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Universal Toeplitz Algebras on Discrete Abelian Groups

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Abstract Let G be a discrete abelian group and (G,G_+) a quasi-partially ordered group. In this note, the universal Toeplitz algebra $UT^{G_+}(G)$ associated to such a quasi-partial ordered group is constructed.

Key words discrete abelian group; quasi-partially ordered group; universal Toeplitz algebra

1 Introduction

Throughout the note, We assume that G is a discrete abelian group. For any subset G_+ of G, we say that (G,G_+) is a quasi-partial ordered group if $0\in G_+$, $G_+ + G_+ \subseteq G_+$ and $G = G_+ - G_+$; further, (G,G_+) is referred to as a quasily ordered group if $G = G_+ \cup (-G_+)$. Note that when $G_+^0 = G_+ \cap (-G_+) = \{0\}$, then a quasi-partial ordered group (resp. quasily ordered group) (G,G_+) is known as a partially ordered (resp. ordered) group.

Murphy G J proved in [1], among other things, that for any partially ordered group (G, G_+), there is a universal Toeplitz algebra that played a key role in his subsequent work. In this note, we generalize such a result to the case when (G,G_+) is only quasi-partially ordered. The ideas in this note are mostly contained in [1].

2 The main results

Let G be a discrete abelian group and $\hat{G}$ denote the dual group of G. Since G is discrete and abelian, $\hat{G}$ is compact and $\hat{G}$ is connected if and only if G is torsion-free. By Stone-Weierstrass Theorem, $C(\hat{G})$ is generated by $\{\varepsilon_g \mid g\in G\}$, where $\varepsilon_g(\gamma)=\gamma(g)$ for $\gamma\in\hat{G}$.

Lemma 1 (cf. [1], **Lemma 1.2**) If $\pi: G\to B$ is a homomorphism from an abelian group

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into the group of unitaries of a unital C^* -algebra B , then there is a unique $*$ -homomorphism $\beta: C(\hat{G}) \to B$ such that $\beta(\varepsilon_x) = \pi(x)$, $x \in G$.

Let (G,G_+) be a quasi-partially ordered group. By a representation (V,H) of G_+ , we mean that H is a Hilbert space and $V: G_+ \to \mathscr{B}(H)$ is a map satisfying

(1) $V(0) = 1$, $V^*(x)V(x) = 1$ for $x \in G_+$;

(2) $V(x+y) = V(x)V(y)$ for $x, y \in G_+$;

(3) $V(x)V(x)^* = 1$ for $x \in G_+^0$.

Now let (V,H) be a representation of G_+ . We call a pair (π, K) a unitary lifting of V if K is a Hilbert space containing H as a closed subspace, $\pi: G \to \mathscr{B}(K)$ is a homomorphism of G into the group of unitaries of $\mathscr{B}(K)$, H is invariant for all $\pi(x), x \in G_+$, and $\pi(x)|_H = V(x)$ for such x .

Lemma 2(cf. [1], Theorem 1.1) Let (G,G_+) be a partially ordered group and $\beta: G_+ \to \mathscr{B}(H)$ a semigroup of isometries on a Hilbert space H . Then β admits a unitary lifting (π, K).

The following proposition is crucial to our construction of the universal Toeplitz algebras.

Proposition 3 Let (G,G_+) be a quasi-partially ordered group and $V: G_+ \to \mathscr{B}(H)$ a representation. Then V admits a unitary lifting (π,K) .

Proof Let (G,G_+) be a quasi-partially ordered group. G_+^* denotes $G_+ \setminus G_+^0$. Then it is easy to show that

$$G_+ + G_+^* = G_+^*.$$

Choose $g_0 \in G_+^*$. Then for any $x \in G$, $x = g_1 - g_2$ with $g_i \in G_+$ for $i = 1,2$, we have $x = (g_1 + g_0) - (g_2 + g_0) \in G_+^* - G_+^*$. It follows that $G = G_+^* - G_+^*$. So if we set $G_1 = G_+^* \cup \{0\}$, then (G,G_1) is actually a partially ordered group.

Now if (V,H) is a representation of G_+ , then needless to say, it is also an isometric representation of G_1 . By Lemm 2, we know that V admits a unitary lifting (π,K) . Choose $g_0 \in G_+^*$. Then for any $x \in G_+^0$, $x = -g_0 + (x + g_0)$, so for any $h \in H$,

$$\begin{aligned}\pi(x)(h) = \pi(g_0)^*\pi(g_0)\pi(x)(h) = \pi(g_0)^*\pi(g_0 + x)(h) = \\ \pi(g_0)^*V(g_0 + x)(h) = \pi(g_0)^*V(g_0)(V(x)(h)) = \\ \pi(g_0)^*\pi(g_0)(V(x)(h)) = V(x)(h).\end{aligned}$$

Therefore, H is also invariant for $V(x), x \in G_+^0$ and $\pi(x)|_H = V(x)$ for such x . So (π, K) is exactly a unitary lifting of V .

We can now define the universal Toeplitz algebra as in [1] and show that it has a certain universal property.

