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A Jacobi Pseudospectral Method for Solving the Nonlinear Klein-Gordon Equation on the Half Line

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Abstract: A Jacobi pseudospectral method is proposed for the nonlinear Klein—Gordon (NLKG) equation on the half line with rough asymptotic behaviors at infinity. The stability and convergence of the proposed scheme are proved. Numerical results illustrate the efficiency of this approach.

Key words: NLKG equation; the half line; Jacobi pseudospectral method

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1 Introduction

As we know, the NLKG equation plays an important role in quantum mechanics[1]. It is of the form

$$\begin{cases}\partial_t^2 V(y,t) - \Delta V(y,t) + V(y,t) + V^3(y,t) = F(y,t), & y\in I,\ t\in(0,T],\\ \partial_t V(y,0) = V_1(y), & y\in\bar{I},\\ V(y,0) = V_0(y), & y\in\bar{I},\end{cases}\qquad(1)$$

where $I\subseteq\mathbf{R}^1$, $\Delta V(y,t) = \partial_y^2 V(y,t)$ and V_0, V_1, F are given functions. Also V satisfies some boundary conditions. There are many papers concerning the existence and uniqueness of the smoothness or weak solution of (1)[2]. Numerical studies of this equation in bounded domains are also considered by many authors[3,4]. But it is more challenging to solve the NLKG equation numerically in unbounded domains. Indeed, there are several ways to deal with this kind of problems. The first one is to restrict the calculations to certain bounded domains and impose some conditions on artificial boundaries. But this treatment usually destroys the accuracy. The second way is to use spectral approximations associated with some orthogonal systems in unbounded domains[5,6]. This kind of method keeps the physical meanings properly but requires some quadratures over unbounded domains. The third approach is to

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change the original problems into certain problems in bounded domains. But the coefficients of the derivatives are degenerating on the boundaries. Recently, GUO[7,8] developed the Jacobi spectral method for numerical solutions of such problems. The main advantages of this treatment are that it avoids quadratures in unbounded domains and keeps the spectral accuracy. But in actual computations, the Jacobi pseudospectral method is more preferable since it only needs to evaluate the numerical solutions at some interpolation nodes. Hence, it saves a lot of work and is easier in dealing with nonlinear problems.

In this paper, we consider the numerical solution of the NKLG equation on the half line $I=(0,+\infty)$ with V satisfying

$$V(0,t)=0,\qquad \lim_{y\to+\infty} e^{-\frac{y}{2}}\partial_y V(y,t)=0,\quad t\in[0,T]. \tag{2}$$

Let $\Lambda=(-1,1)$ and

$$y(x)=\ln 2-\ln(1-x),\qquad x(y)=1-\frac{2}{e^y}. \tag{3}$$

Clearly, $x(0)=-1$, $x(+\infty)=1$, and for all $x\in\Lambda$, $\frac{\mathrm{d}x}{\mathrm{d}y}=1-x>0$. Put $U(x,t)=V(y(x),t)$, $U_0(x)=V_0(y(x))$, $U_1(x)=V_1(y(x))$ and $f(x,t)=F(y(x),t)$. Then the problem (1), (2) is tranformed into

$$\begin{cases}\partial_t^2 U(x,t)-\tilde{\Delta}U(x,t)+U(x,t)+U^3(x,t)=f(x,t), & x\in\Lambda,\ t\in(0,T],\\ U(0,t)=\lim_{x\to 1}(1-x)^{\frac{3}{2}}\partial_x U(x,t)=0, & t\in[0,T],\\ \partial_t U(x,0)=U_1(x), & U(x,0)=U_0(x),\ x\in\bar{\Lambda},\end{cases} \tag{4}$$

where $\tilde{\Delta}U(x,t)=(1-x)\partial_x((1-x)\partial_x U(x))$. In the sequel, we shall propose a Jacobi pseudospectral scheme for (4), and prove the stability and convergence of the proposed scheme. We shall present some numerical results to illustrate the efficiency of this method. The implementations in this paper are easy to be generalized to multiple-dimensional problems, and can also be applied to some other nonlinear differential equations in unbounded domains.

