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Nonlinear Ergodic Retraction Theorem for Lipschitzian Semigroups in Uniformly Convex Spaces

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Abstract Let C be a bounded closed convex subset of a p- uniformly convex Banach space E. We prove an existence theorem of nonlinear ergodic retractions for Lipschitzian semigroups of self-mappings on C. Further, we give an application of such a theorem in $L^p(1 < p < +\infty)$ spaces.

Key words Ergodic retraction; Lipschitzian semigroup; fixed point; invariant submean; p- uniformly convex Banach space

1 Introduction

In 1975, Baillon[2] first established a nonlinear ergodic theorem for nonexpansive mappings: Let C be a bounded closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. Then, for each x in C, the Cesàro means $S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converge weakly to some $y \in F(T)$. In this case, setting $y = Px$ for each x in C, we define a nonexpansive retraction P from C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{co}\{T^n x : n \geqslant 0\}$ for each x in C, where $\overline{co}(A)$ is the closure of the convex hull of A. The analogous results were obtained for nonexpansive semigroups on C by [1, 3, 4, 5, 9]. Recently, Mizoguchi and Takahashi [6] proved an existence theorem of nonlinear ergodic retractions for Lipschitzian semigroups in Hilbert spaces by introducing submeans.

In this paper, let C be a nonempty bounded closed convex subset of a p- uniformly convex Banach space E. Using the method of [6], we show an existence theorem of nonlinear ergodic retractions for Lipschitzian semigroups, which partially extends the corresponding result in [6]

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to the p- uniformly convex Banach space setting. Further we give an application of such a theorem in $L^p(1 < p < +\infty)$ spaces.

2 Preliminaries and Lemmas

It is known that the modulus of convexity of a Banach space E is defined as

$$\delta_E(\varepsilon) = \inf\{1 - \| (x + y)/2 \| : x, y \in B_E \text{ and } \| x - y \| \geqslant \varepsilon\},$$

where $B_E = \{x \in E : \| x \| \leqslant 1\}$ is the closed unit ball of E. Recall that E is said to have the modulus of convexity of power type $p \geqslant 2$ (and E is said to be p- uniformly convex) if there is a constant $d > 0$ such that $\delta_E(\varepsilon) \geqslant d\varepsilon^p$ for $0 < \varepsilon \leqslant 2$. Note that a Hlibert space H is 2-uniformly convex (indeed $\delta_H(\varepsilon) = 1 - (1 - (\frac{1}{2}\varepsilon)^2)^{\frac{1}{2}} \geqslant \frac{1}{8}\varepsilon^2$) and an L^p space $(1 < p < +\infty)$ is max $(2, p)$ -uniformly convex.

Lemma 2. 1[7] If E is a p- uniformly convex Banach space, then there exists a constant $d_p > 0$ such that

$$\| tx + (1-t)y \|^p \leqslant t \| x \|^p + (1-t) \| y \|^p - d_p W_p(t) \| x - y \|^p \qquad (2.1)$$

for all $x, y \in E$ and $0 \leqslant t \leqslant 1$, where $W_p(t) = t(1-t)^p + t^p(1-t)$.

Lemma 2. 2[7] If E is an L^p space with $1 < p < +\infty$, then

$$\| tx + (1-t)y \|^q \leqslant t \| x \|^q + (1-t) \| y \|^q - d_p W_q(t) \| x - y \|^q$$

for all x, y$\in E$ and $0 \leqslant t \leqslant 1$, where $q = \max(2, p)$, $W_q(t) = t^q(1-t) + t(1-t)^q$ and

$$d_p = \begin{cases} (1 + t_p{}^{p-1})/(1 + t_p)^{p-1} & \text{if } 2 < p < +\infty \\ p - 1 & \text{if } 1 < p \leqslant 2 \end{cases} \qquad (2.2)$$

with t_p being the unique solution of the equation

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0, \qquad t \in (0,1).$$

Throughout this paper, we assume that S is a semitopological semigroup, i. e. , a semigroup with a Hausdorff topology such that for every $s \in S$ the mappings $t \to s \cdot t$ and $t \to t \cdot s$ of S into itself are continuous. Let $B(S)$ denote the Banach space of all bounded real valued functions on S with supremum norm. Let X be a subspace of $B(S)$ containing constants. A real valued function μ on X is called a submean on X if the following conditions are satisfied:

(1) $\mu(f + g) \leqslant \mu(f) + \mu(g)$ for every $f, g \in X$;

(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geqslant 0$;

(3) for $f, g \in X$, $f \leqslant g$ implies $\mu(f) \leqslant \mu(g)$;

(4) $\mu(c) = c$ for every constant c.