Let (G,G_+) be a quasi-partially ordered group, p denotes the projection $(1,0)$ in C^2 , and I be the closed ideal in $C^2 * C(\hat{G})$ generated by all $\varepsilon_x p - p\varepsilon_x p, x \in G_+$, where $C^2 * C(\hat{G})$ is the free product of C^2 and $C(\hat{G})$. If π denotes the quotient map from $C^2 * C(\hat{G})$ to $C^2 * C(\hat{G})/I$, then we set $UT^{G_+}(G) = \pi(p)(\mathrm{Im}(\pi))\pi(\mathrm{p})$. Clearly, $UT^{G_+}(G)$ is a unital C^* -algebra with unit $\pi(p)$ and we call it the *universal Toeplitz algebra* with respect to (G,G_+) .

For any $x \in G_+$, define $V(x) = \pi(\varepsilon_x)\pi(p)$. Then

$$V(x)^* V(x) = \pi(p)\pi(\varepsilon_{-x})\pi(\varepsilon_x)\pi(p) = \pi(p),$$

so every $\pi(x), x \in G_+$ is an isometry in $UT^{G_+}(G)$. Further, for any $x, y \in G_+$,

$$V(x)V(y) = \pi(\varepsilon_x)(\pi(p)\pi(\varepsilon_y)\pi(p)) =$$
$$\pi(\varepsilon_x)\pi(\varepsilon_y)\pi(p) = \pi(\varepsilon_{x+y})\pi(p) = V(x+y).$$

When $x \in G_+^0$, we know that

$$p\varepsilon_x - p\varepsilon_x p = (\varepsilon_{-x}p - p\varepsilon_{-x}p)^* \in I^* = I.$$

So $p\varepsilon_x - \varepsilon_x p = (p\varepsilon_x - p\varepsilon_x p) - (\varepsilon_x p - p\varepsilon_x p) \in I$ and therefore $\pi(p)\pi(\varepsilon_x) = \pi(\varepsilon_x)\pi(p)$. It follows that $V(x)V(x)^* = 1$ for all $x \in G_+^0$, so V is in fact a representation of G_+ .

By Lemma 1, Proposition 3 and [1], Theorem 1.3, we have the following proclaim:

Theorem 4 Let (G, G_+) be a quasi-partially ordered group and $\beta: G_+ \to B$ a representation of G_+ in a unital C^* -algebra B . Then there is a unique C^* -algebra morphism $\beta^*: UT^{G_+}(G) \to B$ such that $\beta^* V = \beta$.

Let (G, G_+) be a quasi-partially ordered group. We have another Toeplitz algebra defined in a usual way as follows:

Let $\{e_g | g \in G\}$ be the usual orthonormal basis for $l^2(G)$, where

$$e_g(h) = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{otherwise.} \end{cases} \quad \text{for } g, h \in G.$$

Let $l^2(G_+)$ be the closed subspace of $l^2(G)$ generated by $\{e_g | g \in G_+\}$; its projection is denoted by p^{G_+} . For any $g \in G$, we define a unitary u_g on $l^2(G)$ by $u_g(e_h) = e_{g+h}$ for $h \in G$. The C^* -algebra generated by $\{p^{G_+}u_g p^{G_+} | g \in G\}$ is denoted by $T^{G_+}(G)$, and is called the *Toeplitz algebra* with respect to (G, G_+) .

Now let (G, G_+) be a quasi-partially ordered group. At this point, there are at least two Toeplitz algebras associated with such a group: one is the universal Toeplitz algebra $UT^{G_+}(G)$; another is the usual Toeplitz algebra $T^{G_+}(G)$. By the universal property of $UT^{G_+}(G)$, we know that there is a C^* -algebra morphism $\pi: UT^{G_+}(G) \to T^{G_+}(G)$, $\pi(V(x)) = p^{G_+}u_x p^{G_+}$ for $x \in G_+$. Naturally, one may ask: is π an isomorphism between these two Toeplitz algebras?

Remark (1) When (G, G_+) is a quasily ordered group, then by Theorem 4 and [2], Theorem 3.5, we know that π is exactly an isomorphism.

(2) When $(G, G_+) = (\mathbb{Z}^2, \mathbb{Z}_+^2)$, it is proved in [3] that in this case π is not an isomorphism.

Question Let $G = \mathbb{Z}^2$, $G_1 = \{(m, n) \in \mathbb{Z}^2 | m + n > 0\} \cup \{0\}$ and $G_2 = \{(m, n) \in \mathbb{Z}^2 | m + n \geqslant 0\}$. Whether the natural morphism $\gamma^{G_2, G_1}: T^{G_1}(G) \to T^{G_2}(G)$, $\gamma^{G_2, G_1}(p^{G_1}u_g p^{G_1}) = p^{G_2}u_g p^{G_2}$ for $g \in G$, is well-defined and can be extended to be a C^* -algebra morphism?

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离散交换群上的万有 Toeplitz 算子代数

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提　要　设 G 为一离散交换群,(G,G_+) 为一拟偏序群. 相应于这样的一个拟偏序群 (G, G_+),构造了一个万有 Toeplitz 算子代数.

关键词　离散交换群;拟偏序群;万有 Toeplitz 算子代数

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