2 The Scheme

Let $\chi(x)$ be a certain weight function in the usual sense, and $\mathbf{N}$ be the set of all nonnegative integers. The weighted Sobolev space $H^r_\chi(\Lambda)$ is defined in the usual way and its inner product, semi-norm and norm are denoted by $(u,v)_{r,\chi}$, $|v|_{r,\chi}$ and $\|v\|_{r,\chi}$. In particular, $L^2_\chi(\Lambda)=H^0_\chi(\Lambda)$, $(u,v)_\chi=(u,v)_{0,\chi}$ and $\|v\|_\chi=\|v\|_{0,\chi}$. The space $H^r_{0,\chi}(\Lambda)$ stands for the closure of the set of all infinitely differentiable functions with compact support in Λ. When $\chi(x)\equiv 1$, we omit χ in the notations as usual. If $\chi(x)\equiv 1$, then we drop the script χ in the above definitions. For any $N\in\mathbf{N}$, let $\mathscr{P}_N$ be the set of all algebraic polynomials of degree at most N. Let ${}_0\mathscr{P}_N=\{v\,|\,v\in\mathscr{P}_N, v(-1)=0\}$, and c be a generic constant independent of any function and N.

Let $\chi^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$, and the Jacobi polynomial of degree l is defined by

$$(1-x)^\alpha(1+x)^\beta J_l^{(\alpha,\beta)}(x)=\frac{(-1)^l}{2^l\,l!}\partial_x^l((1-x)^{l+\alpha}(1+x)^{l+\beta}),\quad l=0,1,2,\cdots.$$

If $\alpha,\beta>-1$, then the set of Jacobi polynomials is $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal. Let $\{x_j\}_{j=0}^N$ be the set of distinct zeros of the Jacobi polynomial $(1+x)J_N^{(0,1)}(x)$. Assume that they are arranged in the increasing order. Then there exists a unique set of quadrature weights $\{\omega_j\}_{j=0}^N$ (cf. [9]) such that for any

$v \in \mathscr{P}_{2N}$,

$$\int_{\Lambda} v(x)\mathrm{d}x = \sum_{j=0}^{N} v(x_j)\omega_j. \tag{5}$$

Further, let $\Lambda_N = \{x_j:\ 1 \leqslant j \leqslant N\}$, and define the discrete L^p- norm

$$\| v \|_{L^p,N} = \begin{cases} (\sum\limits_{j=0}^{N} |v(x_j)|^p \omega_j)^{\frac{1}{p}}, & 1 \leqslant p \leqslant \infty, \\ \operatorname{ess}\max\limits_{x\in\Lambda_N} |v(x)|, & p = \infty. \end{cases}$$

In particular, for $p = 2$, $\| v \|_N = \| v \|_{L^2,N}$, $|v|_{\infty,N} = \| v \|_{L^\infty,N}$ and the discrete inner product is denoted by $(u,v)_N = \sum\limits_{j=0}^{N} u(x_j)v(x_j)\omega_j$. Let I_N^c: $C([-1,1)) \to \mathscr{P}_N$ be the interpolation operator such that

$$I_N^c v(x_j) = v(x_j), \quad 0 \leqslant j \leqslant N.$$

The semi-discrete Jacobi pseudospectral scheme for (4) is to find $u \in {}_0\mathscr{P}_N$ such that

$$\begin{cases} \partial_t^2 u(x,t) - I_N^c(\widetilde{\Delta} u)(x,t) + u(x,t) + u^3(x,t) = f(x,t), & x \in \Lambda_N,\quad t \in (0,T], \\ \partial_t u(x,0) = u_1(x) = I_N^c U_1(x), & x \in \Lambda_N, \\ u(x,0) = u_0(x) = I_N^c U_0(x), & x \in \Lambda_N. \end{cases} \tag{6}$$

For simplicity, let $w(x) = (1-x)^2, \widetilde{w}(x) = 1 - x$ and for any $u,v \in C^1([-1,1))$,

$$a_{w,N}^{\nu}(u,v) = (\widetilde{w}\partial_x u,\ \partial_x(\widetilde{w}\, v))_N + \nu(u,v)_N, \quad \nu > \frac{1}{2}.$$

Taking the discrete inner product on both sides of (6), we find that the scheme (6) is equivalent to

$$\begin{cases} (\partial_t^2 u(t) + (1-\nu)\ u(t) + u^3(t),v)_N + a_{w,N}^{\nu}(u(t),v) = (f(t),v)_N, & \forall\ v \in {}_0\mathscr{P}_N,\ t \in (0,T], \\ (\partial_t u(0),v)_N = (U_1,v)_N, & \forall\ v \in {}_0\mathscr{P}_N, \\ (u(0),v)_N = (U_0,v)_N, & \forall\ v \in {}_0\mathscr{P}_N. \end{cases} \tag{7}$$