Let μ be a submean on X and $f \in X$. Then according to time and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$.

Lemma 2. 3[7] Suppose that E is a p- uniformly convex Banach space, C is a bounded closed convex subset of E, and $\{x_t : t \in S\}$ is a bounded family of elements of E. Suppose also that for each $x \in C$, the function f on S defined by $f(t) = \| x_t - x \|^p$, $t \in S$, belongs to X. Set $r(x) = \mu_t \| x_t - x \|^p$, $x \in C$ and $r = \inf\{r(x) : x \in C\}$. Then there exists a unique point z in C such that $r(z) = r$.

For $s \in S$ and $f \in B(S)$, we define $r_s f(t) = f(ts)$ for all $t \in S$. Let X be a subspace of $B(S)$ containing constants which is r_s-invariant, i. e., $r_s(X) \subset X$ for each $s \in S$. Then a submean μ on X is right invariant if $\mu(f) = \mu(r_s f)$ for all $s \in S$, $f \in X$.

Let C be a nonempty closed convex subset of the Banach space E. Then a family $\mathscr{P} = \{T_s: s \in S\}$ of mappings of C into itself is said to be a Lipschitzian semigroup on C if it satisfies the following:

(1) $T_{st}x = T_s T_t x$, for all $s, t \in S$ and $x \in C$;

(2) for each $x \in C$, the mapping $s \to T_s x$ is continuous on S;

(3) for each $t \in S$, T_t is a Lipschitzian mapping of C into itself, i. e., there is $k_t \geqslant 0$ such that $\| T_t x - T_t y \| \leqslant k_t \| x - y \|$ for all $x, y \in C$. Let $F(\mathscr{P})$ denote the set of common fixed points of T_s, $s \in S$, i. e.,

$$F(\mathscr{P}) = \{x \in C: T_s x = x \text{ for all } s \in S\}.$$

Lemma 2. 4[8] Let C be a nonempty closed convex subset of a p-uniformly convex Banach space E and let $\mathscr{P} = \{T_s: s \in S\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^p \leqslant d_p$. Then $F(\mathscr{P})$ is closed and convex.

3 The main result

Theorem 3. 1 Let C be a nonempty bounded closed convex subset of a p-uniformly convex Banach space E with $0 < d_p < (1 + 2^{p-1})^{-\frac{1}{2}}$ and let X be a r_s-invariant subspace of $B(S)$ containing constants which has a right invariant submean μ. Let $\mathscr{P} = \{T_t: t \in S\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^p \leqslant d_p$ and $F(\mathscr{P}) \neq \varnothing$. If for every x, y in C, the function f on S defined by $f(t) = \| T_t x - y \|^p$ for all $t \in S$ and the function g on S defined by $g(t) = k_t^p$ for all $t \in S$, belong to X, then the following are equivalent:

(1) $\bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x: t \in S\} \cap F(\mathscr{P}) \neq \varnothing$ for each $x \in C$;

(2) There is a uniformly continuous ergodic retraction P of C onto $F(\mathscr{P})$ such that $PT_s = T_s P = P$ for every $s \in S$ and $Px \in \overline{\text{co}}\{T_t x: t \in S\}$ for every x in C.

Proof For simplicity, d_p is denoted by c.

(2)⇒(1)

Let $x \in C$. Then, it is obvious that $Px \in F(\mathscr{P})$. Since

$$Px = PT_s x \in \overline{\text{co}}\{T_t T_s x: t \in S\} = \overline{\text{co}}\{T_{ts}x: t \in S\}$$

for each $s \in S$, we have $Px \in \bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x: t \in S\}$.

(1)⇒(2)

Let $x \in C$. Then by virtue of Lemma 2. 3, there is a unique element z in $F(\mathscr{P})$ such that

$$\mu_t \| T_t x - z \|^p = \min\{\mu_t \| T_t x - y \|^p: y \in F(\mathscr{P})\}.$$

Zeng and Yang[8] have proved that

$$\mu_t(k_t^p) \leqslant \inf_s \sup_t k_{ts}^p, \ \sup_s \inf_t k_{ts} \leqslant \sqrt[p]{c}, \ \bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x: t \in S\} \cap F(\mathscr{P}) = \{z\},$$

and

$$\inf_s \sup_t \| T_{ts}x - f \|^p \leqslant c \cdot \mu_t \| T_t x - f \|^p \leqslant c^2 \cdot \inf_t \| T_{ts}x - f \|^p \qquad (*)$$

for every $s \in S$, $f \in F(\mathscr{S})$ and $x \in C$.

