modulation



Block Data into Frames

The data block is divided into M frames of length T

T is chosen by the user to be the period of the periodic component

The *t-th* observation in the *mth* frame is

$$x\left((m-1)T + n\delta\right) \quad \mathrm{n} = 0, \cdots, N-1$$

Frame Rate Synchronization

The frame length T is chosen by the user to be the hypothetical period of the randomly modulated periodic signal.

If T is not an integer multiple of the true period then coherence is lost.

Signal Coherence Spectrum

$$\gamma_x(k) = \sqrt{\frac{|s_k|^2}{|s_k|^2 + \sigma_u^2(k)}}$$

The signal-to-noise ratio is

$$\rho_x(k) = |s_k|^2 \sigma_u^{-2}(k)$$

$$\rho_x(k) = \frac{\gamma_x^2(k)}{1 - \gamma_x^2(k)}$$

Estimating Signal Coherence

$\left\{\hat{x}\left(t_n\right): n = 0, \cdots, N-1\right\}$ *is the mean frame averaged over the M frames*

$$\hat{X}(k) = \sum_{n=0}^{N-1} \hat{x}(t_n) \exp(-i2\pi f_k t_n)$$

$$\hat{\gamma}_x(k) = \sqrt{\frac{\left|\hat{X}(k)\right|^2}{\left|\hat{X}(k)\right|^2 + \hat{\sigma}_u^2(k)}}$$

$$\hat{\sigma}_u^2(k) = M^{-1} \sum_{m=1}^{M} \left|X_m(k) - \hat{X}(k)\right|^2$$

Statistical Measure of Modulation SNR

$$Z(k) = \frac{M}{N} \frac{\left|\hat{X}(k)\right|^2}{\hat{\sigma}_x^2(k)}$$

$$= \frac{M}{N} \hat{\rho}_x^2(k)$$

$$\hat{\sigma}_x^2(k) = \left|\hat{X}(k)\right|^2 + \hat{\sigma}_u^2(k)$$

Chi Squared Statistic

$$Z(k) = \frac{M}{N}\hat{\rho}_x^2(k)$$

If the modulation *is stationary the distribution of each* $Z(k)$ *is approximately* $\chi_2^2(\lambda_k)$ & *they are independently distri*buted.

