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# The Faithful Representations of Toeplitz Algebras

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**Abstract:** Let $(G, G_+)$ be a quasi-lattice ordered group, $H$ a directed and hereditary subset of $G_+$. Let $G_H = G_+ \cdot H^{-1}$ and $\mathscr{T}^{G_H}$ the associated Toeplitz algebra. In this paper, the faithful representations of $\mathscr{T}^{G_H}$ are clarified.

**Key words:** Toeplitz algebra; quasi-lattice ordered group; faithful representation

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## 0 Introduction

For any discrete group $G$, any subset $E$ of $G$, a Toeplitz algebra $\mathscr{T}^E$ can be fixed. To study the algebraic structure of the Toeplitz algebras, one important way is to clarify the faithful representations of the associated Toeplitz algebras. Some former works can be found in this field. For example, when $(G, E)$ is a quasi-lattice ordered group, a necessary and sufficient condition was given by Laca and Raeburn in [2]; when $G$ is abelian and $(G, E)$ is a quasily ordered group, a similar condition was given by the author in [3]. Recently Lorch and the author introduced the concept of quasi-lattice quasi-ordered group. More precisely, for any quasi-lattice ordered group $(G, G_+)$, any directed and hereditary subset $H$ of $G_+$, if $G_H = G_+ \cdot H^{-1}$ is a semigroup of $G$, then $(G, G_H)$ is referred to as a quasi-lattice quasi-ordered group. Note that when $H = \{e\}$, a quasi-lattice quasi-ordered group reduces to a quasi-lattice ordered group; when $G$ is abelian and $G = G_H \cup G_H^{-1}$, a quasi-lattice quasi-ordered group reduces to a quasily ordered group. So, a generalization of both [2] and [3] has been obtained by clarifying the faithful representations of the Toeplitz algebras on quasi-lattice quasi-ordered groups. Following the same lines as [2] ~ [4], in this paper we study further the faithful representations of the Toeplitz algebras. The main result of this paper is Theorem 1, where a faithful representation of $\mathscr{T}^{G_H}$ has been given without an additional assumption that $G_H$ is a semigroup (in other words, $(G, G_H)$ need not to be a quasi-lattice quasi-ordered group). Thus, the technique result of [4] appears as a special case of our result. Moreover, our proof given here is simpler than the original one even in the special case.

## 1 Quasi-lattice ordered groups

In this section, we recall some facts about quasi-lattice ordered groups stated in [1], [2] and [4].

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Let $G$ be a discrete group, and $G_+$ a semigroup of $G$ such that $G_+ \cap G_+^{-1} = \{e\}$. There is a partial order on $G$ defined by $x \leqslant y \Leftrightarrow x^{-1}y \in G_+$, which is left invariant in the sense that, if $x \leqslant y$, then $zx \leqslant zy$ for any $x$, $y, z \in G$.

**Definition 1** $(G, G_+)$ is said to be a quasi-lattice ordered group, if every finite subset of $G$ with an upper bound in $G_+$ has a least upper bound in $G_+$.

Let $(G, G_+)$ be a quasi-lattice ordered group. We follow the notations provided in [1] and [2]. For any $x \in G$, it is easy to show that $x$ has an upper bound in $G_+$ if and only if $x \in G_+ \cdot G_+^{-1}$, and when $x \in G_+ \cdot G_+^{-1}$, its least upper bound in $G_+$ will be denoted by $\sigma(x)$. More generally, for any subset $A \subseteq G_+$, if $A$ has an upper bound in $G_+$, then its least upper bound in $G_+$ will be denoted by $\sigma(A)$. When $x \in G_+ \cdot G_+^{-1}$, $x^{-1}$ also belongs to $G_+ \cdot G_+^{-1}$, and if we let $\tau(x) = x^{-1}\sigma(x)$, then it is easy to show that $\sigma(x) = \tau(x^{-1})$, $\sigma(x^{-1}) = \tau(x)$ and $x = \sigma(x) \cdot \tau(x)^{-1}$. To simplify the notation given in [2], for any pair $x, y \in G_+$, if they have no common upper bound in $G_+$, then let $\sigma(x, y) = \infty$.