Setting $Px = z$ for each $x \in C$, we have, for each $s \in S$,

$$\mu_t \| T_t x - PT_s x \|^p = \mu_t \| T_{ts}x - PT_s x \|^p =$$
$$\mu_t \| T_t T_s x - PT_s x \|^p \leqslant \mu_t \| T_t T_s x - Px \|^p =$$
$$\mu_t \| T_{ts}x - Px \|^p = \mu_t \| T_t x - Px \|^p.$$

From the uniqueness of Px it follows that $PT_s x = Px$ for every $s \in S$. It is clear that $T_s P = P$. Finally we show that P is uniformly continuous. Let $w \in F(\mathscr{S})$ and $0 < \lambda < 1$. Then we have, for each $s, t \in S$,

$$\| T_{ts}x - ((1-\lambda)z + \lambda w) \|^p \leqslant$$
$$\lambda \| T_{ts}x - w \|^p + (1-\lambda) \| T_{ts}x - z \|^p - c \cdot W_p(\lambda) \cdot \| w - z \|^p \leqslant$$
$$\sup_a \| T_{as}x - z \|^p + \lambda(\| T_{ts}x - w \|^p - \| T_{ts}x - z \|^p) - c \cdot W_p(\lambda) \cdot \| w - z \|^p$$

and hence

$$\inf_t \| T_{ts}x - ((1-\lambda)z + \lambda w) \|^p \leqslant$$
$$\sup_t \| T_{ts}x - z \|^p + \lambda \inf_t (\| T_{ts}x - w \|^p - \| T_{ts}x - z \|^p) - c \cdot W_p(\lambda) \cdot \| w - z \|^p.$$

From $(*)$, we also have

$$\mu_t \| T_t x - ((1-\lambda)z + \lambda w) \|^p \leqslant c \cdot \inf_t \| T_{ts}x - ((1-\lambda)z + \lambda w) \|^p$$

and

$$\inf_s \sup_t \| T_{ts}x - z \|^p \leqslant c\mu_t \| T_t x - z \|^p.$$

Then, we obtain

$$c^{-1}\mu_t \| T_t x - ((1-\lambda)z + \lambda w) \|^p - \lambda \inf_t (\| T_{ts}x - w \|^p - \| T_{ts}x - z \|^p) \leqslant$$
$$\sup_t \| T_{ts}x - z \|^p - c \cdot W_p(\lambda) \cdot \| w - z \|^p$$

and hence

$$c^{-1}\mu_t \| T_t x - z \|^p - \lambda \sup_s \inf_t (\| T_{ts}x - w \|^p - \| T_{ts}x - w \|^p) \leqslant$$
$$c^{-1}\mu_t \| T_t x - ((1-\lambda)z + \lambda w) \|^p - \lambda \sup_s \inf_t (\| T_{ts}x - w \|^p - \| T_{ts}x - z \|^p) \leqslant$$
$$\inf_s \sup_t \| T_{ts}x - z \|^p - c \cdot W_p(\lambda) \cdot \| w - z \|^p \leqslant$$
$$c\mu_t \| T_t x - z \|^p - c \cdot W_p(\lambda) \cdot \| w - z \|^p.$$

Note that $0 < c < \sqrt{\frac{1}{1 + 2^{p-1}}}$ implies $c - c^{-1} < 0$. Hence we have

$$-\lambda \sup_s \inf_t (\| T_{ts}x - w \|^p - \| T_{ts}x - z \|^p) < -c \cdot W_p(\lambda) \cdot \| w - z \|^p,$$
$$\sup_s \inf_t (\| T_{ts}x - w \|^p - \| T_{ts}x - z \|^p) > c \cdot [(1-\lambda)^p + (1-\lambda)\lambda^{p-1}] \cdot \| w - z \|^p.$$

Now, letting $\lambda \to 0$, we obtain

$$\sup_s \inf_t (\| T_{ts}x - w \|^p - \| T_{ts}x - z \|^p) \geqslant c \cdot \| w - z \|^p.$$

Therefore, for $y \in C$, we have

$$\sup_s \inf_t (\| T_{ts}x - Py \|^p - \| T_{ts}x - Px \|^p) \geqslant c \cdot \| Px - Py \|^p.$$