$$\lambda_k = \frac{M}{N}\rho_x^2(k)$$

Spectrum of the Variance

$s(t)$ - *a stationary random process*

Fourier Series Expansion

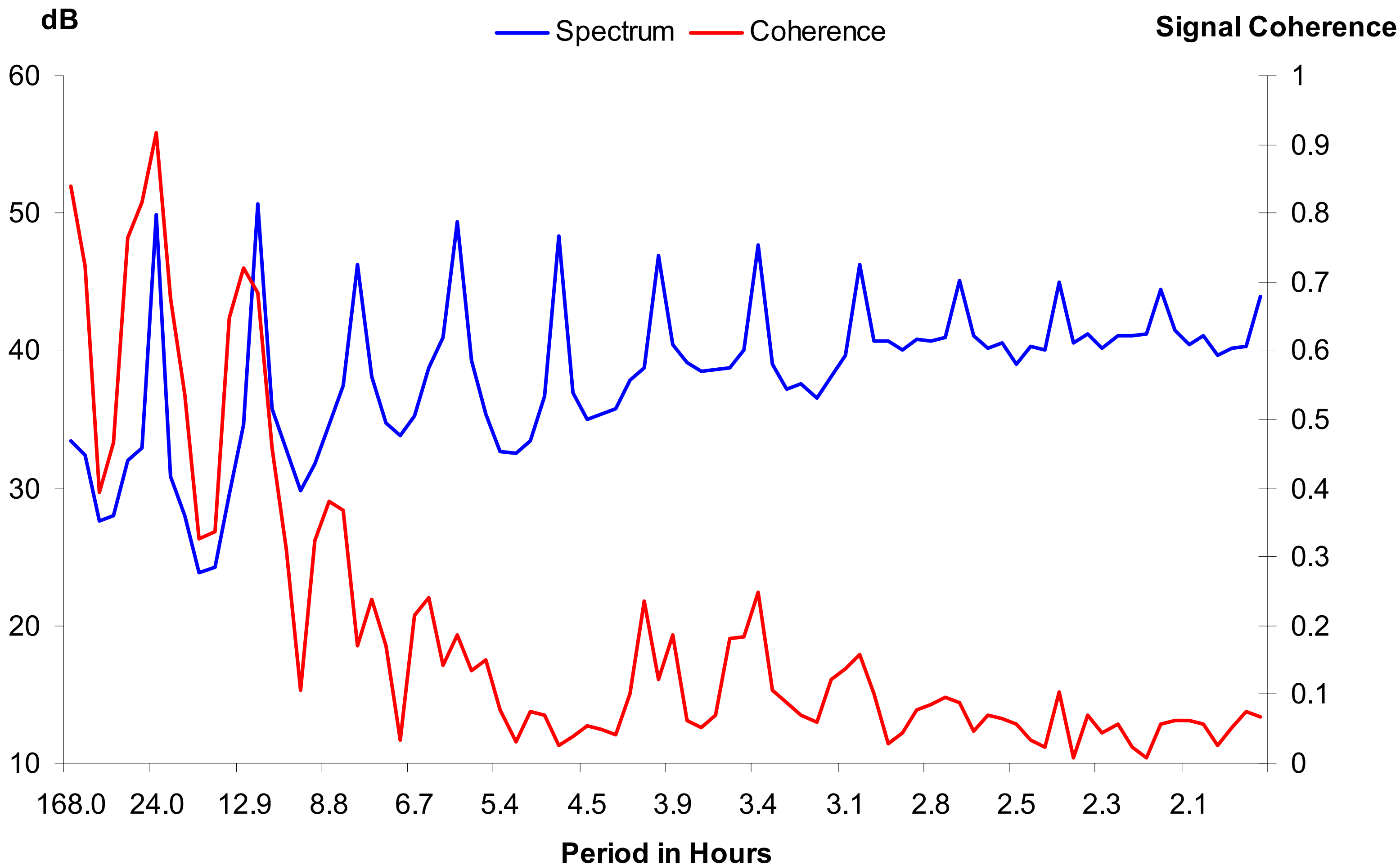
$$s(t_n) = a_0 + 2\sum_{k=1}^{K} a_k \cos(2\pi f_k t_n) + 2\sum_{k=1}^{K} b_k \sin(2\pi f_k t_n)$$