**Definition 2** Let $(G, G_+)$ be a quasi-lattice ordered group and $H \subseteq G_+$. $H$ is said to be hereditary, if for any $x, y \in G_+$, $x \leqslant y \in H$ implies that $x \in H$ ; and $H$ is said to be directed, if any two elements of $H$ have a common upper bound in $H$.

**Definition 3** Let $(G, G_+)$ be a quasi-lattice ordered group, $H$ a directed and hereditary subset of $G_+$. And let $G_H = G_+ \cdot H^{-1}$. If $G_H$ is a semigroup of $G$, then $(G, G_H)$ is called a quasi-lattice quasi-ordered group.

## 2 Toeplitz algebras on quasi-lattice ordered groups

Let $G$ be a discrete group and $\{\delta_g \mid g \in G\}$ be the usual orthonormal basis for $l^2(G)$. For any $g \in G$, we define a unitary operator $u_g$ on $l^2(G)$ by $u_g(\delta_h) = \delta_{gh}$ for $h \in G$. For any subset $E$ of $G$, let $l^2(E)$ be the closed subspace of $l^2(G)$ generated by $\{\delta_g \mid g \in E\}$ ; the projection from $l^2(G)$ onto $l^2(E)$ is denoted by $p^E$.

**Definition 4** The $C^*$-algebra generated by $\{T_g^E \triangleq p^E u_g p^E \mid g \in G\}$ is denoted by $\mathscr{T}^E$ and is called the Toeplitz algebra, with respect to $E$.

Throughout the rest of this section, $(G, G_+)$ is a quasi-lattice ordered group. With the convention that $T_\infty^{G_+} = 0$. By([1], Section 2 and Section 3) we know that

$$T_x^{G_+} = \begin{cases} T_{\sigma(x)}^{G_+} \cdot T_{\tau(x)^{-1}}^{G_+} & \text{if } x \in G_+ \cdot G_+^{-1}; \\ 0, & \text{if } x \notin G_+ \cdot G_+^{-1}. \end{cases} \tag{2.1}$$

and if $x, y \in G_+$, then

$$(T_x^{G_+})^* = T_{x^{-1}}^{G_+},\ T_{x^{-1}}^{G_+} \cdot T_x^{G_+} = 1,\ T_x^{G_+} T_y^{G_+} = T_{xy}^{G_+}, \tag{2.2}$$

$$(T_x^{G_+} \cdot T_{x^{-1}}^{G_+}) \cdot (T_y^{G_+} \cdot T_{y^{-1}}^{G_+}) = T_{\sigma(x,y)}^{G_+} \cdot T_{\sigma(x,y)^{-1}}^{G_+}, \tag{2.3}$$

$$(T_x^{G_+})^* T_y^{G_+} = T_{x^{-1}\sigma(x,y)}^{G_+} (T_{y^{-1}\sigma(x,y)}^{G_+})^*. \tag{2.4}$$

Let $\mathscr{T}^\infty(G_+) = \text{span}\{T_g^{G_+} T_{h^{-1}}^{G_+} \mid g, h \in G_+\}$. Then by (2.1) ~ (2.4), $\mathscr{T}^\infty(G_+)$ is a dense $*$-subalgebra of $\mathscr{T}^{G_+}$. Let $D^{G_+} = \overline{\text{span}\{T_g^{G_+} T_{g^{-1}}^{G_+} \mid g \in G_+\}}$. It's a commutative $C^*$-subalgebra of $\mathscr{T}^{G_+}$. Moreover, there is a faithful compress linear operator $\theta^{G_+}$ from $\mathscr{T}^{G_+}$ onto $D^{G_+}$ such that $\theta^{G_+}(T_g^{G_+} T_{h^{-1}}^{G_+}) = \delta_{g,h} T_g^{G_+} T_{h^{-1}}^{G_+}$.

