

关于二元延迟 3 步前馈逆有限自动机的结构*

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On the Structure of Binary Feedforward Inverse Finite Automata with Delay 3

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Abstract: The structure of feedforward inverses is a fundamental problem in the invertibility theory of finite automata. The characterization of the structure of feedforward inverses with delay steps ≥ 3 is a long-term unsolved problem. This paper deals with this topic. For a binary weakly invertible semi-input memory finite automaton $C(M_a, f)$ with delay 3, where the state graph of M_a is cyclic, the characterizations of the structures are given when its minimal 3-output weight is 1, 2, and 8, respectively. Because $C(M_a, f)$ is weakly invertible with delay 3 iff it is weakly inverse with delay 3, a partial characterization of the structure of binary feedforward inverses with delay 3 is obtained.

Key words: finite automata; semi-input memory; feedforward inverses; invertibility

摘要: 前馈逆有限自动机的结构是有限自动机可逆性理论中的基本问题.对延迟步数 ≥ 3 的前馈逆结构的刻画,则是一个长期的未解决问题.研究了二元延迟 3 步前馈逆有限自动机的结构.对于自治有限自动机 M_a 的状态图为圈的二元延迟 3 步弱可逆半输入存储有限自动机 $C(M_a, f)$,给出了其长 3 极小输出权分别为 1, 2, 8 三种情形下结构的一种刻画.由于 $C(M_a, f)$ 延迟 3 步弱可逆当且仅当它是延迟 3 步弱逆,因此,得到了二元延迟 3 步前馈逆有限自动机结构的一种部分刻画.

关键词: 有限自动机;半输入存储;前馈逆;可逆性

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1 Introduction

A semi-input memory (SIM) finite automaton (FA) is called feedforward inverse if it is weakly inverse^[1]. A fundamental problem of feedforward inverses is to characterize their structures^[2]. However, this is not trivial. The previous systematic results on this topic are in the case of delay steps ≤ 2 ^[3-7], while keeping unsolved for the case of delay steps ≥ 3 for a long-term. This paper studies the structure of binary feedforward inverses with delay 3.

Reference [6] shows that the binary weakly invertible(WI) SIM finite automata $C(M_a, f)$ with delay 3, where the state graph of M_a is cyclic, can be divided into four classes by the minimal 3-output weight $w_{3,M}$, i.e., $w_{3,M}=1, 2, 4, 8$. Because the binary WI finite automata with delay 3 and the weak inverse finite automata with delay 3 are the same in some sense^[8], we investigate the structure of binary feedforward inverses with delay 3 via WI finite automata in case of $w_{3,M}=1, 2$, and 8, respectively, and give their corresponding characterizations.

We briefly recall some definitions and notations. Let $M=(X, Y, S, \delta, \lambda)$ be an FA, $s \in S$. If for any $\alpha=x_0x_1\dots x_l$ in X^* of length $l+1$, x_0 can be uniquely determined by s and $\lambda(s, \alpha)$, then s is called a $\leq l$ -step state, $l \geq 0$. If s is a $\leq l$ -step state and not a $\leq (l-1)$ -step state, then s is called an l -step state. Especially, if s is a ≤ 0 -step state, then s is called a 0-step state. Denote $S_0=\{s|s \in S, |W_{3,s}^M|=w_{3,M}\}$. Throughout this paper, an FA M is referred to $M=(X, Y, S, \delta, \lambda)$, which has the property of $|X|=|Y|=2$. $C(M_a, f)=(X, Y, X^c \times S_a, \delta, \lambda)$ is called a c -order SIM FA, if $\delta(x_{-c}, \dots, x_{-1}, s_a, x_0)=\langle x_{-c+1}, \dots, x_0, \delta_a(s_a) \rangle$, $\lambda(\langle x_{-c}, \dots, x_{-1}, s_a \rangle, x_0)=f(x_{-c}, \dots, x_0, \lambda_a(s_a))$, where $M_a=(S_a, Y_a, \delta_a, \lambda_a)$ is an autonomous FA, f is a mapping from $X^{c+1} \times \lambda_a(S_a)$ to Y . M_a is called cyclic, if $S_a=\{s_{a,1}, \dots, s_{a,n_a}\}$, $\delta_a(s_{a,i})=s_{a,i+1}$ for $i=1, \dots, n_a-1$, $\delta_a(s_{a,n_a})=s_{a,1}$, and $\lambda_a(s_a)=s_a$ for any $s_a \in S_a$. For those terminologies not explained here, readers are referred to Refs.[1,8].

2 Binary WI SIM Finite Automata with Delay 3 of Which $w_{3,M}=2$

Let Ω stand for the condition: Let M be an WIFA, $w_{3,M}=2$, s_i and s_{ij} be the successor states of $s \in S_0$ and s_i ($i, j=1, 2$), respectively, $s_1 \neq s_2$.

Lemma 1. Assume that Ω holds. If s is a 0-step state, then $|\lambda(s_i, X)|=|\lambda(s_{ij}, X)|=1$, $\lambda(s_{i1}, X)=\lambda(s_{i2}, X)$.

Proof: Since s is a 0-step state, $|\lambda(s, X)|=2$. Since $s \in S_0$, $|\lambda(s_i, X)|=|\lambda(s_{ij}, X)|=1$, $\lambda(s_{i1}, X)=\lambda(s_{i2}, X)$ ($i, j=1, 2$).

Lemma 2. Assume that Ω holds, then s is a 1-step state iff $|\lambda(s, X)|=|\lambda(s_i, X)|=1$ ($i=1, 2$), $\lambda(s_1, X) \neq \lambda(s_2, X)$. Furthermore, $|\lambda(s_{ij}, X)|=1$, $\lambda(s_{i1}, X)=\lambda(s_{i2}, X)$ ($i, j=1, 2$).

Proof: “ \Leftarrow ” It is obvious. “ \Rightarrow ” Since s is a 1-step state, $|\lambda(s, X)|=1$. First, $|\lambda(s_i, X)|=1$ ($i=1, 2$) (Otherwise, there exist x_0x_1 and $x'_0x'_1$ such that $\lambda(s, x_0x_1)=\lambda(s, x'_0x'_1)$, $x_0 \neq x'_0$, a contradiction). By the definition of 1-step state, $\lambda(s_1, X) \neq \lambda(s_2, X)$. Since $s \in S_0$, $|\lambda(s_{ij}, X)|=1$, $\lambda(s_{i1}, X)=\lambda(s_{i2}, X)$ ($i, j=1, 2$).

Lemma 3. Assume that Ω holds, then s_1 is a 0-step state iff s_2 is a 0-step state.

Proof: Since $s \in S_0$, using Proposition 2 in Ref.[6], $s_i \in S_0$ ($i=1, 2$). By symmetry we need only to prove “ \Rightarrow ”. Suppose that s_1 is a 0-step state while s_2 isn't. By Lemma 1, s isn't a 0-step state. Thus $\lambda(s_1, X)=Y$, $|\lambda(s_2, X)|=|\lambda(s, X)|=1$. Since $s \in S_0$, $|\lambda(s_{2i}, X)|=1$ ($i=1, 2$), $\lambda(s_{21}, X)=\lambda(s_{22}, X)$. Note $s_2 \in S_0$, $\cup_{i,j=1,2} \lambda(s_{2ij}, X)=Y$. Since $\lambda(s_2, X) \subset \lambda(s_1, X)$, there exists $x_1 \in X$ such that $\lambda(s_1, x_1)=\lambda(s_2, x'_1)$ for any $x'_1 \in X$. Denote $s_{11}=\delta(s_1, x_1)$. Since $s \in S_0$, $\lambda(s_{21}, X)=\lambda(s_{22}, X)=\{\lambda(s_{11}, x_2)\}$. Let $s_{11}=\delta(s_{11}, x_2)$, then $\lambda(s_{11}, X) \subseteq \cup_{i,j=1,2} \lambda(s_{2ij}, X)$. There exist x_3 and x'_3 such that $\lambda(s_{11}, x_3)=\lambda(s_{2ij}, x'_3)$. Let $s_1=\delta(s, x_0)$, $s_2=\delta(s, x'_0)$, $s_{2ij}=\delta(s_{2i}, x'_j)$, $x_0 \neq x'_0$. Then $\lambda(s, x_0x_1x_2x_3)=\lambda(s, x'_0x'_1x'_2x'_3)$, $x_0 \neq x'_0$, which contradicts that M is weakly invertible with delay 3. Hence “ \Rightarrow ” follows.