$$Var(a_k) = Var(b_k) \propto S(f_k) \text{ - } Spectrum$$

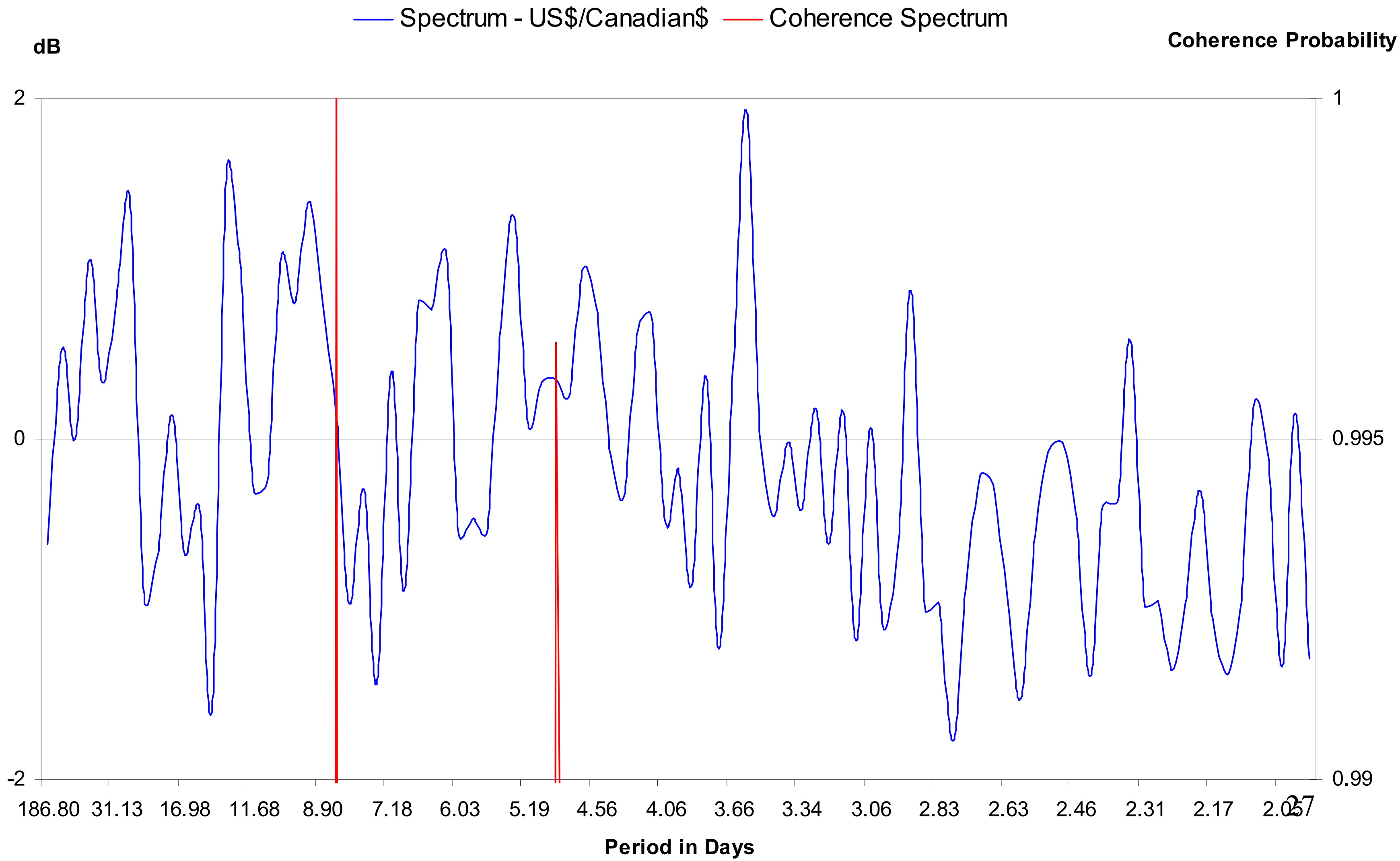
$$\sigma_s^2 = \sum_{k=1}^{K} S(f_k)$$

Power & Signal Coherence Spectra - Demand

Power & Signal Coherence Spectra of the Residuals from an AR(12) Fit of the Alberta Electricity Hourly Spot Demand



Canadian$/US$ Daily Data Spectra



Bicorrelations of a Random Signal

$$c_{xxx}(t_1,t_2,t_3) = E\left[x(t_1)x(t_2)x(t_3)\right]$$

If $\{x(t)\}$ is stationary then

$$c_{xxx}(t_1,t_2,t_3) = E\left[x(t+t_1-t_3)x(t+t_2-t_3)x(t)\right]$$

$$c_{xxx}(\tau_1,\tau_2) = E\left[x(t)x(t+\tau_1)x(t+\tau_2)\right]$$

The Bispectrum

$$S_{xxx}(f_1, f_2) =$$

$$\int_{-\infty}^{\infty} c_{xxx}(\tau_1, \tau_2)\exp\left[-i2\pi(f_1\tau_1 + f_2\tau_2)\right]d\tau_1 d\tau_2$$

If the noise is gaussian then

$$S_{xxx}(f_1, f_2) = 0$$

Example of a Simple Nonlinear Model

$$x(t_n) + a_1(x(t_{n-2}))x(t_{n-1}) + a_2(x(t_{n-2}))x(t_{n-2})$$
$$= \sigma u(t_n)$$