**Lemma 1**[4] Let $(G, G_+)$ be a quasi-lattice ordered group and $E$ a subset of $G$ such that $G_+ \subseteq E$. Then the following two conditions are equivalent:

(1) There is a $C^*$ -morphism $\gamma^{E,G_+}: \mathscr{T}^{G_+} \to \mathscr{T}^E$ satisfying $\gamma^{E,G_+}(T_g^{G_+}) = T_g^E$ for any $g \in G$;

(2) There exists a hereditary and directed subset $H$ of $G_+$ such that $E = G_H$.

By the preceding Lemma, we know that given any direct and hereditary subset $H$ of $G_+$, $\mathscr{T}^{G_H} = \operatorname{span}\{T_g^{G_H}T_{h^{-1}}^{G_H} \mid g,h \in G_+\}$ and $D^{G_H} \triangleq \gamma^{G_H,G_+}(D^{G_+}) = \operatorname{span}\{T_g^{G_H}T_{g^{-1}}^{G_H} \mid g \in G_+\}$ is a commutative $C^*$ -subalgebra of $\mathscr{T}^{G_H}$. Similarly there is a faithful compress linear operator $\theta^{G_H}$ from $\mathscr{T}^{G_H}$ onto $D^{G_H}$ with $\theta^{G_H}(T_g^{G_H}T_{h^{-1}}^{G_H}) = \delta_{g,h}T_g^{G_H}T_{h^{-1}}^{G_H}$. Since $\gamma^{G_H,G_+}: \mathscr{T}^{G_+} \to \mathscr{T}^{G_H}$ is a $C^*$ -morphism, replacing $G_+$ by $G_H$, we know that the same equations as (2.1) ~ (2.4) also hold.

## 3 The faithful representation of Toeplitz algebras

Throughout this section, $(G,G_+)$ is a quasi-lattice ordered group, $H$ is a directed and hereditary subset of $G_+$.

**Lemma 2**[1] Let $(G,G_+)$ be a quasi-lattice ordered group and $\{L(t) \mid t \in G_+\}$ be a family of projections of a unital $C^*$ -algebra $B$ satisfying $L(e) = 1$ and

$$L(s)\,L(t) = \begin{cases} L(\sigma(s,t)), & \text{if } \sigma(s,t) \in G_+; \\ 0, & \text{if } \sigma(x,y) = \infty \end{cases}$$

Then for any finite subset $F = \{t_1,t_2,\cdots,t_n\}$ of $G_+$, any $\lambda_1,\lambda_2,\cdots,\lambda_n \in C$, we have

$$\Big\|\sum_{j=1}^n \lambda_j L(t_j)\Big\| = \max\Big\{\Big|\sum_{j\in A}\lambda_j\Big| \,\Big|\, \varnothing \neq A \subseteq \{1,2,\cdots,n\}, \prod_{j\in A}L(t_j)\cdot\prod_{k\notin A}(1-L(t_k)) \neq 0\Big\}.$$

**Remark 1** (1) If $A = F$, then the product $\prod_{j\in A}L(t_j)\cdot\prod_{k\notin A}(1-L(t_k))$ in the preceding lemma should be understood as $\prod_{j\in F}L(t_j)$, and if for every $\varnothing \neq A \subseteq \{1,2,\cdots,n\}$, $\prod_{j\in A}L(t_j)\cdot\prod_{k\notin A}(1-L(t_k)) = 0$, then $\sum_{j=1}^n \lambda_j L(t_j) = 0$.

(2) For convenience's sake, in the following we take $L(\infty) = 0$.