Lemma 4. Assume that Ω holds, then s is a 2-step state iff (a), (b) and (c) hold, where (a) $|\lambda(s, X)|=1$; (b) $|\lambda(s_i, X)|=1$ ($i=1, 2$), $\lambda(s_1, X)=\lambda(s_2, X)$; (c) $|\lambda(s_{ij}, X)|=1$ ($i, j=1, 2$), $\lambda(s_{i1}, X)=\lambda(s_{i2}, X)$ ($i=1, 2$), $\lambda(s_{11}, X) \neq \lambda(s_{22}, X)$.

Proof: “ \Leftarrow ” It is obvious. “ \Rightarrow ” Since s is a 2-step state, $|\lambda(s, X)|=1$. (a) Follows; (b) Suppose that $|\lambda(s_i, X)| \neq 1$ for some $i \in \{1, 2\}$, then s_i is a 0-step state. By Lemma 3, s_1 and s_2 are 0-step states. Thus $\lambda(s_1, X)=\lambda(s_2, X)=Y$. Then there

exist $x_1, x'_1 \in X$ such that $\lambda(s_1, x_1) = \lambda(s_2, x'_1)$. Let $s_{11} = \delta(s_1, x_1)$, $s_{21} = \delta(s_2, x'_1)$. Since $s \in S_0$, by Lemma 1, there exist $x_2, x'_2 \in X$ such that $\lambda(s_{11}, x_2) = \lambda(s_{21}, x'_2)$. Let $s_1 = \delta(s, x_0)$, $s_2 = \delta(s, x'_0)$, $x_0 \neq x'_0$. Then $\lambda(s, x_0, x_1, x_2) = \lambda(s, x'_0, x'_1, x'_2)$, $x_0 \neq x'_0$, a contradiction. Thus $|\lambda(s_i, X)| = 1$ ($i=1, 2$). Since s is a 2-step state, by Lemma 2, $\lambda(s_1, X) = \lambda(s_2, X)$; (c) Since s is a 2-step state, using (a) and (b) $|\lambda(s_{ij}, X)| = 1$ and $\lambda(s_{i1}, X) = \lambda(s_{i2}, X)$ ($i, j=1, 2$). Note $s \in S_0$, $\lambda(s_{11}, X) \neq \lambda(s_{22}, X)$. Thus (c) follows.

Lemma 5. Assume that Ω holds. If s is a 3-step state, then s_1 is not a 2-step state.

Proof: Suppose that s_1 is a 2-step state. By Lemma 3 s_2 is not a 0-step state, then $|\lambda(s_2, X)| = 1$. Since s is not a 1-step state, by Lemma 2, $|\lambda(s, X)| = |\lambda(s_i, X)| = 1$ ($i=1, 2$) and $\lambda(s_1, X) = \lambda(s_2, X)$. Since s_1 is a 2-step state, by Lemma 4, $|\lambda(s_{1j}, X)| = 1$ ($j=1, 2$), $\lambda(s_{11}, X) = \lambda(s_{12}, X)$ and $\cup_{i,j=1,2} \lambda(s_{1ij}, X) = Y$. It is easy to see whether s_{11} is a 0-step state or not, $\lambda(s_{11}, X) = \lambda(s_{12}, X) \subset \lambda(s_{21}, X) \cup \lambda(s_{22}, X) = Y$. Thus there exist x_2 and x'_2 such that $\{\lambda(s_{2k}, x'_2)\} = \lambda(s_{11}, X) = \lambda(s_{12}, X)$ for some $k \in \{1, 2\}$. Let $s_{2k1} = \delta(s_{2k}, x'_2)$, since $\lambda(s_{2k1}, x'_3) \in Y = \cup_{i,j=1,2} \lambda(s_{1ij}, X)$, there exists $x_3 \in X$ such that $\lambda(s_{2k1}, x'_3) = \lambda(s_{1ij}, x_3)$ for some $i, j \in \{1, 2\}$. Let $s_1 = \delta(s, x_0)$, $s_2 = \delta(s, x'_0)$, $s_{1i} = \delta(s_1, x_i)$, $s_{2k} = \delta(s_2, x'_i)$, $s_{1ij} = \delta(s_{1i}, x_2)$, $x_0 \neq x'_0$. Then $\lambda(s, x_0, x_1, x_2, x_3) = \lambda(s, x'_0, x'_1, x'_2, x'_3)$, $x_0 \neq x'_0$, which contradicts that s is a 3-step state.

Lemma 6. Assume that Ω holds. If s_1 is a 0-step state, then s is a 3-step state, $|\lambda(s_{ij}, X)| = |\lambda(s_{ijk}, X)| = 1$ and $\lambda(s_{ij1}, X) = \lambda(s_{ij2}, X)$ ($i, j, k=1, 2$). Denote $\{e_1\} = \lambda(s_{11}, X)$, $\{e_2\} = \lambda(s_{12}, X)$, $\{e_3\} = \lambda(s_{21}, X)$, $\{e_4\} = \lambda(s_{22}, X)$, $\{e_5\} = \lambda(s_{111}, X)$, $\{e_6\} = \lambda(s_{121}, X)$, $\{e_7\} = \lambda(s_{211}, X)$, $\{e_8\} = \lambda(s_{221}, X)$, then the following statements hold. (1) If $\lambda(s_1, x) = \lambda(s_2, x')$, then $e_1 = e_3$, $e_5 \neq e_7$; (2) $e_1 = e_2$ iff $e_3 = e_4$, and if $e_1 = e_2$, then $e_1 = e_2 = e_3 = e_4$; if $e_1 \neq e_2$, then $\lambda(s_1, x) = \lambda(s_2, x')$ iff $e_1 = e_3$; (3) $e_5 = e_6$ iff $e_7 = e_8$. And if $e_5 = e_6$, then $e_5 \neq e_7$; if $e_5 \neq e_6$, then $e_5 \neq e_7$ iff $\lambda(s_1, x) = \lambda(s_2, x')$ and $e_1 = e_3$, where $s_{11} = \delta(s_1, x)$, $s_{21} = \delta(s_2, x')$.

Proof: Since s_1 is a 0-step state, by Lemma 3, s_2 is a 0-step state. By Lemma 1, $|\lambda(s_{ij}, X)| = |\lambda(s_{ijk}, X)| = 1$, $\lambda(s_{ij1}, X) = \lambda(s_{ij2}, X)$ ($i, j, k=1, 2$). Since M is WI with delay 3, s is an l -step state ($0 \leq l \leq 3$). By Lemmas 1, 2 and 4, s is a 3-step state. Assume $\lambda(s_1, x) = \lambda(s_2, x')$, and let $s_{11} = \delta(s_1, x)$, $s_{21} = \delta(s_2, x')$. Since $s \in S_0$, $e_1 = e_2$. Since s is a 3-step state, $e_5 \neq e_7$. Thus (1) follows.

(2) By symmetry we need only to prove " \Rightarrow ". Suppose $e_1 = e_2$ and $e_3 \neq e_4$, then $|W_{3,s}^M| = 3$. This contradicts $s \in S_0$. Thus " \Leftarrow " follows. Clearly, by (1) if $e_1 = e_2$ then $e_1 = e_2 = e_3 = e_4$. Let $e_1 \neq e_2$, $e_1 = e_3$, then $e_3 \neq e_4$. Now suppose $\lambda(s_1, x) \neq \lambda(s_2, x')$, where $s_{11} = \delta(s_1, x)$, $s_{21} = \delta(s_2, x')$. Then $\lambda(s_1, x) = \lambda(s_2, x'')$, $x' \neq x''$. Using (1), $e_1 = e_4$. Thus $e_3 = e_4$, a contradiction. Combining with (1), $\lambda(s_1, x) = \lambda(s_2, x')$ iff $e_1 = e_3$.

(3) By symmetry we need only to prove " \Rightarrow ". Suppose $e_5 = e_6$ and $e_7 \neq e_8$. Since $e_5 = e_6 \in \{e_7, e_8\}$, $e_5 = e_6 = e_7$ or e_8 . Without loss of generality, let $e_5 = e_6 = e_7$. Let $s_{11} = \delta(s_1, x)$, $s_{12} = \delta(s_1, x_1)$, $s_{21} = \delta(s_2, x')$, $x \neq x_1$. By (1), $\lambda(s_1, x) \neq \lambda(s_2, x')$, $\lambda(s_1, x_1) \neq \lambda(s_2, x')$. Thus $\lambda(s_2, x') \notin \{\lambda(s_1, x), \lambda(s_1, x_1)\} = Y$, a contradiction. Thus " \Leftarrow " follows. Since s is a 3-step state, clearly, if $e_5 = e_6$ then $e_5 \neq e_7$. Now assume $e_5 \neq e_6$. To prove " \Leftarrow ", it is immediate from (1). To prove " \Rightarrow ", suppose $e_5 \neq e_7$, $\lambda(s_1, x) \neq \lambda(s_2, x')$, where $s_{11} = \delta(s_1, x)$, $s_{21} = \delta(s_2, x')$. Then $e_6 = e_7$. Let $s_{12} = \delta(s_1, x_1)$, $x_1 \neq x$. By (1), $\lambda(s_1, x_1) \neq \lambda(s_2, x')$. Thus $\lambda(s_1, x_1) \notin \{\lambda(s_1, x), \lambda(s_1, x_1)\} = Y$, a contradiction. Combining with (1) " \Rightarrow " follows.