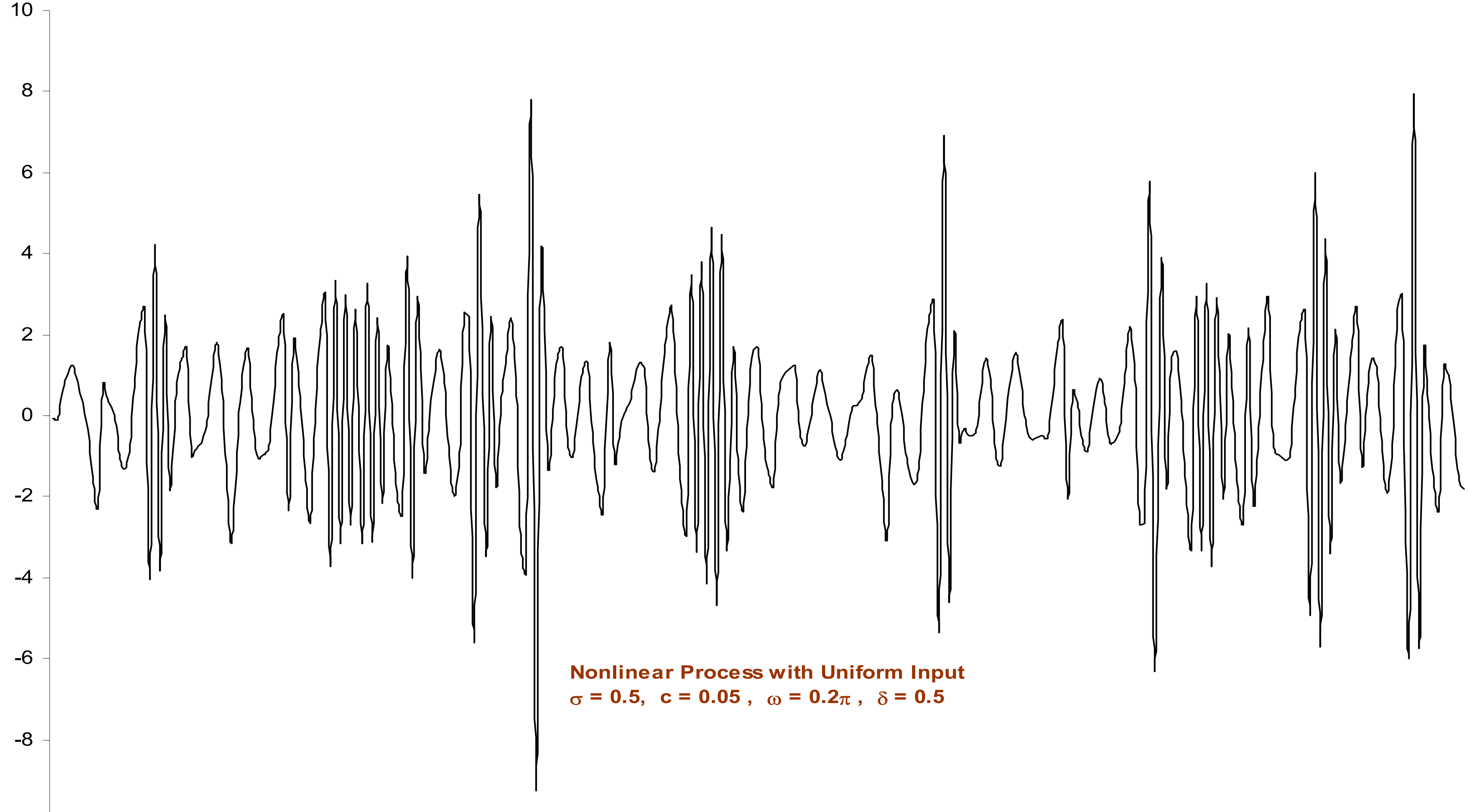
$$a_2(x(t_{n-2})) = e^{-c(x(t_{n-2}))} \quad ,$$