Now suppose that $B$ is a unital $C^*$ -algebra, $\pi$ is a unital $C^*$ -morphism from $\mathscr{T}^{G_H}$ to $B$. Let $V(t) = \pi(T_t^{G_H})$ and $L(t) = \pi(T_t^{G_H}T_{t^{-1}}^{G_H})$ for any $t \in G_+$. For any finite collection $g_1,g_2,\cdots,g_m \in (G_+\setminus\{e\}) \cup \{\infty\}$ with $\prod_{j=1}^m(1-L(g_j)) \neq 0$, $x_i,y_i \in G_+$ such that $x_i \neq y_i$ for all $1 \leqslant i \leqslant n$, and $\lambda_0,\lambda_1,\cdots,\lambda_n \in C$, let

$$T(g,x,y,\lambda) = \prod_{j=1}^m(1-L(g_j))\Big(\lambda_0 + \sum_{i=1}^n\lambda_i V(x_i)V(y_i)^*\Big)\prod_{j=1}^m(1-L(g_j)). \tag{3.1}$$

**Theorem 1** Let $\pi: \mathscr{T}^{G_H} \to B$ be a unital $C^*$ -morphism. Then $\pi$ is faithful if and only if the following two conditions hold:

(1) The restriction of $\pi$ on $D^{G_H}$ is faithful;

(2) For all such $T(g,x,y,\lambda)$ defined as (3.1), $|\lambda_0| \leqslant \|T(g,x,y,\lambda)\|$.

**Proof** First assume that $\pi$ is faithful. Then obviously its restriction on $D^{G_H}$ is faithful. Since by assumption $\prod_{j=1}^m(1-T_{g_j}^{G_H}T_{g_j^{-1}}^{G_H}) \neq 0$, we know that for any $g_j \in (G_+\setminus\{e\}) \cup \{\infty\}$, $x_i,y_i \in G_+$ with $x_i \neq y_i$, $\lambda_i \in C$,

$$|\lambda_0| = \Big\|\lambda_0\prod_{j=1}^m(1-T_{g_j}^{G_H}T_{g_j^{-1}}^{G_H})\Big\| = \Big\|\prod_{j=1}^m(1-T_{g_j}^{G_H}T_{g_j^{-1}}^{G_H})\cdot\lambda_0\prod_{j=1}^m(1-T_{g_j}^{G_H}T_{g_j^{-1}}^{G_H})\Big\|$$

$$= \left\| \theta^{G_H}\left( \prod_{j=1}^{m} (1 - T_{g_j}^{G_H} T_{g_j^{-1}}^{G_H})(\lambda_0 + \sum_{i=1}^{n} \lambda_i T_{x_i}^{G_H} (T_{y_i}^{G_H})^*) \prod_{j=1}^{m} (1 - T_{g_j}^{G_H} T_{g_j^{-1}}^{G_H}) \right) \right\|$$

$$\leqslant \left\| \prod_{j=1}^{m} (1 - T_{g_j}^{G_H} T_{g_j^{-1}}^{G_H})(\lambda_0 + \sum_{i=1}^{n} \lambda_i T_{x_i}^{G_H} (T_{y_i}^{G_H})^*) \prod_{j=1}^{m} (1 - T_{g_j}^{G_H} T_{g_j^{-1}}^{G_H}) \right\|.$$

The inequality $|\lambda_0| \leqslant \| T(g,x,y,\lambda) \|$ then follows from the faithfulness of $\pi$.

For the reverse implication, assume that conditions (1) and (2) are satisfied. As in [3], it suffices to prove that there is a contractive and linear operator $\phi$ from $C^*(\{V(g)V(h)^* \mid g,h \in G_+\})$ to $C^*(\{V(g)V(g)^* \mid g \in G_+\})$ such that

$$\phi(V(g)V(h)^*) = \begin{cases} V(g)V(h)^*, & \text{if } g = h, \\ 0, & \text{otherwise.} \end{cases}$$

We focus on proving the existing of such a morphism $\phi$. First we note that $\mathscr{T}^\infty(G_H) = \operatorname{span}\{T_g^{G_H} T_{h^{-1}}^{G_H} \mid g,h \in G_+\}$ is dense in $\mathscr{T}^{G_H}$, so to show that $\phi$ is well-defined and can be extended as a contractive and linear operator, it suffices to verify that for any finite subset $F \subseteq G_+$, the following inequality holds:

$$\left\| \sum_{x \in F} \lambda_{x,x} L(x) \right\| \leqslant \left\| \sum_{x,y \in F} \lambda_{x,y} V(x) V(y)^* \right\|,$$

where $\lambda_{x,y} \in C$ for $x,y \in F$.