Lemma 7. Assume that Ω holds. If s_1 is a 1-step state, then s is a 3-step state, s_2 is not a 0-step state, $|\lambda(s_1, X)| = |\lambda(s_{1j}, X)| = |\lambda(s_{1jk}, X)| = 1$ and $\lambda(s_{1j1}, X) = \lambda(s_{1j2}, X)$ ($j, k=1, 2$). Furthermore, (1) and (2) hold. (1) If some s_{2k} is not a 0-step state, then s_2 is a 1-step state, $|\lambda(s_2, X)| = |\lambda(s_{2j}, X)| = |\lambda(s_{2jk}, X)| = 1$ and $\lambda(s_{2j1}, X) = \lambda(s_{2j2}, X)$ ($j, k=1, 2$). Denote $\{e_1\} = \lambda(s_{11}, X)$, $\{e_2\} = \lambda(s_{12}, X)$, $\{e_3\} = \lambda(s_{21}, X)$, $\{e_4\} = \lambda(s_{22}, X)$, $\{e_5\} = \lambda(s_{111}, X)$, $\{e_6\} = \lambda(s_{121}, X)$, $\{e_7\} = \lambda(s_{211}, X)$, $\{e_8\} = \lambda(s_{221}, X)$, then (a) If $e_1 = e_3$ then $e_5 \neq e_7$; (b) $e_5 = e_6$ iff $e_7 = e_8$. And if $e_5 = e_6$ then $e_5 \neq e_7$; if $e_5 \neq e_6$, then $e_1 = e_3$ iff $e_5 \neq e_7$; (2) If some s_{2k} is a 0-step state, then s_2 is a 3-step state, s_{2j} is a 0-step state ($j=1, 2$), $|\lambda(s_{2jk}, X)| = 1$ ($j, k=1, 2$). Denote $\{e_9\} = \lambda(s_{211}, X)$, $\{e_{10}\} = \lambda(s_{221}, X)$, $\{e_{11}\} = \lambda(s_{212}, X)$, $\{e_{12}\} = \lambda(s_{222}, X)$, let $e_1 = \lambda(s_{21}, x) = \lambda(s_{22}, x')$, then (a) $e_5 \neq e_9$, $e_9 = e_{10}$, $e_{11} = e_{12}$, where $s_{211} = \delta(s_{21}, x)$, $s_{221} = \delta(s_{22}, x')$; (b) $e_5 = e_6$ iff $e_9 = e_{10} = e_{11} = e_{12}$. And if $e_5 = e_6$, then $e_5 \neq e_9$; if $e_5 \neq e_6$, then $e_1 = \lambda(s_{21}, x) = \lambda(s_{22}, x')$ iff $e_5 \neq e_9$.

Proof: Since s_1 is a 1-step state, by Lemma 3 s_2 is not a 0-step state, then $|\lambda(s_2, X)| = 1$, by Lemma 2, $|\lambda(s_1, X)| = |\lambda(s_{1j}, X)| = |\lambda(s_{1jk}, X)| = 1$ and $\lambda(s_{1j1}, X) = \lambda(s_{1j2}, X)$ ($j, k=1, 2$). Since M is weakly invertible with delay 3, s is a

l -step state ($0 \leq l \leq 3$). By Lemmas 1, 2, and 4, s is a 3-step state. Then $|\lambda(s, X)|=1$. By Lemma 2, $\lambda(s_1, X)=\lambda(s_2, X)$. Since $s \in S_0$, by Proposition 2 in Ref.[6], $s_2 \in S_0$.

(1). Assume some s_{2k} is not a 0-step state, by Lemma 3, s_{2j} is not a 0-step state ($j=1,2$), then $|\lambda(s_{2j}, X)|=1$ ($j=1,2$). Suppose s_2 is not a 1-step state. Note $s_2 \in S_0$, $\cup_{i,j=1,2} \lambda(s_{2ij}, X)=Y$, then $\{e_5, e_6\} \subseteq \cup_{i,j=1,2} \lambda(s_{2ij}, X)$. On the other hand, $\lambda(s_1, X)=\lambda(s_2, X) \subset Y = \{e_1, e_2\}$. Then there exist $x_0 x_1 x_2 x_3$ and $x'_0 x'_1 x'_2 x'_3$ such that $\lambda(s, x_0 x_1 x_2 x_3) = \lambda(s, x'_0 x'_1 x'_2 x'_3)$, $x_0 \neq x'_0$, a contradiction. Thus s_2 is a 1-step state. By Lemma 2, $|\lambda(s_2, X)|=|\lambda(s_{2j}, X)|=|\lambda(s_{2jk}, X)|=1$, $\lambda(s_{2j1}, X)=\lambda(s_{2j2}, X)$ ($j, k=1,2$), and $\lambda(s_{21}, X) \neq \lambda(s_{22}, X)$. Since s is a 3-step state, it is easy to see that (a) follows; (b) By symmetry we need only to prove “ \Rightarrow ”. Suppose $e_5=e_6$, $e_7 \neq e_8$, then $e_5=e_6 \in Y = \{e_7, e_8\}$. Thus $e_5=e_6=e_7$ or e_8 . Without loss of generality let $e_5=e_6=e_7$, then $e_1 \neq e_3$, $e_2 \neq e_3$, i.e., $e_3 \notin \{e_1, e_2\} = Y$, a contradiction. “ \Leftarrow ” follows. Since s is a 3-step state, by its definition, it is not difficult to see that the remainder of (b) hold.

(2). Assume that some s_{2k} is a 0-step state, by Lemma 3, s_{2j} is a 0-step state ($j=1,2$). By Lemma 1, $|\lambda(s_{2jk}, X)|=1$ ($j, k=1,2$), by Lemma 6, s_2 is a 3-step state. Let $e_1 = \lambda(s_{21}, X) = \lambda(s_{22}, X)$. Since s is a 3-step state, $e_5 \neq e_9$. By Lemma 1, $e_9=e_{10}$, $e_{11}=e_{12}$, where $s_{211} = \tilde{\alpha}(s_{21}, X)$, $s_{221} = \tilde{\alpha}(s_{22}, X)$. Since $e_9=e_{10}$, $e_{11}=e_{12}$, to prove “ \Rightarrow ”, suppose $e_5=e_6$, $e_9 \neq e_{11}$, then $e_5=e_6 \in \{e_9, e_{11}\} = Y$. On the other hand, $\{e_1, e_3\} = \lambda(s_{21}, X) = Y$. There exist $x_0 x_1 x_2 x_3$ and $x'_0 x'_1 x'_2 x'_3$ such that $\lambda(s, x_0 x_1 x_2 x_3) = \lambda(s, x'_0 x'_1 x'_2 x'_3)$, $x_0 \neq x'_0$, a contradiction. Thus “ \Rightarrow ” follows. By the same arguments “ \Leftarrow ” follows. Since s is a 3-step state, clearly, if $e_5=e_6$ then $e_5 \neq e_9$. Now assume $e_5 \neq e_6$, by the same arguments as Lemma 6, the remainder of (b) hold.

Lemma 8. Assume that Ω holds. Then s, s_i ($i=1,2$) are 3-step states iff s_{ij} is a 0-step state ($i, j=1,2$). Assume s, s_i ($i=1,2$) are 3-step states, then $|\lambda(s_{ijk}, X)|=1$ ($i, j, k=1,2$). Denote $\{e_{ijk}\} = \lambda(s_{ijk}, X)$ ($i, j, k=1,2$), then (a) $e_{111} = e_{112}$ iff $e_{121} = e_{122}$ ($i=1,2$); (b) $e_{111} = e_{112} = e_{121} = e_{122}$ iff $e_{211} = e_{212} = e_{221} = e_{222}$. If $e_{111} = e_{112} = e_{121} = e_{122}$, then $e_{111} \neq e_{211}$; if $e_{111} \neq e_{112}$, then $\lambda(s_{1j}, X) = \lambda(s_{2k}, X)$ iff $e_{1j1} \neq e_{2k1}$.