$$a_1(x(t_{n-2})) = -2a_2(x(t_{n-2}))\cos\omega(x(t_{n-2}))$$

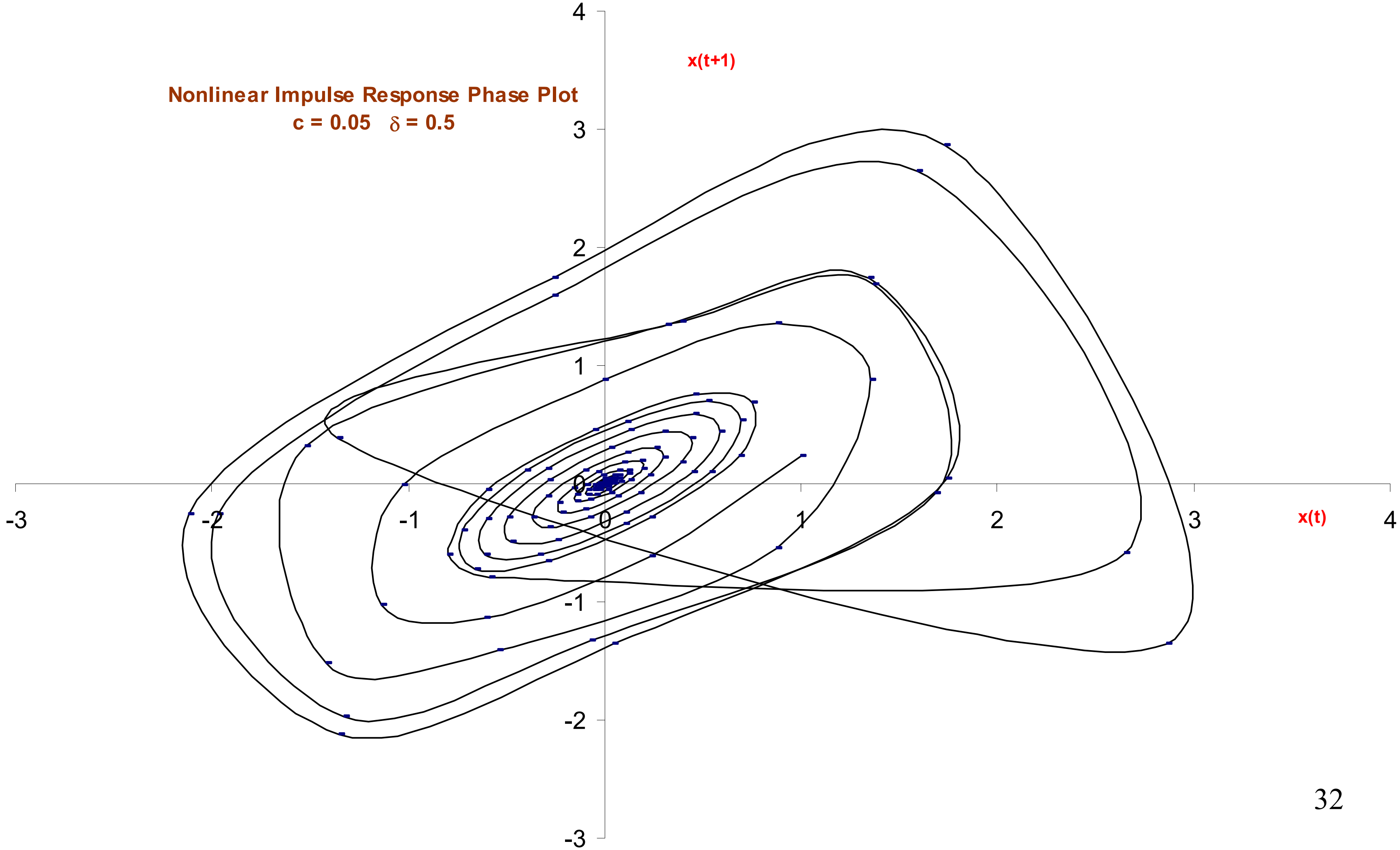
$$c(x(t_{n-2})) = c\left(1 + \delta x^2(t_{n-2})\right)$$