Let $T \in \mathscr{T}^\infty(G_H)$. Then $T$ can be decomposed as $T = T_1 + T_2$ with

$$T_1 = \sum_{x \in F_1} \lambda_x L(x), \qquad T_2 = \sum_{x,y \in F_2, x \neq y} \lambda_{x,y} V(x) V(y)^*, \tag{3.2}$$

where $F_1$ and $F_2$ are two finite subsets of $G_+$. It suffices to prove that $\| T_1 \| \leqslant \| T \|$.

By Lemma 2, we know there is a non-empty subset $A$ of $F_1$ such that

$$\prod_{j \in A} L(t_j) \cdot \prod_{k \in F_1 \setminus A} (1 - L(t_k)) \neq 0 \quad \text{and} \quad \left\| \sum_{x \in F_1} \lambda_x L(x) \right\| = \left| \sum_{x \in A} \lambda_x \right|.$$

Define

$$Q_A = \prod_{j \in A} L(t_j) \cdot \prod_{k \in F_1 \setminus A} (1 - L(t_k)).$$

Then since $Q_A \neq 0$, we know that $\sigma(A) \in G_+$ (equivalently, $\sigma(A) \neq \infty$). Let $a = \sigma(A)$. Then by (2.3), we know

$$Q_A = \prod_{y \in F_1 \setminus A} (L(a) - L(\sigma(a,y))). \tag{3.3}$$

First, we note that for any $x \in A$, $\sigma(x,a) = a$ and $\sigma(x,\sigma(a,y)) = \sigma(a,y)$, which implies $L(x) \cdot Q_A = Q_A$ for any $x \in A$. By the same reason, $L(y) Q_A = 0$ for any $y \in F_1 \setminus A$. It follows that

$$Q_A T_1 Q_A = Q_A \left( \sum_{x \in A} \lambda_x \right) Q_A = \left( \sum_{x \in A} \lambda_x \right) Q_A. \tag{3.4}$$

Next, note that

$$Q_A = V(a) \cdot \prod_{y \in F_1 \setminus A} (1 - L(a^{-1} \sigma(a,y))) \cdot V(a)^*. \tag{3.5}$$

So for any $s,t \in G_+$, if $Q_A V(s) V(t)^* Q_A \neq 0$, then $V(a)^* V(s) V(t)^* V(a) \neq 0$. It follows from (2.2) and (2.4) that in this case $V(a)^* V(s) V(t)^* V(a) = V(u) V(v)^*$ for some $u,v \in G_+$, and $s \neq t$ implies that $u \neq v$. Hence, for such $s,t$,

$$Q_A V(s) V(t)^* Q_A = V(a) P(a,y) V(u) V(v)^* P(a,y) V(a)^*, \tag{3.6}$$

where $P(a,y) = \prod_{y \in F_1 \setminus A} (1 - L(a^{-1} \sigma(a,y)))$.

Finally, by (3.4) and (3.6) we know that

$$Q_ATQ_A = Q_A(T_1 + T_2)Q_A = V(a)T(g,x,y,\lambda)V(a)^* \tag{3.7}$$

for some $g_j \in (G_+\setminus\{e\}) \cup \{\infty\}$, $x_i, y_i \in G_+$ with $x_i \neq y_i$, $\lambda_i \in C$ and $\lambda_0 = \sum_{x\in A}\lambda_x$. Since $V(a)^*V(a) = 1$, we know $\|V(a)SV(a)^*\| = \|S\|$ for any $S \in \mathbf{B}(l^2(G_H))$. It follows that

$$\|T_1\| = |\lambda_0| \leqslant \|T(g,x,y,\lambda)\| = \|V(a)T(g,x,y,\lambda)V(a)^*\| = \|Q_ATQ_A\| \leqslant \|T\|.$$

**Remark 2** In the proof above, we assume that $A \subsetneqq F_1$. If it happens that $A = F_1$, then just replace $Q_A$ by 1.