3 Numerical Results

In actual computations, we need to discretize the problem in time t. Let τ be the mesh size of t,

$$\mathscr{R}_\tau = \{t = k\tau \mid k = 1,2,\cdots,[\frac{T}{\tau}]\},\ \hat{v}(x,t) = \frac{1}{2}(v(x,t+\tau) + v(x,t-\tau)),$$

$$D_{\tau\tau}v(x,t) = \frac{1}{\tau^2}\ (v(x,t+\tau) - 2v(x,t) + v(x,t-\tau)).$$

To increase the computational stability, we discretize the nonlinear term as

$$\hat{G}(u(x,t)) = \frac{1}{4}\sum_{j=0}^{3} u^j(x,t+\tau)u^{3-j}(x,t-\tau).$$

The fully discrete Jacobi pseudospectral scheme for (4) is

$$\begin{cases} D_{\tau\tau}u(x,t) - I_N^c(\widetilde{\Delta} u)(x,t) + \hat{u}(x,t) + \hat{G}(u(x,t)) = \hat{f}(x,t), & x \in \Lambda_N,\ t \in \mathscr{R}_\tau, \\ u(x,0) = I_N^c U_0(x),\ u(x,\tau) = I_N^c U_0(x) + \tau I_N^c U_1(x), & x \in \Lambda_N. \end{cases} \tag{8}$$

We next present some numerical results. Let $y(x)$ and $x(y)$ be the same as those in (3). We consider two typical examples.

Example 1 The exact solution of the original problem is $V(y,t) = y\operatorname{sech}(ay - bt - c)$, $a = 1.8$, $b = 0.5$, $c = 4.0$. Clearly, it is exponential decay at infinity.

Example 2 The exact solution of the original problem is

$$V(y,t) = \frac{1}{2^{\eta}} e^{\eta y} \sin\left(\frac{1}{2}(1 + x(y))(t+1)\right).$$

Clearly, if $\eta > 0$, then $V(y,t) \to \infty$ as $y \to +\infty$, i. e., the exact solution of the transformed problem (4) also tends to infinity as $x \to 1$.

We now use the scheme (8) to solve (1) with (2) numerically. To illustrate the accuracy in the spatial direction, the time step is chosen to be sufficiently small so that the error is dominated by the spatial disretization errors. Let $y_j = y(x_j)$, $x_j \in \Lambda_N$, $u(x,t)$ be the solution of (8) and $v(y,t) = u(x(y),t)$. For description of the errors, let

$$E_{\mathrm{abs}}(t) = \max_{0 \leqslant j \leqslant N} |V(y_j,t) - v(y_j,t)|, \quad E_{\mathrm{rel}}(t) = \max_{0 \leqslant j \leqslant N} \left| \frac{V(y_j,t) - v(y_j,t)}{V(y_j,t)} \right|, \ t \in \mathscr{R}_\tau.$$

In Figure 1, we plot the errors of Examples 1 and 2 at $t = 1$ with $\tau = 10^{-4}$, $\eta = 10^{-4}$ and different N. We notice that the scheme (8) converges very fast for smooth solution decaying exponentially at infinity, and achieves the spectral accuracy of exponential order. Moreover, this method provides good numerical results for solutions with weak singularities, but the convergence rate is relatively slower. We plot in Figure 2 the exact solution and pseudospectral solution obtained with $N = 64$, $\eta = 0.1$ (Exapmle 2) in space-time domain $[0,20] \times [0,2]$. We find that the exact solution and the approximation solution are indistinguished in both cases.

4 The Stability and Convergence of the Proposed Scheme

This section is for the main theoretical results of this paper. We shall show the stability and convergence of the scheme (6). To do this, we first list some useful lemmas.

Lemma 1 For any $v \in {}_0\mathscr{P}_N$, and $1 \leqslant p < \infty$,

$$\| v \|_{L^p,N} \leqslant c(p) \| v \|_{L^p} \leqslant d(p) N^{1-\frac{2}{p}} \| v \|, \tag{9}$$

where $c(p)$ and $d(p)$ are positive constants depending only on p.

The validity of the first statement is ensured by a similar argument that can be found in the proof of Lemma 4.11 in [10], and the second result follows from an inverse inequality in [11].