$$\omega(x(t_{n-2})) = \omega\left(1 + \delta x^2(t_{n-2})\right)$$

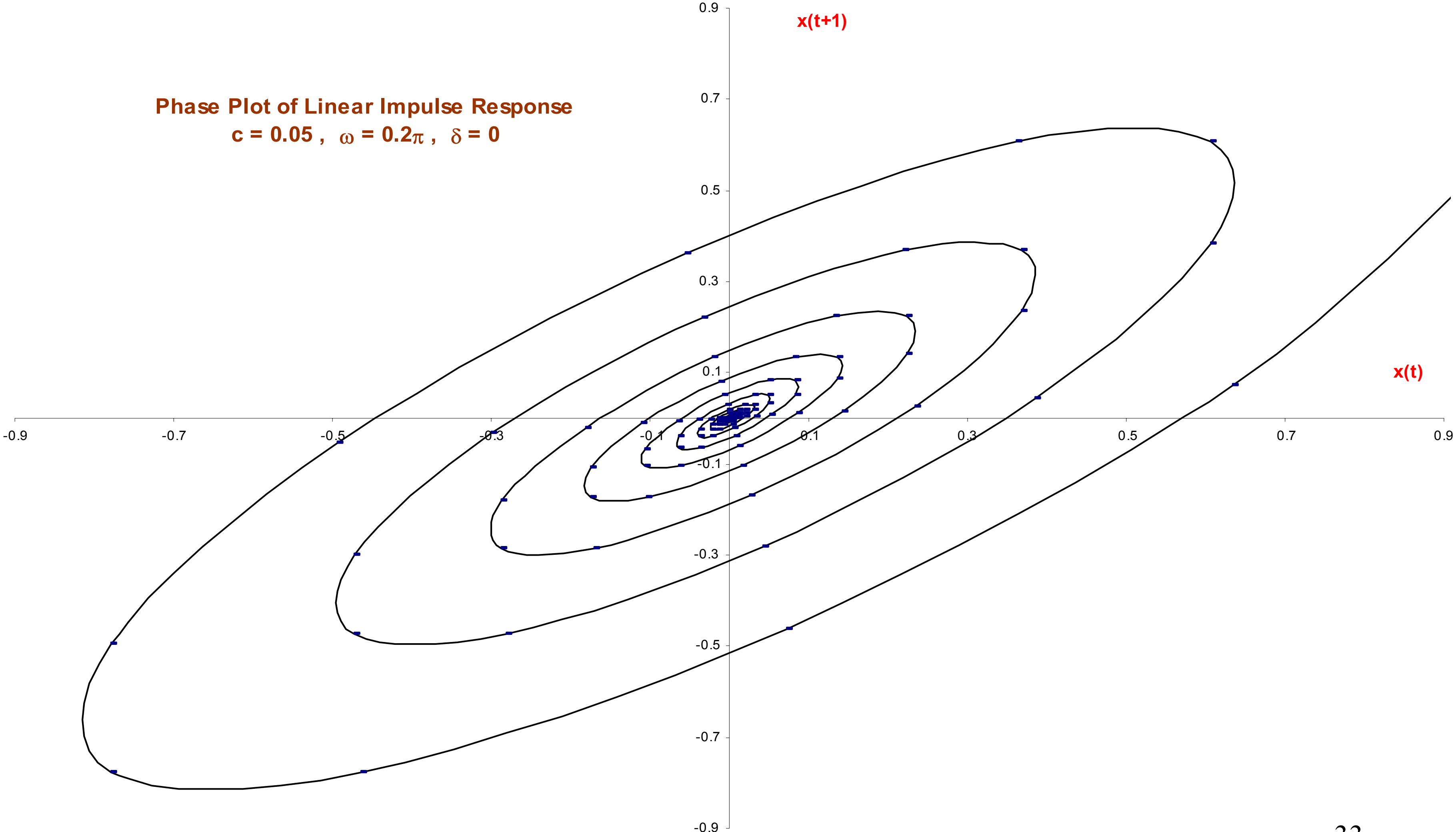
Nonlinear Model - Uniform Input
$\sigma = 0.5, \; c = 0.05, \; \omega = 0.2\pi, \; \delta = 0.5$