Proof: “ \Rightarrow ” Since s, s_i ($i=1,2$) are 3-step states, $|\lambda(s, X)|=|\lambda(s_i, X)|=1$ ($i=1,2$), $\lambda(s_1, X)=\lambda(s_2, X)$. Since $s \in S_0$, by Proposition 2 in Ref.[6], $s_i \in S_0$ ($i=1,2$). By Lemma 3, we consider three cases. Case 1. s_{ij} is a 0-step state ($i, j=1,2$); Case 2. s_{ij} ($i, j=1,2$) are not 0-step states; Case 3. Some s_{mj} is a 0-step state while some s_{nk} is not a 0-step state, $m \neq n$. Next we prove Cases 2 and 3 don't occur. Suppose Case 2 holds. Since s_j ($j=1,2$) are 3-step states, by Lemma 2, $\lambda(s_{i1}, X) = \lambda(s_{i2}, X)$ ($i=1,2$). Note $s \in S_0$, by Lemma 4, s is a 2-step state, a contradiction. Suppose Case 3 holds and let s_{mj} be a 0-step state, s_{nk} not be. Since s_n is a 3-step state, $\lambda(s_{n1}, X) = \lambda(s_{n2}, X)$. Note $s_n \in S_0$, $\cup_{j,k=1,2} \lambda(s_{nj}, X) = Y$. Clearly, $\lambda(s_{n1}, X) = \lambda(s_{n2}, X) \subset Y = \lambda(s_{m1}, X) = \lambda(s_{m2}, X)$, $\lambda(s_{mjk}, X) \subset \cup_{j,k=1,2} \lambda(s_{nj}, X)$. There exist $x_0 x_1 x_2 x_3$ and $x'_0 x'_1 x'_2 x'_3$ such that $\lambda(s, x_0 x_1 x_2 x_3) = \lambda(s, x'_0 x'_1 x'_2 x'_3)$, $x_0 \neq x'_0$, a contradiction. Thus “ \Rightarrow ” follows. To prove “ \Leftarrow ”, assume that s_{ij} is a 0-step state ($i, j=1,2$). By Lemma 6, s_i ($i=1,2$) are 3-step states. Since s is an l -step state ($0 \leq l \leq 3$), by Lemmas 1, 2, and 4, s is a 3-step state. Assume that s, s_i ($i=1,2$) are 3-step states, then s_{ij} ($i, j=1,2$) are 0-step states, by Lemma 1, $|\lambda(s_{ijk}, X)|=1$ ($i, j, k=1,2$). (a) It is immediate from Lemma 6. (b) By the same arguments as Lemma 7, (b) follows.

Lemma 9. Assume that Ω holds. If some s_{mj} is a 0-step state, then s and s_m are 3-step states, s_n is a 1-step state or a 3-step state ($m \neq n$).

Proof: Assume that s_{mj} is a 0-step state, by Lemma 6, s_m is a 3-step state. By Lemma 3, s_n is not a 0-step state. Since M is weakly invertible with delay 3, s is an l -step state ($0 \leq l \leq 3$). By Lemmas 1, 2, and 4, s is a 3-step state. Since s_n is an l -step state ($1 \leq l \leq 3$), by Lemma 5, s_n is not a 2-step state. Therefore by Lemmas 7 and 8, s_n is a 1-step state or a 3-step state.

Lemma 10. Assume that Ω holds. Then $|\lambda(s_{1j}, X)|=1$ ($j=1,2$) and $\lambda(s_{11}, X) = \lambda(s_{12}, X)$ iff $|\lambda(s_{2j}, X)|=1$ ($j=1,2$) and $\lambda(s_{21}, X) = \lambda(s_{22}, X)$.

Proof: By symmetry we need only to prove “ \Rightarrow ”. Assume that $|\lambda(s_{1j}, X)|=1$ ($j=1,2$), $\lambda(s_{11}, X) = \lambda(s_{12}, X)$. By Lemma 2, s_1 is not a 1-step state. Next we consider three cases of s_1 . Case 1. s_1 is a 0-step state. By Lemma 6, the

conclusion follows; Case 2. s_1 is a 2-step state. By Lemmas 1, 2, 4, and 5, there are three subcases of s to discuss: s is a 0-step state or a 1-step state, or 2-step state. It is immediate from Lemmas 1 or 2, or 4, respectively; Case 3. s_1 is a 3-step state. By Lemmas 1, 2, 4, 7, and 8, there are three subcases of s to discuss: s is a 0-step state or a 1-step state, or 2-step state. It is immediate from Lemmas 1 or 2, or 4, respectively.

Lemma 11. Assume that Ω holds. Then $|\lambda(s_{1ij}, X)|=1$ ($i, j=1, 2$) and $\lambda(s_{111}, X)=\lambda(s_{112}, X)=\lambda(s_{121}, X)=\lambda(s_{122}, X)$ iff $|\lambda(s_{2ij}, X)|=1$ ($i, j=1, 2$) and $\lambda(s_{211}, X)=\lambda(s_{212}, X)=\lambda(s_{221}, X)=\lambda(s_{222}, X)$. If this case occurs, then $\lambda(s_{111}, X)=\lambda(s_{112}, X)=\lambda(s_{121}, X)=\lambda(s_{122}, X)\neq\lambda(s_{211}, X)=\lambda(s_{212}, X)=\lambda(s_{221}, X)=\lambda(s_{222}, X)$.

Proof: By symmetry we need only to prove “ \Rightarrow ”. Assume $|\lambda(s_{1ij}, X)|=1$ ($i, j=1, 2$), $\lambda(s_{111}, X)=\lambda(s_{112}, X)=\lambda(s_{121}, X)=\lambda(s_{122}, X)$, by Lemma 4, s_1 is not a 2-step state. Next we consider three cases of s_1 . Case 1. s_1 is a 0-step state; Case 2. s_1 is a 1-step state; Case 3. s_1 is a 3-step state. In the first two cases, it is immediate from Lemmas 8 and 7, respectively. In Case 3, since $s_1 \in S_0$, $\lambda(s_{11}, X) \cup \lambda(s_{12}, X) = Y$. By Lemma 3, s_{1j} ($j=1, 2$) are 0-step states. By Lemma 9, s is a 3-step state, and s_2 is a 3-step state or a 1-step state. Thus it is immediate from Lemmas 8 and 7, respectively.

Lemma 12. Let M be a c -order SIM FA $C(M_a, f)$, $M_a = \langle S_a, Y_a, \delta_a, \lambda_a \rangle$ be cyclic, $X=Y=\{0, 1\}$, $w_{3,M}=2$, $c \geq 3$, if M is WI with delay 3, then there exist mappings h_0 from $X^{c-1} \times S_a$ to $\{0, 1\}$, h_1 from $X^{c-2} \times S_a$ to $\{0, 1\}$, h_2 from $X^{c-3} \times S_a$ to $\{0, 1\}$, f_0 from $X^c \times S_a$ to Y , f_1 from $X^{c-1} \times S_a$ to Y , f_2 from $X^{c-2} \times S_a$ to Y , f_3 from $X^{c-3} \times S_a$ to Y , such that

$$(2.1) \quad h_0(x_{-c}, \dots, x_{-2}, s_a) = 0 \rightarrow h_0(x_{-c+1}, \dots, x_{-1}, \delta_a(s_a)) = 1 \wedge h_1(x_{-c+2}, \dots, x_{-1}, \delta_a^2(s_a)) = 1, \quad h_0(x_{-c}, \dots, x_{-2}, s_a) = 1 \wedge h_0(x_{-c+1}, \dots, x_{-1}, \delta_a(s_a)) = 1 \wedge h_1(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) = 0 \rightarrow h_1(x_{-c+2}, \dots, x_{-1}, \delta_a^2(s_a)) = 1, \quad h_0(x_{-c}, \dots, x_{-2}, s_a) = 1 \wedge h_1(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) = 1 \rightarrow h_2(x_{-c+2}, \dots, x_{-2}, \delta_a^2(s_a)) = 0, \quad h_0(x_{-c}, \dots, x_{-2}, s_a) = 0 \wedge h_1(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) = 1 \rightarrow h_2(x_{-c+1}, \dots, x_{-3}, \delta_a(s_a)) = 1, \quad h_0(x_{-c}, \dots, x_{-3}, 0, s_a) = 0 \wedge h_0(x_{-c}, \dots, x_{-3}, 1, s_a) = 0 \wedge h_1(x_{-c+1}, \dots, x_{-3}, x_{-2}, \delta_a(s_a)) = 0 \rightarrow h_1(x_{-c+1}, \dots, x_{-3}, x'_{-2}, \delta_a(s_a)) = 0 (x_{-2} \neq x'_{-2}), \quad h_2(x_{-c}, \dots, x_{-4}, s_a) = 1 \rightarrow h_1(x_{-c}, \dots, x_{-3}, s_a) = 1, \quad h_1(x_{-c}, \dots, x_{-3}, s_a) = 1 \rightarrow h_0(x_{-c}, \dots, x_{-2}, s_a) = 1.$$