In numerical analysis of the Jacobi pseudospectral method, we need to consider some orthogonal projections. For description of the approximation results, we introduce the Hilbert space $H^r_{\omega,A}(\Lambda)$ as appears in [7]. For any $r \in \mathbf{N}$,

$$H^r_{\omega,A}(\Lambda) = \{ v \mid v \text{ is measurable and } \| v \|_{r,\omega,A} < \infty \}$$

where

$$\| v \|_{r,\omega,A} = \left(\sum_{k=0}^{[\frac{r-1}{2}]} \| (1-x^2)^{\frac{r}{2}-k} \partial_x^{r-k} v \|_\omega^2 + \| v \|^2_{[\frac{r}{2}],\omega} \right)^{\frac{1}{2}}.$$

Similarly, we define the spaces $H^r_{\tilde{\omega},A}(\Lambda)$, $H^r_{\tilde{\omega},*}(\Lambda)$ and their norms by replacing ω with $\tilde{\omega}$ in the above definitions. For any real $r > 0$, the space $H^r_{\omega,A}(\Lambda)$ is defined by space interpolation. For $r \geqslant 1$,

$$H^r_{\omega,*}(\Lambda) = \{ v \mid \partial_x v \in H^{r-1}_{\omega,A}(\Lambda) \text{ and } \| v \|_{r,\omega,*} = \| \partial_x v \|_{r-1,\omega,A} < \infty \}.$$

We also define the Hilbert space $\tilde{H}^1_\omega(\Lambda)$ as

$$\tilde{H}^1_\omega(\Lambda) = \{ v \mid v \text{ is measurable on } \Lambda \text{ and } \| v \|_{1,\omega,\sim} = (\| \partial_x v \|_\omega^2 + \| v \|^2)^{\frac{1}{2}} < \infty \}.$$

Further, let ${}_0\tilde{H}^1_\omega(\Lambda) = \{ v \mid v \in \tilde{H}^1_\omega(\Lambda) \text{ and } v(-1) = 0 \}$. For any $u, v \in \tilde{H}^1_\omega(\Lambda)$,

$$a^\nu_\omega(u,v) = (\tilde{\omega} \partial_x u, \partial_x(\tilde{\omega} v)) + \nu(u,v).$$

The orthogonal projection ${}_0P_N^1 : {}_0\tilde{H}_\omega^1(\Lambda) \to {}_0\mathscr{P}_N$ is a mapping such that for any $v \in {}_0\tilde{H}_\omega^1(\Lambda)$,

$$a_\omega^\nu({}_0P_N^1 v - v, \varphi) = 0, \ \forall \ \varphi \in {}_0\mathscr{P}_N \ . \tag{10}$$

Lemma 2 Let $\nu > \frac{1}{2}$. Then for any $u, v \in \tilde{H}_\omega^1(\Lambda)$,

$$a_\omega{}^\nu(u,u) \geqslant \min(1, \nu - \frac{1}{2}) \| u \|_{1,\omega,\sim}^2, \quad a_\omega{}^\nu(u,v) \leqslant c \| u \|_{1,\omega,\sim} \| v \|_{1,\omega,\sim}.$$

Proof Integrating by parts yields that

$$\tilde{\omega}\partial_x u, \ \partial_x(\tilde{\omega}u)) = (\partial_x u, \ \partial_x u)_\omega - \frac{1}{2}\int_\Lambda \tilde{\omega} d \ u^2(x) =$$

$$\| \partial_x u \|_\omega^2 - \frac{1}{2} \| u \|^2 + u^2(-1) \geqslant \| \partial_x u \|_\omega^2 - \frac{1}{2} \| u \|^2.$$

Hence $a_\omega^\nu(u,u) \geqslant \min(1, \nu - \frac{1}{2}) \| u \|_{1,\omega,\sim}^2$. The second result is clear. □

Following the same lines as those in the proof of Theorems 3.3 and 3.4 in [13], we can get the following results.