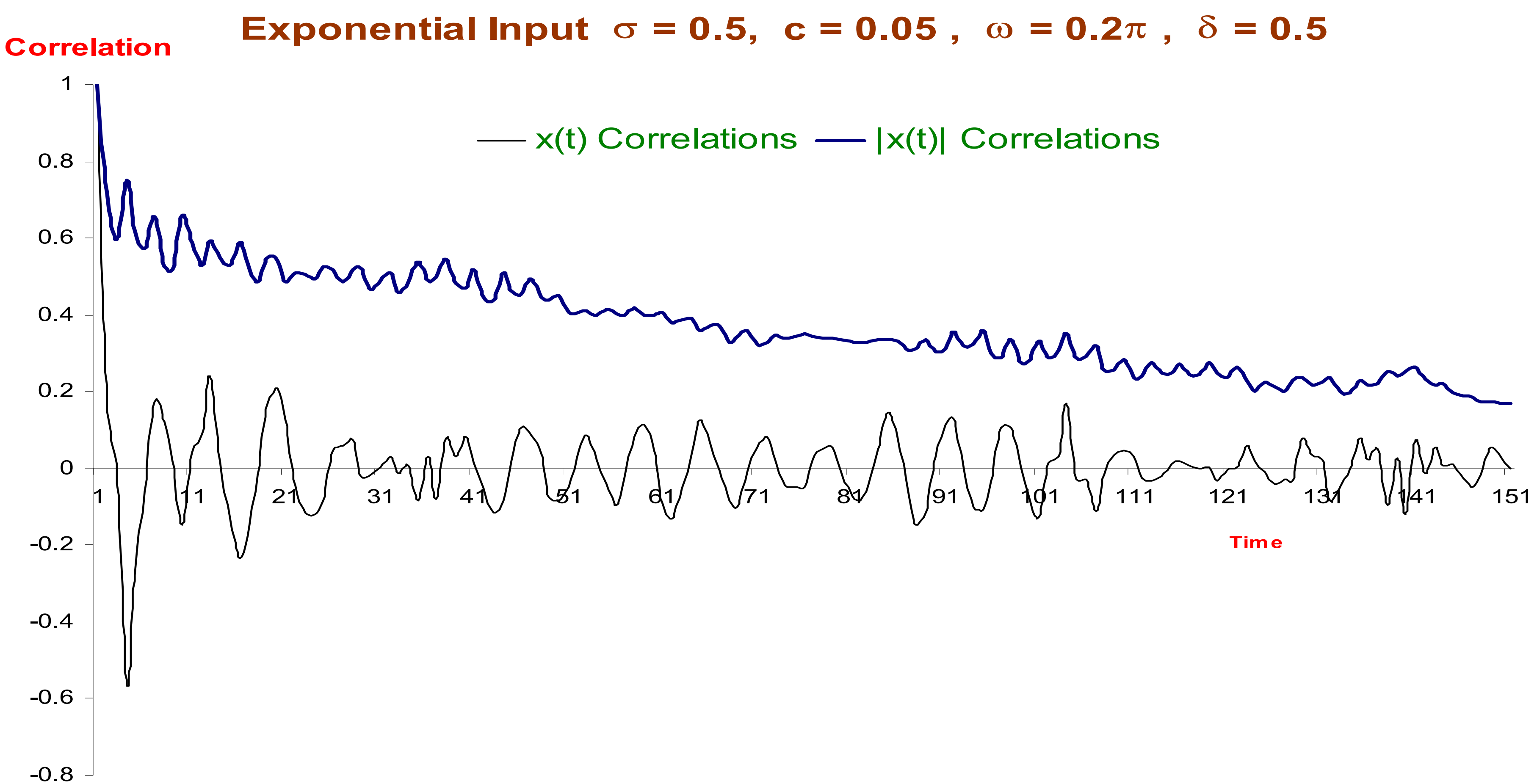
Phase Plot of the Nonlinear Impulse Response



Phase Plot of the Linear Impulse Response



Correlations Functions of x(t) & |x(t)|



Bispectrum - Nonlinear AR(2) Signal