$$(2.2) \quad f_0(x_{-c}, \dots, x_{-2}, 0, s_a) + f_0(x_{-c}, \dots, x_{-2}, 1, s_a) = h_3(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)), \quad \text{if } h_0(x_{-c}, \dots, x_{-2}, s_a) = 0 \wedge h_1(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) = 0; \\ h_3(x_{-c}, \dots, x_{-3}, s_a) = h_4(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) + 1, \quad \text{if } h_0(x_{-c}, \dots, x_{-3}, 0, s_a) = 1 \wedge h_0(x_{-c}, \dots, x_{-3}, 1, s_a) = 1 \wedge h_1(x_{-c}, \dots, x_{-3}, s_a) = 0 \wedge h_2(x_{-c+1}, \dots, x_{-3}, \delta_a(s_a)) = 0; \\ f_0(x_{-c}, \dots, x_{-3}, 0, 0, s_a) + f_0(x_{-c}, \dots, x_{-3}, 1, 0, s_a) = f_1(x_{-c+1}, \dots, x_{-3}, 0, 0, \delta_a(s_a)) + f_1(x_{-c+1}, \dots, x_{-3}, 1, 0, \delta_a(s_a)) + 1, \quad \text{if } h_0(x_{-c}, \dots, x_{-3}, 0, s_a) = 0 \wedge h_0(x_{-c}, \dots, x_{-3}, 1, s_a) = 0 \wedge h_1(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) = 0; \\ f_0(x_{-c}, \dots, x_{-3}, x_{-2}, 0, s_a) + f_1(x_{-c}, \dots, x_{-3}, x'_{-2}, s_a) = f_1(x_{-c+1}, \dots, x_{-2}, 0, \delta_a(s_a)) + f_2(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) + 1, \quad \text{if } h_0(x_{-c}, \dots, x_{-3}, x_{-2}, s_a) = 0 \wedge h_0(x_{-c}, \dots, x_{-3}, x'_{-2}, s_a) = 1 \wedge h_1(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) = 0 (x_{-2} \neq x'_{-2}); \\ f_0(x_{-c}, \dots, x_{-2}, 0, s_a) + f_0(x_{-c}, \dots, x_{-2}, 1, s_a) = h_4(x_{-c+2}, \dots, x_{-2}, \delta_a^2(s_a)) + 1, \quad \text{if } h_0(x_{-c}, \dots, x_{-2}, s_a) = 0 \wedge h_1(x_{-c+1}, \dots, x_{-2}, \delta_a(s_a)) = 1 \wedge h_2(x_{-c+2}, \dots, x_{-2}, \delta_a^2(s_a)) = 0.$$

$$(2.3) \quad f(x_{-c}, \dots, x_0, s_a) = \begin{cases} f_0(x_{-c}, \dots, x_{-1}, s_a) + x_0, & \text{if } h_0(x_{-c}, \dots, x_{-2}, s_a) = 0; \\ f_1(x_{-c}, \dots, x_{-2}, s_a) + x_{-1}, & \text{if } h_0(x_{-c}, \dots, x_{-2}, s_a) = 1 \wedge h_1(x_{-c}, \dots, x_{-3}, s_a) = 0; \\ f_2(x_{-c}, \dots, x_{-3}, s_a) + x_{-2}, & \text{if } h_1(x_{-c}, \dots, x_{-3}, s_a) = 1 \wedge h_2(x_{-c}, \dots, x_{-4}, s_a) = 0; \\ f_3(x_{-c}, \dots, x_{-4}, s_a) + x_{-3}, & \text{if } h_2(x_{-c}, \dots, x_{-4}, s_a) = 1. \end{cases}$$

where (2.4) $h_3(x_{-c}, \dots, x_{-3}, s_a) = f_1(x_{-c}, \dots, x_{-3}, 0, s_a) + f_1(x_{-c}, \dots, x_{-3}, 1, s_a)$, $h_4(x_{-c}, \dots, x_{-4}, s_a) = f_2(x_{-c}, \dots, x_{-4}, 0, s_a) + f_2(x_{-c}, \dots, x_{-4}, 1, s_a)$.

Proof: Denote $T_1(x_{-2}) = \langle x_{-c}, \dots, x_{-2}, s_a \rangle$, $T_1(x_{-2}x_{-1}) = \langle x_{-c+1}, \dots, x_{-1}, \delta_a(s_a) \rangle$, $T_1(x_{-2}x_{-1}x_0) = \langle x_{-c+2}, \dots, x_0, \delta_a^2(s_a) \rangle$, $T_1(x_{-2}x_{-1}x_0x_1) = \langle x_{-c+3}, \dots, x_1, \delta_a^3(s_a) \rangle$, $T_2(x_{-3}) = \langle x_{-c}, \dots, x_{-3}, s_a \rangle$, $T_2(x_{-3}x_{-2}) = \langle x_{-c+1}, \dots, x_{-2}, \delta_a(s_a) \rangle$, $T_2(x_{-3}x_{-2}x_{-1}) = \langle x_{-c+2}, \dots, x_{-1}, \delta_a^2(s_a) \rangle$, $T_2(x_{-3}x_{-2}x_{-1}x_0) = \langle x_{-c+3}, \dots, x_0, \delta_a^3(s_a) \rangle$, $T_3(x_{-4}) = \langle x_{-c}, \dots, x_{-4}, s_a \rangle$, $T_3(x_{-4}x_{-3}) = \langle x_{-c+1}, \dots, x_{-3}, \delta_a(s_a) \rangle$, $T_3(x_{-4}x_{-3}x_{-2}) = \langle x_{-c+2}, \dots, x_{-2}, \delta_a^2(s_a) \rangle$, $T_3(x_{-4}x_{-3}x_{-2}x_{-1}) = \langle x_{-c+3}, \dots, x_{-1}, \delta_a^3(s_a) \rangle$, $s(x_{-2}x_{-1}) = \langle x_{-c}, \dots, x_{-1}, s_a \rangle$, $s(x_{-2}x_{-1}x_0) = \langle x_{-c+1}, \dots, x_0, \delta_a(s_a) \rangle$, $s(x_{-2}x_{-1}x_0x_1) = \langle x_{-c+2}, \dots, x_1, \delta_a^2(s_a) \rangle$, $s(x_{-2}x_{-1}x_0x_1x_2) = \langle x_{-c+3}, \dots, x_2, \delta_a^3(s_a) \rangle$. Define $f_0(x_{-c}, \dots, x_{-1}, s_a) = f(x_{-c}, \dots, x_{-1}, 0, s_a)$, $f_1(x_{-c}, \dots, x_{-2}, s_a) = f(x_{-c}, \dots, x_{-2}, 0, 0, s_a)$, $f_2(x_{-c}, \dots, x_{-3}, s_a) = f(x_{-c}, \dots, x_{-3}, 0, 0, 0, s_a)$, $f_3(x_{-c}, \dots, x_{-4}, s_a) = f(x_{-c}, \dots, x_{-4}, 0, 0, 0, 0, s_a)$. $h_0(T_1(x_{-2})) = 1$ iff $s(x_{-2}^0)$ is not a 0-step state, $h_1(T_2(x_{-3})) = 1$ iff $f(x_{-c}, \dots, x_{-3}, 0, x_{-1}, x_0, s_a)$ doesn't rely on x_{-1} and x_0 , $h_2(T_3(x_{-4})) = 1$ iff $f(x_{-c}, \dots, x_{-4}, 0, x_{-2}, x_{-1}, x_0, s_a)$ doesn't rely on x_{-2}, x_{-1}, x_0 . Clearly, $h_2(T_3(x_{-4})) = 1 \rightarrow h_1(T_2(x_{-3})) = 1$,

$h_1(T_2(x_{-3}))=1 \rightarrow h_0(T_1(x_{-2}))=1$. Since M_a is cyclic, $M=C(M_a, f)$ is strongly connected. Then $|W_{3,s}^M|=2, \forall s \in S$.

Clearly, by Lemmas 3, 10, 11, $h_0(T_1(x_{-2}))=1$ iff $s(x_{-2}^1)$ is not a 0-step state, $h_1(T_2(x_{-3}))=1$ iff $f(x_{-c}, \dots, x_{-3}, 1, x_{-1}, x_0, s_a)$ doesn't rely on x_{-1} and x_0 , $h_2(T_3(x_{-4}))=1$ iff $f(x_{-c}, \dots, x_{-4}, 1, x_{-2}, x_{-1}, x_0, s_a)$ doesn't rely on x_{-2}, x_{-1}, x_0 .