Lemma 3 For any $v \in {}_0\tilde{H}_\omega^1(\Lambda) \cap H_{\omega,*}^r(\Lambda)$ and $r \geqslant 1$,

$$\| {}_0P_N^1 v - v \|_{1,\omega,\sim} \leqslant cN^{1-r} \| v \|_{r,\omega,*}. \tag{11}$$

Moreover, for any $v \in {}_0\tilde{H}_\omega^1(\Lambda) \cap H_{\omega,*}^{1+d}(\Lambda) \cap H^d(\Lambda)$ and $d > 1$,

$$\| {}_0^1P_N v \|_\infty \leqslant c(\| v \|_{1+d,\omega,*} + \| v \|_d). \tag{12}$$

By Theorem 4.8 of [9], we have that

Lemma 4 For any $v \in H_{\tilde{\omega},*}^r(\Lambda)$ and $r \geqslant 1$,

$$\| \partial_x(I_N^c v - v) \|_{\tilde{\omega}} + N^2 \| I_N^c v - v \| \leqslant cN^{2-r} \| v \|_{r,\tilde{\omega},*}.$$

We now analyze the stability of scheme (6). To this end, suppose that the data u_0, u_1 and f are disturbed by $\tilde{u}_0$, $\tilde{u}_1$ and $\tilde{f}$, respectively, which induce the error of u, denoted by $\tilde{u}$. Assume that all functions are valued at t. Let $\nu > \frac{1}{2}$, $q(\nu) = min(1, \nu - \frac{1}{2})$ and

$$E(\tilde{u},t) = q(\nu) \| \tilde{u}(t) \|_{1,\omega,\sim,N}^2 + \frac{1}{2} \| \tilde{u}(t) \|_{L^4,N}^4 + \| \partial_t \tilde{u}(t) \|_N^2,$$

$$\rho(\tilde{u}_0, \tilde{u}_1, \tilde{f}, t) = q(\nu) \| \tilde{u}_0 \|_{1,\omega,\sim,N} + \frac{1}{2} \| \tilde{u}_0 \|_{L^4,N}^4 + \| \tilde{u}_1 \|_N{}^2 + \int_0^t \| \tilde{f}(s) \|_N^2 ds,$$

where $\| v \|_{1,\omega,\sim,N} = (\| \omega^{\frac{1}{2}} \partial_x v \|_N^2 + \| v \|_N^2)^{\frac{1}{2}}$. Then we have the following stability result.

Theorem 1 Let $\nu > \frac{1}{2}$, u be the solution of (6), and $\tilde{u}$ be its error induced by $\tilde{u}_0$, $\tilde{u}_1$ and $\tilde{f}$. Then for all $0 \leqslant t \leqslant T$,

$$E(\tilde{u},t) \leqslant \rho(\tilde{u}_0, \tilde{u}_1, \tilde{f}, t) e^{M(\nu,u)t}$$

where $M(\nu,u)$ is given in the proof below.

Proof By (7), we have that

$$\begin{cases} (\partial_t^2 \tilde{u} + (1-\nu)\tilde{u} + \tilde{u}^3 + \tilde{G}_0, v)_N + a_{\omega,N}^\nu(\tilde{u}, v) = (\tilde{f}, v)_N, & \forall \ v \in {}_0\mathscr{P}_N \ , t \in (0,T], \\ \partial_t \tilde{u}(0) = \tilde{u}_1, \quad \tilde{u}(0) = \tilde{u}_0, \end{cases} \tag{13}$$

where $\tilde{G}_0 = 3\tilde{u}^2 u + 3\tilde{u}u^2$. Taking $\varphi = 2\partial_t \tilde{u}$ in (13), we get from (5) and Lemma 2 that

$$\frac{d}{dt}(q(\nu) \| \tilde{u} \|_{1,\omega,N}^2 + \frac{1}{2} \| \tilde{u} \|_{L^4,N}^4 + \| \partial_t \tilde{u} \|_N^2) \leqslant |W(t)|, \tag{14}$$

where $W(t) = (1-\nu)(\tilde{u}, 2\partial_t \tilde{u})_N + (\tilde{G}_0, 2\partial_t \tilde{u})_N + (\tilde{f}, 2\partial_t \tilde{u})_N$. Clearly,

$$|2(\tilde{G}, \partial_t \tilde{u})_N| \leqslant \frac{1}{2} \| \tilde{u} \|_{L^4,N}^4 + \frac{1}{2} \| \tilde{u} \|_N^2 + 18(|u|_{\infty,N}^2 + |u|_{\infty,N}^4) \| \partial_t \{\tilde{u}\} \|_N^2,$$

$$|2(1-\nu)(\tilde{u},\ \partial_t\tilde{u})_N| \leqslant \frac{1}{2}\ \|\tilde{u}\|_N^2 + 2(1-\nu)^2\,\|\partial_t\tilde{u}\|_N^2.$$

We get from the above inequalities and (14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\tilde{u},t) \leqslant M(\nu,u)\ E(\tilde{u},t) + \|\tilde{\mathcal{F}}(t)\|_N^2 \tag{15}$$

where $M(\nu,u) = 1 + 2(1-\nu)^2 + 18\max_{0\leqslant t\leqslant T}(|u|_{\infty,N}^2 + |u|_{\infty,N}^4)$. This implies the desired result.