As the first application of Theorem 1, we get the following main technique result of [2].

**Corollary 1**[2] Let $\pi: \mathcal{T}^{G_+} \to B$ be a unital $C^*$-morphism. Then $\pi$ is faithful if and only if for any $g_1, g_2, \cdots, g_n \in G_+\setminus\{e\}$,

$$\prod_{j=1}^{n}(1 - L(g_j)) \neq 0. \tag{3.8}$$

**Proof** For any $g_1, g_2, \cdots, g_n \in G_+\setminus\{e\}$, since $\prod_{j=1}^{n}(1 - T^{G_+}_{g_j}T^{G_+}_{g_j^{-1}})\delta_e = \delta_e \neq 0$, the necessary part follows.

For the revise direction, suppose (3.8) holds. First we show that $\pi|_{D^{G_+}}$ is faithful. Let $F = \{t_1, t_2, \cdots, t_n\}$ be any finite subset of $G_+$, and $A$ be any subset of $F$,

$$\begin{aligned}&\prod_{t_j\in A}T^{G_+}_{t_j}T^{G_+}_{t_j^{-1}}\cdot\prod_{t_i\in F\setminus A}(1 - T^{G_+}_{t_i}T^{G_+}_{t_i^{-1}}) \neq 0\\ \Leftrightarrow\ &\sigma(A) \in G_+ \text{ and } (F\setminus A) \subseteq G_+\setminus\{e\}\\ \Leftrightarrow\ &\prod_{t_j\in A}L(t_j)\prod_{t_i\in F\setminus A}(1 - L(t_i)) \neq 0.\end{aligned}$$

So by Lemma 2, $\pi|_{D^{G_+}}$ is faithful. Let $0 \neq Q_A$ be as in the proof of Theorem 1. For any $x, y \in F_2$ with $x \neq y$, if $V(a)^*V(x)V(y)^*V(a) \neq 0$, then $V(a)^*V(x)V(y)^*V(a) = V(s)V(t)^*$ for some $s, t \in G_+$. Since $x \neq y$, $s$ is not equal to $t$, so one can choose $n_{x,y} \in \{s,t\}$ such that $n_{x,y} \neq e$. Then $Q_A(1 - L(n_{x,y})) \neq 0$. If $n_{x,y} = s$, then $Q_A(1 - L(n_{x,y}))V(s) = Q_A(V(s) - L(s)V(s)) = Q_A(V(s) - V(s)) = 0$; otherwise, $V(t)^*(1 - L(n_{x,y}))Q_A = (Q_A(1 - L(n_{x,y}))V(t))^* = 0$. In any case, we have $Q_A(1 - L(n_{x,y}))V(s)V(t)^*Q_A(1 - L(n_{x,y})) = 0$.

Let

$$Q_1 = Q_A\cdot V(a)\cdot\Big(\prod_{\substack{x,y\in F_2,\ x\neq y\\ V(a)^*V(x)V(y)^*V(a)\neq 0}}(1 - L(n_{x,y}))\Big)\cdot V(a)^*.$$

Then $Q_1 \neq 0$ and there exist some $g_1, g_2, \cdots, g_n \in G_+\setminus\{e\}$ such that

$$Q_1TQ_1 = \lambda_0 V(a)\cdot\prod_{j=1}^{n}(1 - L(g_j))\cdot V(a)^*.$$

Therefore,

$$\begin{aligned}\|T_1\| = |\lambda_0| &= \Big\|\lambda_0\prod_{j=1}^{n}(1 - L(g_j))\Big\|\\ &= \Big\|\lambda_0 V(a)\cdot\prod_{j=1}^{n}(1 - L(g_j))\cdot V(a)^*\Big\| = \|Q_1TQ_1\| \leqslant \|T\|.\end{aligned}$$