To prove (2.1), assume $h_0(T_1(x_{-2}))=0$, then By Lemma 1, $h_0(T_1(x_{-2}x_{-1}))=1 \wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=1$. Assume $h_0(T_1(x_{-2}))=1 \wedge h_0(T_1(x_{-2}x_{-1}))=1 \wedge h_1(T_2(x_{-3}x_{-2}))=0$, then $|\lambda(s(x_{-2}x_{-1}), X)|=|\lambda(s(x_{-2}x_{-1}x_0), X)|=1, \lambda(s(x_{-2}x_{-1}^0), 0) \neq \lambda(s(x_{-2}x_{-1}^1), 0)$. Thus $s(x_{-2}x_{-1})$ is a 1-step state. By Lemma 2, $h_1(T_2(x_{-3}x_{-2}x_{-1}))=1$. Assume $h_0(T_1(x_{-2}))=1 \wedge h_1(T_2(x_{-3}x_{-2}))=1$, then $|\lambda(s(x_{-2}x_{-1}), X)|=1, \lambda(s(x_{-2}x_{-1}x_0), x_1)=\lambda(s(x_{-2}x_{-1}^0), 0), \forall x_0, x_1 \in X$. Since $|W_{3,s(x_{-2}, x_{-1})}^M|=2, \cup_{x_0, x_1 \in X} \lambda(s(x_{-2}x_{-1}), X)=Y$, thus $h_2(T_3(x_{-4}x_{-3}x_{-2}))=0$. Assume $h_0(T_1(x_{-2}))=0 \wedge h_1(T_2(x_{-3}x_{-2}))=1$, then $s(x_{-2}^0), s(x_{-2}^1)$ are 0-step states, $\lambda(s(x_{-2}x_{-1}x_0), x_1)=\lambda(s(x_{-2}x_{-1}^0), 0), \forall x_{-1}, x_0, x_1 \in X$. By Lemma 6, $h_2(T_3(x_{-4}x_{-3}))=1$. Assume $h_0(T_1(0))=0 \wedge h_0(T_1(1))=0 \wedge h_1(T_2(x_{-3}x_{-2}))=0, s(x_{-2}x_{-1}), \forall x_{-2}, x_{-1} \in X$ are 0-step states. By Lemma 8, $h_1(T_2(x_{-3}x_{-2}'))=0 (x_{-2} \neq x_{-2}')$.

To prove (2.2), assume $h_0(T_1(x_{-2}))=0 \wedge h_1(T_2(x_{-3}x_{-2}))=0$, then $s(x_{-2}^0), s(x_{-2}^1)$ are 0-step states. By Lemma 1, $|\lambda(s(x_{-2}x_{-1}x_0), X)|=1, \forall x_{-1}, x_0 \in X$. Since $h_1(T_2(x_{-3}x_{-2}))=0, \lambda(s(x_{-2}x_{-1}^0), 0) \neq \lambda(s(x_{-2}x_{-1}^1), 0), \forall x_{-1} \in X$. By Lemma 6, $\lambda(s(x_{-2}^0), 0) = \lambda(s(x_{-2}^1), 0)$ iff $\lambda(s(x_{-2}^{00}), 0) = \lambda(s(x_{-2}^{10}), 0)$. Then $\lambda(s(x_{-2}^0), 0) + \lambda(s(x_{-2}^1), 0) = \lambda(s(x_{-2}^{00}), 0) + \lambda(s(x_{-2}^{10}), 0)$. Thus $f_0(x_{-c}, \dots, x_{-2}, 0, s_a) + f_0(x_{-c}, \dots, x_{-2}, 1, s_a) = h_3(T_2(x_{-3}x_{-2}))$. Assume $h_0(T_1(0))=1 \wedge h_0(T_1(1))=1 \wedge h_1(T_2(x_{-3}))=0 \wedge h_2(T_3(x_{-4}x_{-3}))=0$, then $|\lambda(s(x_{-2}x_{-1}), X)|=1, \forall x_{-2}, x_{-1} \in X$. Since $h_1(T_2(x_{-3}))=0$, by Lemma 10, $\lambda(s(x_{-2}^0), 0) \neq \lambda(s(x_{-2}^1), 0), \forall x_{-2} \in X$. By Lemma 3, $\langle x_{-c-1}, \dots, x_{-3}, i, \delta_a^{-1}(s_a) \rangle (i=0, 1)$ are 0-step states, or neither is, where $s(x_{-2}x_{-1})$ is the successor state of $\langle x_{-c-1}, \dots, x_{-3}, i, \delta_a^{-1}(s_a) \rangle, \forall x_{-2}, x_{-1} \in X$. In the first case, since $h_2(T_3(x_{-4}x_{-3}))=0$, by Lemma 6, $\lambda(s(x_{-2}x_{-1}x_0), x_1) = \lambda(s(x_{-2}x_{-1}^0), 0), \forall x_{-1} \in X (k=-2, -1, 0, 1), \lambda(s(x_{-2}^0), 0) \neq \lambda(s(x_{-2}^1), 0)$, and $\lambda(s(00), 0) = \lambda(s(10), 0)$ iff $\lambda(s(000), 0) \neq \lambda(s(100), 0)$. Then $\lambda(s(00), 0) + \lambda(s(10), 0) = \lambda(s(000), 0) + \lambda(s(100), 0) + 1$. Thus $f(x_{-c}, \dots, x_{-3}, 0, 0, 0, s_a) + f(x_{-c}, \dots, x_{-3}, 1, 0, 0, s_a) = f(x_{-c+1}, \dots, x_{-3}, 0, 0, 0, \delta_a(s_a)) + f(x_{-c+1}, \dots, x_{-3}, 1, 0, 0, \delta_a(s_a)) + 1$, i.e., $h_3(T_2(x_{-3})) = h_4(T_2(x_{-3}x_{-2})) + 1$, where $h_4(T_3(x_{-4})) = f_2(T_2(0)) + f_2(T_2(1))$. In the second case, by Lemma 2, $\langle x_{-c-1}, \dots, x_{-3}, i, \delta_a^{-1}(s_a) \rangle (i=0, 1)$ are 1-step states. By the same arguments and using Lemma 7, $h_3(T_2(x_{-3})) = h_4(T_2(x_{-3}x_{-2})) + 1$. Assume $h_0(T_1(0))=0 \wedge h_0(T_1(1))=0 \wedge h_1(T_2(x_{-3}x_{-2}))=0$, using (2.1), $h_1(T_2(x_{-3}x_{-2}'))=0 (x_{-2}' \neq x_{-2})$. Since $h_0(T_1(x_{-2}))=0, \forall x_{-2} \in X, s(x_{-2}^0)$ and $s(x_{-2}^1)$ are 0-step states. By Lemma 1, $|\lambda(s(x_{-2}x_{-1}x_0), X)|=1, \forall x_{-2}, x_{-1}, x_0 \in X$. Since $h_1(T_2(x_{-3}x_{-2}))=0$, by Lemma 6, $\lambda(s(x_{-2}x_{-1}^0), 0) \neq \lambda(s(x_{-2}x_{-1}^1), 0)$. By Lemma 8, $\lambda(s(00), 0) = \lambda(s(10), 0)$ iff $\lambda(s(000), 0) \neq \lambda(s(100), 0)$. Then $\lambda(s(00), 0) + \lambda(s(10), 0) = \lambda(s(000), 0) + \lambda(s(100), 0) + 1$. Thus $f_0(x_{-c}, \dots, x_{-3}, 0, 0, s_a) + f_0(x_{-c}, \dots, x_{-3}, 1, 0, s_a) = f_1(T_1(00)) + f_1(T_1(10)) + 1$. Assume $h_0(T_1(x_{-2}))=0 \wedge h_0(T_1(x_{-2}'))=1 \wedge h_1(T_2(x_{-3}x_{-2}))=0 (x_{-2}' \neq x_{-2})$, then $s(x_{-2}^0)$ and $s(x_{-2}^1)$ are 0-step states. By Lemma 9, $\langle x_{-c-1}, \dots, x_{-2}, \delta_a^{-1}(s_a) \rangle$ is a 3-step state, $\langle x_{-c-1}, \dots, x_{-2}', \delta_a^{-1}(s_a) \rangle$ is a 3-step state or a 1-step state ($x_{-2}' \neq x_{-2}$), where $s(x_{-2}x_{-1})$ is the successor state of $\langle x_{-c-1}, \dots, x_{-3}, i, \delta_a^{-1}(s_a) \rangle, \forall x_{-2}, x_{-1} \in X$. Since $h_0(T_1(x_{-2}'))=1, s(x_{-2}^0), s(x_{-2}^1)$ are not 0-step states. By Lemma 8, $\langle x_{-c-1}, \dots, x_{-2}', \delta_a^{-1}(s_a) \rangle$ is a 1-step state. By Lemma 2, $\lambda(s(x_{-2}'x_{-1}x_0), x_1) = \lambda(s(x_{-2}'x_{-1}^0), 0), |\lambda(s(x_{-2}'x_{-1}), X)|=1, \forall x_{-1}, x_0, x_1 \in X$. Since $h_1(T_2(x_{-3}x_{-2}))=0$, by Lemma 6, $\lambda(s(x_{-2}x_{-1}^0), 0) \neq \lambda(s(x_{-2}x_{-1}^1), 0), \forall x_{-1} \in X$. By Lemma 11, $\lambda(s(x_{-2}^0), 0) \neq \lambda(s(x_{-2}^1), 0)$. By Lemma 7, $\lambda(s(x_{-2}^0), 0) = \lambda(s(x_{-2}^1), 0)$ iff $\lambda(s(x_{-2}^{00}), 0) \neq \lambda(s(x_{-2}^{10}), 0)$. Then $\lambda(s(x_{-2}^0), 0) + \lambda(s(x_{-2}^1), 0) = \lambda(s(x_{-2}00), 0) + \lambda(s(x_{-2}^{10}), 0) + 1$. Thus $f_0(x_{-c}, \dots, x_{-2}, 0, s_a) + f_1(T_1(x_{-2}')) = f_1(T_1(x_{-2}^0)) + f_2(T_2(x_{-3}x_{-2}')) + 1$. Assume $h_0(T_1(x_{-2}))=0 \wedge h_1(T_2(x_{-3}x_{-2}))=1 \wedge h_2(T_3(x_{-4}x_{-3}x_{-2}))=0$, then $s(x_{-2}^0)$ and $s(x_{-2}^1)$ are 0-step states. By Lemma 1, $\lambda(s(x_{-2}x_{-1}x_0x_1), x_2) = \lambda(s(x_{-2}x_{-1}x_0), 0), |\lambda(s(x_{-2}x_{-1}x_0), X)|=1, \forall x_{-1}, x_0, x_1, x_2 \in X$. Since $h_1(T_2(x_{-3}x_{-2}))=1$, by Lemma 6, $\lambda(s(x_{-2}x_{-1}x_0), x_1) = \lambda(s(x_{-2}^0), 0), \forall x_{-1}, x_0, x_1 \in X$. Since $h_2(T_3(x_{-4}x_{-3}x_{-2}))=0$, by Lemma 11, $\lambda(s(x_{-2}x_{-1}^0), 0) \neq \lambda(s(x_{-2}x_{-1}^1), 0), \forall x_{-1} \in X$. Then by Lemma 6, $\lambda(s(x_{-2}^0), 0) = \lambda(s(x_{-2}^1), 0)$ iff $\lambda(s(x_{-2}^{000}), 0) \neq \lambda(s(x_{-2}^{100}), 0)$. Thus $f_0(x_{-c}, \dots, x_{-2}, 0, s_a) + f_0(x_{-c}, \dots, x_{-2}, 1, s_a) = h_4(T_3(x_{-4}x_{-3}x_{-2})) + 1$.