Theorem 2 Let U and u be the solutions of (4) and (6), respectively. Assume that for $r, s_2, s_3 \geqslant 0$, $s_1 \geqslant 1$ and $d > 1$, $U \in H^2(0,T;{}_0\tilde{H}^1_\omega(\Lambda) \cap H^{r+1}_{\omega,*}(\Lambda)) \cap L^\infty(0,T;H^{1+d}_{\omega,*}(\Lambda) \cap H^d(\Lambda))$, $f \in L^2(0,T;\ H^{s_1}_{\omega,*}(\Lambda))$, $U_0 \in H^{s_2+2}_{\omega,*}(\Lambda)$ and $U_1 \in H^{s_3+1}_{\omega,*}(\Lambda)$. Then for all $0 \leqslant t \leqslant T$,

$$E(U-u,t) \leqslant M^*(N^{-2r} + N^{-2s_1} + N^{-2s_2} + N^{-2s_3})$$

where M^* is a positive constant depending only on ν and the norms of U in the mentioned spaces.

Proof To obtain a better error estimate, compare the numerical solution with $u^* = {}_0\mathscr{P}_N\,U$. By the exactness (5) and the definition (10), we have that

$$a^\nu_{\omega,N}(u^*,v) = a^\nu_\omega(u^*,v) = a^\nu_\omega(U,v),\quad \forall\ \ v \in {}_0\mathscr{P}_N\ .$$

So by (4),

$$(\partial_t^2 u^* + (1-\nu)u^* + u^{*3},v)_N + a^\nu_{\omega,N}(u^*,v) + \sum_{j=1}^4 G_j(v) = (f,v)_N,\quad \forall\ \ v \in {}_0\mathscr{P}_N\ , \tag{16}$$

where

$$G_1(v) = (\partial_t^2 U,v) - (\partial_t^2 u^*,v)_N,\quad G_2(v) = (U^3,v) - (u^{*3},v)_N,$$
$$G_3(v) = (1-\nu)(U,v) - (1-\nu)(u^*,v)_N,\quad G_4(v) = (f,v)_N - (f,v).$$

Further let $\tilde{u}^* = u - u^*$. Then by (7) and (16),

$$(\partial_t^2\tilde{u}^* + (1-\nu)\tilde{u}^* + (\tilde{u}^*)^3 + \tilde{G}_0^*,v)_N + a^\nu_{\omega,N}(\tilde{u}^*,v) = \sum_{j=1}^4 G_j(v),\ \forall\ \ v \in {}_0\mathscr{P}_N,\ 0 < t \leqslant T \tag{17}$$

where $\tilde{G}_0^* = 3(\tilde{u}^*)^2 u^* + 3\tilde{u}^*(u^*)^2$. In addition,

$$\tilde{u}^*(0) = I^c_N U_0 - {}_0P^1_N U_0,\qquad \partial_t\tilde{u}^*(0) = I^c_N U_1 - {}_0\mathscr{P}^1_N\ U_1.$$

Taking $\varphi = 2\partial_t\tilde{u}^*$ in (17) and comparing (17) with (13), we derive an error estimation similar to (15). But u, $\tilde{u}$, $\tilde{u}_0$, $\tilde{u}_1$ and $\|u\|_{\infty,N}$ are now replaced by u^*, $\tilde{u}^*$, $\tilde{u}^*(0)$, $\partial_t\tilde{u}^*(0)$ and $\|u^*\|_{\infty,N}$, respectively. Thus it remains to estimate $|G_j(\partial_t\tilde{u}^*)|$ $(1 \leqslant j \leqslant 4)$, $\|\tilde{u}^*(0)\|_{1,\omega,\sim,N}$, $\|\tilde{u}^*(0)\|_{L^4,N}$ and $\|\partial_t\tilde{u}^*(0)\|_N$. Let ε be a suitably small positive constant. By (5) and (11),

$$|G_1(\partial_t\tilde{u}^*)| \leqslant \varepsilon\ \|\partial_t\tilde{u}^*\|_N^2 + \frac{c}{\varepsilon}N^{-2r}\,\|\partial_t^2 U(t)\|_{r+1,\omega,*}^2,$$

$$|G_3(\partial_t\tilde{u}^*)| \leqslant \varepsilon \sim\ \|\partial_t\tilde{u}^*\|_N^2 + \frac{c(1-\nu)^2}{\varepsilon}N^{-2r}\,\|U(t)\|_{r+1,\omega,*}^2.$$