The second application of Theorem 1 can be stated as follows:

**Corollary 2**[4] Suppose that $(G, G_H)$ is a quasi-lattice quasi-ordered group such that $H \neq \{e\}$, $\pi: \mathcal{T}^{G_H} \to B$ is a unital $C^*$-morphism. Then $\pi$ is faithful if and only if, for any finite collection $g_1, g_2, \cdots, g_m \in G_+\setminus H$, $x_1, y_1, x_2, y_2, \cdots, x_n, y_n$ in $G_+$ with $x_i, y_i \in H$, $x_i \neq y_i$ for all $i$, and $\lambda_0, \lambda_1, \cdots, \lambda_n \in C$, the following inequality holds:

$$|\lambda_0| \leqslant \left\| \prod_{j=1}^{m} (1 - L(g_j))(\lambda_0 + \sum_{i=1}^{n} \lambda_i V(x_i) V(y_i)^*) \prod_{j=1}^{m} (1 - L(g_j)) \right\|. \quad (3.9)$$

**Proof** By assumption, $G_H$ is a semigroup of $G$, so for any $g \in H$, $T_g^{G_H} T_{g^{-1}}^{G_H} = 1$. It follows that for any $g_1, g_2, \cdots, g_m \in G_+$, $\prod_{i=1}^{m} (1 - T_{g_i}^{G_H} T_{g_i^{-1}}^{G_H}) \neq 0$ if and only if $g_i \in G_+ \backslash H$ for all $i$. Note if (3.9) holds, then $\prod_{i=1}^{m} (1 - L(g_i)) \neq 0$ for any $g_1, g_2, \cdots, g_m \in G_+ \backslash H$. Hence $\pi|_{D^{G_H}}$ is faithful.

Let $F_1, F_2$ and $\sigma(A) = a$ be as in the proof of Theorem 1. For any $x, y \in F_2$ with $x \neq y$, if $V(a)^* V(x) V(y)^* V(a) \neq 0$, then $V(a)^* V(x) V(y)^* V(a) = V(s) V(t)^*$ for some $s, t \in G_+$ with $s \neq t$. If either $s$ or $t$ belongs to $G_+ \backslash H$, say $s \in G_+ \backslash H$, then $Q_A(1 - L(s)) V(s) V(t)^* Q_A = 0$ and $Q_A(1 - L(s)) \neq 0$. Multiplying $Q_A$ by some projections as above, we know that there exists a projection $Q_2 = V(a) \cdot (\prod_{j=1}^{m} (1 - L(g_j))) \cdot V(a)^*$ with some $g_1, g_2, \cdots, g_m \in G_+ \backslash H$, such that

$$Q_2 T Q_2 = V(a) \prod_{j=1}^{m} (1 - L(g_j))(\lambda_0 + \sum_{i=1}^{n} \lambda_i V(x_i) V(y_i)^*) \prod_{j=1}^{m} (1 - L(g_j)) V(a)^*$$

with some $x_i, y_i \in H$, $x_i \neq y_i$ for all $i$. The inequality $\| T_1 \| \leqslant \| T \|$ then follows as in the proof of Corollary 1.

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# Toeplitz 算子代数的忠实表示

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**摘要**: 设 $(G, G_+)$ 为一个拟格序群, $H$ 为 $G_+$ 的可传定向子集. 令 $G_H = G_+ \cdot H^{-1}$, $\mathcal{T}^{G_H}$为相应的 Toeplitz 算子代数. 作者刻划了 $\mathcal{T}^{G_H}$ 的忠实表示.

**关键词**: Toeplitz 算子代数; 拟格序群; 忠实表示