To prove (2.3), assume $h_0(T_1(x_{-2}))=0$, then $s(x_{-2}^0)$ and $s(x_{-2}^1)$ are 0-step states. Clearly, $\lambda(s(x_{-2}x_{-1}),x_0)=\lambda(s(x_{-2}x_{-1}),0)+x_0, \forall x_{-1},x_0 \in X$. Then $f(x_{-c},\dots,x_0,s_a)=f_0(x_{-c},\dots,x_{-1},s_a)+x_0$. Assume $h_0(T_1(x_{-2}))=1 \wedge h_1(T_2(x_{-3}))=0$, then $\lambda(s(x_{-2}x_{-1}),x_0)=\lambda(s(x_{-2}x_{-1}),0), \forall x_{-1} \in X$. Let $e=\lambda(s(x_{-2}x_{-1}),0), x'_{-1} \neq x_{-1}$. Since $h_1(T_2(x_{-3}))=0, \lambda(s(x_{-2}x_{-1}),0) \neq \lambda(s(x_{-2}x'_{-1}),0)$. Thus $\lambda(s(x_{-2}x'_{-1}),0)=e+1$. Then $f(x_{-c},\dots,x_0,s_a)=\lambda(s(x_{-2}x_{-1}),0)=\lambda(s(x_{-2}^0),0)+x_{-1}=f_1(T_1(x_{-2}))+x_{-1}$. Assume $h_1(T_2(x_{-3}))=1 \wedge h_2(T_3(x_{-4}))=0$, then $\lambda(s(x_{-2}x_{-1}),x_0)=\lambda(s(x_{-2}^0),0), \forall x_{-1},x_0 \in X$. Let $e=\lambda(s(x_{-2}^0),0), x'_{-2} \neq x_{-2}$. Since $h_2(T_3(x_{-4}))=0, \lambda(s(x_{-2}^0),0) \neq \lambda(s(x_{-2}^0),0)$. Thus $\lambda(s(x_{-2}^0),0)=e+1$. Then $f(x_{-c},\dots,x_0,s_a)=\lambda(s(x_{-2}^0),0)=\lambda(s(00),0)+x_{-2}=f_2(T_2(x_{-3}))+x_{-2}$. Assume $h_2(T_3(x_{-4}))=1$, then $\lambda(s(x_{-2}x_{-1}),x_0)=\lambda(s(00),0), \forall x_{-2},x_{-1},x_0 \in X$. Let $e=\lambda(s(00),0), x'_{-3} \neq x_{-3}$. By Lemma 11, $\lambda(\langle x_{-c},\dots,x_{-3},0,0,s_a \rangle,0) \neq \lambda(\langle x_{-c},\dots,x'_{-3},0,0,s_a \rangle,0)$. Thus $\lambda(\langle x_{-c},\dots,x'_{-3},0,0,s_a \rangle,0)=e+1$. Then $f(x_{-c},\dots,x_0,s_a)=\lambda(s(x_{-2}x_{-1}),x_0)=f_3(T_3(x_{-4}))+x_{-3}$.

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Lemma 13. Let M be a c -order SIM FA $C(M_a, f)$, $M_a=\langle S_a, Y_a, \delta_a, \lambda_a \rangle$ be cyclic, $X=Y=\{0,1\}$. If $c \geq 3, w_{3,M}=1, M$ is weakly invertible with delay 3, then there exists a mapping f_3 from $X^{c-3} \times S_a$ to Y such that

$$f(x_{-c},\dots,x_0,s_a)=f_3(x_{-c},\dots,x_{-4},s_a)+x_{-3}.$$

Proof: Since M_a is cyclic, $C(M_a, f)$ is strongly connected. Then $|W_{3,s}^M|=w_{3,M}=1, \forall s \in S$. Thus $\lambda(s(x_{-2}x_{-1}),x_0)=\lambda(s(00),0), \forall x_{-2},x_{-1},x_0 \in X$. Since M is weakly invertible with delay 3, $\lambda(\langle x_{-c},\dots,x_{-4},0,0,0,s_a \rangle,0) \neq \lambda(\langle x_{-c},\dots,x_{-4},1,0,0,s_a \rangle,0)$. Thus $f(x_{-c},\dots,x_0,s_a)=\lambda(\langle x_{-c},\dots,x_{-4},0,0,0,s_a \rangle,0)+x_{-3}=f_3(x_{-c},\dots,x_{-4},s_a)+x_{-3}$, where $f_3(x_{-c},\dots,x_{-4},s_a)=f(x_{-c},\dots,x_{-4},0,0,0,0,s_a)$.

Lemma 14. Let M be a c -order SIM FA $C(M_a, f)$, $M_a=\langle S_a, Y_a, \delta_a, \lambda_a \rangle$ be cyclic, $X=Y=\{0,1\}$. If $c \geq 3, w_{3,M}=8, M$ is weakly invertible with delay 3, then there exists a mapping f_0 from $X^c \times S_a$ to Y such that

$$f(x_{-c},\dots,x_0,s_a)=f_0(x_{-c},\dots,x_{-1},s_a)+x_0.$$

Proof: Since M_a is cyclic, $C(M_a, f)$ is strongly connected. Then $|W_{3,s}^M|=8, \forall s \in S$. Thus $\lambda(s(x_{-2}x_{-1}),0) \neq \lambda(s(x_{-2}x_{-1}),1)$. Then $f(x_{-c},\dots,x_0,s_a)=\lambda(s(x_{-2}x_{-1}),0)+x_0=f_0(x_{-c},\dots,x_{-1},s_a)+x_0$, where $f_0(x_{-c},\dots,x_{-1},s_a)=f(x_{-c},\dots,x_{-1},0,s_a)$.