Let $\varepsilon > 0$. Then there exists $s_1 \in S$ such that

$$\inf_t(\| T_{ts_1}x - Py \|^p - \| T_{ts_1}x - Px \|^p) > c \cdot \| Px - Py \|^p - \varepsilon. \qquad (**)$$

For such an $s_1 \in S$, we also have

$$\sup_s \inf_t(\| T_{ts}T_{s_1}y - Px \|^p - \| T_{ts}T_{s_1}y - PT_{s_1}y \|^p) \geqslant c \cdot \| PT_{s_1}y - Px \|^p,$$

and hence there exists $s_2 \in S$ such that

$$\inf_t(\| T_{ts_2}T_{s_1}y - Px \|^p - \| T_{ts_2}T_{s_1}y - PT_{s_1}y \|^p) > c \cdot \| PT_{s_1}y - Px \|^p - \varepsilon.$$

Then, from $PT_{s_1}y = Py$, we have

$$\inf_t(\| T_{ts_2s_1}y - Px \|^p - \| T_{ts_2s_1}y - Py \|^p) > c \cdot \| Px - Py \|^p - \varepsilon.$$

On the other hand, it follows from $(**)$ that

$$\inf_t(\| T_{ts_2s_1}x - Py \|^p - \| T_{ts_2s_1}x - Px \|^p) > c \cdot \| Px - Py \|^p - \varepsilon.$$

Combining these two inequalities, we get

$$2c \cdot \| Px - Py \|^p - 2\varepsilon <$$

$$\inf_t(\| T_{ts_2s_1}y - Px \|^p - \| T_{ts_2s_1}y - Py \|^p) + \inf_t(\| T_{ts_2s_1}x - Py \|^p - \| T_{ts_2s_1}x - Px \|^p) \leqslant$$

$$\inf_t\{ \| T_{ts_2s_1}x - Py \|^p - \| T_{ts_2s_1}y - Py \|^p + \| T_{ts_2s_1}y - Px \|^p - \| T_{ts_2s_1}x - Px \|^p\} \leqslant$$

$$\inf_t\{p \cdot (\mathrm{diam}C)^{p-1} \cdot | \| T_{ts_2s_1}y - Py \| - \| T_{ts_2s_1}x - Py \| | +$$

$$p \cdot (\mathrm{diam}C)^{p-1} \cdot | \| T_{ts_2s_1}x - Px \| - \| T_{ts_2s_1}y - Px \| |\} \leqslant$$

$$2p \cdot (\mathrm{diam}C)^{p-1} \cdot \inf_t \| T_{ts_2s_1}y - T_{ts_2s_1}x \| \leqslant$$

$$2p \cdot (\mathrm{diam}C)^{p-1} \cdot \inf_t k_{ts_2s_1} \| y - x \| \leqslant$$

$$2p \cdot (\mathrm{diam}C)^{p-1} \cdot (\sup_s \inf_t k_{ts}) \cdot \| x - y \| \leqslant$$

$$2p \cdot (\mathrm{diam}C)^{p-1} \cdot \sqrt[p]{c} \cdot \| x - y \|.$$

Since $\varepsilon > 0$ is arbitrary, this implies

$$\| Px - Py \|^p \leqslant p \cdot (\mathrm{diam}C)^{p-1} \cdot c^{\frac{1}{p}-1} \cdot \| x - y \|.$$

So, it follows that P is uniformly continuous.

Corollary 3.1 Let C be a nonempty bounded closed convex subset of an L^p space with $1 < p < \frac{3+\sqrt{3}}{3}$ and let X be a r_s-invariant subspace of $B(S)$ containing constants which has a right invariant submean μ. Let $\mathscr{S} = \{T_s: s \in S\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^2 \leqslant p - 1$ and $F(\mathscr{S}) \neq \varnothing$. If for every x, y in C, the function f on S defined by $f(t) = \| T_t x - y \|^2$ for all $t \in S$ and the function g on S defined by $g(t) = k_t^2$ for all $t \in S$, belong to X, then the statements (1) and (2) in Theorem 3.1 are equivalent.

Remark 3.1 Obviously, for an L^p space with $2 < p < +\infty$ we can also obtain the corollary of Theorem 3.1.

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一致凸空间中Lipschitz半群的非线性遍历收缩定理

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提　要　设C是p一致凸Banach空间E的一个非空有界闭凸子集. 在证明了C上自映象的Lipschitz半群的一个非线性遍历收缩定理的基础上,进一步给出了如此定理在L^p空间$(1<p<+\infty)$中的应用.

关键词　遍历收缩; Lipschitz半群; 不动点; 不变次平均; p一致凸Banach空间

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