Let $K = [\frac{N}{3}]$. Then by (5), (9), (11), Lemma 1 and Lemma 4, we get that for $d > 1$,

$$\begin{aligned}
|G_2(\partial_t\tilde{u}^*)| &= |(U^3,\partial_t\tilde{u}^*) - (\{u^*\}^3,\partial_t\tilde{u}^*)_N| = |(U^3 - ({}_0P_K{}^1U)^3,\partial_t\tilde{u}^*)| + |(({}_0P_K{}^1U)^3 - u^{*3},\partial_t\tilde{u}^*)_N| \\
&\leqslant (\ \|U^3 - ({}_0P_K{}^1U)^3\| + \|({}_0P_K{}^1U)^3 - I_N{}^c u^{*3}\|_N)\,\|\partial_t\tilde{u}^*\|_N \\
&\leqslant c(\|U\|_\infty{}^2 + \|{}_0P_K{}^1U\|_\infty{}^2 + \|u^*\|_\infty{}^2)N^{-r}\,\|U(t)\|_{r+1,\omega,*}\,\|\partial_t\tilde{u}^*\|_N \\
&\leqslant \varepsilon\ \|\partial_t\tilde{u}^*\|_N^2 + \frac{c}{\varepsilon}\tilde{M}(U)\ N^{-2r}\,\|U(t)\|_{r+1,\omega,*}^2,
\end{aligned}$$

where $\tilde{M}(U) = \max\limits_{0\leqslant t\leqslant T}((\|U(t)\|_{1+d,\omega,*}^2 + \|U(t)\|_d^2)^2)$. Next, by (5) and Lemma 4,

$$|G_4(\partial_t\tilde{u}^*)| \leqslant \|I^c_N f - f\|\ \|\partial_t\tilde{u}^*\|_N \leqslant \varepsilon\ \|\partial_t\tilde{u}^*\|_N^2 + \frac{c}{\varepsilon}N^{-2s_1}\,\|f\|_{s_1,\omega,*}^2.$$

Moreover, by (5), Lemma 3 and Lemma 4,

$$\tilde{\ }^{*}(0)\|_{1,\omega,\sim,N}=\|\tilde{u}^{*}(0)\|_{1,\omega,\sim}\leqslant c(\|\partial_x(I_N^c U_0-U_0)\|_{\tilde{\omega}}+\|\partial_x(I_N^c U_0-U_0)\|+\|{}_0P_N^1U_0-U_0\|_{1,\omega}$$
$$\leqslant cN^{-s_2}\|U_0\|_{s_2+2,\tilde{\omega},*}.$$

Using Lemma 1, Lemma 3 and Lemma 4, we obtain that

$$\|\tilde{u}^{*}(0)\|_{L^4,N}^4\leqslant c\|\tilde{u}^{*}(0)\|_{L^4}^4\leqslant cN^2\|I_N^c U_0-{}_0P_N^1U_0\|^4\leqslant cN^{-2-4s_2}\|U_0\|_{s_2+2,\tilde{\omega},*}^4.$$

Finally we get from (5), Lemma 3 and Lemma 4 that

$$\|\partial_t\tilde{u}^{*}(0)\|_N\leqslant\|I_N^c U_1-{}_0P_N^1U_1\|\leqslant cN^{-2s_3}\|U_1\|_{s_3+1,\omega,*}.$$

A combination of the above estimates leads to the desired result. □

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用 Jacobi 拟谱方法求解半直线上的非线性 Klei-Gordon 方程

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摘要：研究半直线上的非线性 Klein-Gordon 方程的数值求解方法. 通过适当的变换，将此问题变成有限区间上的某类奇异问题. 然后利用 Jacobi 拟谱方法来求解. 证明了拟谱格式的稳定性和收敛性. 数值结果也说明了该方法的有效性.

关键词：非线性 Klein-Gordon 方程；半直线；Jacobi 拟谱方法