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Theorem 1. Let M be a c -order SIM FA $C(M_a, f)$, $M_a=\langle S_a, Y_a, \delta_a, \lambda_a \rangle$ be cyclic, $X=Y=\{0,1\}, c \geq 3$. Then M is weakly invertible with delay 3, if one of the following conditions holds:

(a) There exists a mapping f_0 from $X^c \times S_a$ to Y such that

$$f(x_{-c},\dots,x_0,s_a)=f_0(x_{-c},\dots,x_{-1},s_a)+x_0;$$

(b) There exists a mapping f_3 from $X^{c-3} \times S_a$ to Y such that

$$f(x_{-c},\dots,x_0,s_a)=f_3(x_{-c},\dots,x_{-4},s_a)+x_{-3};$$

(c) There exist mappings h_0 from $X^{c-1} \times S_a$ to $\{0,1\}, h_1$ from $X^{c-2} \times S_a$ to $\{0,1\}, h_2$ from $X^{c-3} \times S_a$ to $\{0,1\}, f_0$ from $X^c \times S_a$ to Y, f_1 from $X^{c-1} \times S_a$ to Y, f_2 from $X^{c-2} \times S_a$ to Y, f_3 from $X^{c-3} \times S_a$ to Y , such that (2.1), (2.2) and (2.3) hold, where h_3 and h_4 are defined by (2.4).

Proof: Assume that one of conditions (a), (b) and (c) holds. In case of (a), M is weakly invertible with delay 0; In case of (b), it is easy to verify that M is weakly invertible with delay 3. Below we discuss the case of (c); Assume that (c) holds, let $s=\langle x_{-c},\dots,x_{-1},s_a \rangle, s_i=\langle x_{-c+1},\dots,x_{-1},i,\delta_a(s_a) \rangle, s_{i,j}=\langle x_{-c+2},\dots,x_{-1},i,j,\delta_a^2(s_a) \rangle, s_{i,j,k}=\langle x_{-c+3},\dots,x_{-1},i,j,k,\delta_a^3(s_a) \rangle (i,j,k=0,1)$, other notations used below are referred to the proof of Lemma 12. Since M_a is cyclic, $\lambda(s,x_0)=f(x_{-c},\dots,x_0,\lambda_a(s_a))=f(x_{-c},\dots,x_0,s_a)$. By (2.1), $h_2(T_3(x_{-4}))=1 \rightarrow h_1(T_2(x_{-3}))=1, h_1(T_2(x_{-3}))=1 \rightarrow h_0(T_1(x_{-2}))=1$. Thus, to prove s is a t -step state with $0 \leq t \leq 3$, there are two main cases to consider.

Case 1. $h_0(T_1(x_{-2}))=0$. By (2.3), $\lambda(s,x_0)=f_0(x_{-c},\dots,x_{-1},s_a)+x_0$. Thus $\lambda(s,0) \neq \lambda(s,1)$. Hence s is a 0-step state;

Case 2. $h_0(T_1(x_{-2}))=1$. By (2.3), $\lambda(s,0)=\lambda(s,1)$. Next we further consider $h_0(T_1(x_{-2}x_{-1}))$.

Subcase 2-1. $h_0(T_1(x_{-2}x_{-1}))=0$. By (2.3), $\lambda(s_{i,0})\neq\lambda(s_{i,1})$ ($i=0,1$). By (2.1), $h_0(T_1(x_{-2}x_{-1}x_0))=1$, $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$, $\forall x_0\in X$. Then by (2.3), $\lambda(s_{i,j,0})=\lambda(s_{i,j,1})$, $\lambda(s_{i,j,k,0})=\lambda(s_{i,j,k,1})$, $\lambda(s_{i,j,0,0})=\lambda(s_{i,j,1,0})$ ($i,j,k=0,1$). Since $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1\rightarrow h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$, we further consider $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))$.

Subcase 2-1-1. $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1$. By (2.3), $\lambda(s_{i,j,k,x})=\lambda(s_{i,0,0,0})$, $\lambda(s_{0,0,0,0})\neq\lambda(s_{1,0,0,0})$ ($i,j,k=0,1$), $\forall x\in X$. Hence s is a t -step state ($2\leq t\leq 3$).

Subcase 2-1-2. $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1\wedge h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$, by (2.3), $\lambda(s_{i,0,0,0})\neq\lambda(s_{i,1,0,0})$ ($i=0,1$). Since $h_1(T_2(x_{-3}x_{-2}x_{-1}))=1\rightarrow h_0(T_1(x_{-2}x_{-1}x_0))=1$, we further consider $h_1(T_2(x_{-3}x_{-2}x_{-1}))$.

Subcase 2-1-2-1. $h_1(T_2(x_{-3}x_{-2}x_{-1}))=1$. Since $h_0(T_1(x_{-2}x_{-1}))=0\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=1$, by (2.1), $h_2(T_3(x_{-4}x_{-3}x_{-2}))=1$, $\lambda(s_{0,0,0,0})=\lambda(s_{0,1,0,0})=\lambda(s_{1,0,0,0})=\lambda(s_{1,1,0,0})$. It suffices to show that whether $\lambda(s_{0,0,0,0})=\lambda(s_{1,0,0,0})$ holds, if $\lambda(s_0,0)=\lambda(s_1,0)$. Since $h_0(T_1(x_{-2}x_{-1}))=0\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=1\wedge h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$, by (2.2), $f_0(x_{-c+1},\dots,x_{-1},0,\delta_a(s_a))+f_0(x_{-c+1},\dots,x_{-1},1,\delta_a(s_a))=h_4(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1$. By (2.4), $f_0(x_{-c+1},\dots,x_{-1},0,\delta_a(s_a))+f_0(x_{-c+1},\dots,x_{-1},1,\delta_a(s_a))=f_2(T_2(x_{-3}x_{-2}x_{-1}^0))+f_2(T_2(x_{-3}x_{-2}x_{-1}^1))+1$. On the other hand, since $h_0(T_1(x_{-2}x_{-1}))=0$, by (2.3), $f_0(x_{-c+1},\dots,x_{-1},x_0,\delta_a(s_a))=f(x_{-c+1},\dots,x_{-1},x_0,0,\delta_a(s_a))$, $\forall x_0\in X$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1\wedge h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$, $\forall x_0\in X$. By (2.3), $f_2(T_2(x_{-3}x_{-2}x_{-1}x_0))=f(x_{-c+3},\dots,x_0,0,0,0,\delta_a^3(s_a))$, $\forall x_0\in X$. Then $\lambda(s_0,0)+\lambda(s_1,0)=\lambda(s_{0,0,0,0})+\lambda(s_{1,0,0,0})+1$. Thus $\lambda(s_0,0)=\lambda(s_1,0)$ iff $\lambda(s_{0,0,0,0})\neq\lambda(s_{1,0,0,0})$. Hence s is a 3-step state.

Subcase 2-1-2-2. $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0\wedge h_0(T_1(x_{-2}x_{-1}x_0))=1$, $\forall x_0\in X$, by (2.3), $\lambda(s_{i,0,0})\neq\lambda(s_{i,1,0})$ ($i=0,1$). It suffices to show whether $\lambda(s_{0,0,0})=\lambda(s_{1,0,0})$ and $\lambda(s_{0,0,0,0})=\lambda(s_{1,0,0,0})$ hold, respectively, if $\lambda(s_0,0)=\lambda(s_1,0)$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0\wedge h_0(T_1(x_{-2}x_{-1}))=0$, by (2.2), $f_0(x_{-c+1},\dots,x_{-1},0,\delta_a(s_a))+f_0(x_{-c+1},\dots,x_{-1},1,\delta_a(s_a))=h_3(T_2(x_{-3}x_{-2}x_{-1}))$. By (2.3), $f_0(x_{-c+1},\dots,x_{-1},x_0,\delta_a(s_a))=f(x_{-c+1},\dots,x_{-1},x_0,0,\delta_a(s_a))$, $\forall x_0\in X$. Using (2.4), $h_3(T_2(x_{-3}x_{-2}x_{-1}))=f_1(T_1(x_{-2}x_{-1}^0))+f_1(T_1(x_{-2}x_{-1}^1))$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0\wedge h_0(T_1(x_{-2}x_{-1}x_0))=1$, by (2.3), $f_1(T_1(x_{-2}x_{-1}x_0))=f(x_{-c+2},\dots,x_{-1},x_0,0,0,\delta_a^2(s_a))$. Thus $\lambda(s_0,0)+\lambda(s_1,0)=\lambda(s_{0,0,0})+\lambda(s_{1,0,0})$. Hence $\lambda(s_{0,0,0})=\lambda(s_{1,0,0})$ if $\lambda(s_0,0)=\lambda(s_1,0)$. Since $h_0(T_1(x_{-2}x_{-1}^0))=1\wedge h_0(T_1(x_{-2}x_{-1}^1))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=0\wedge h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$, by (2.2), $h_3(T_2(x_{-3}x_{-2}x_{-1}))=h_4(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1$. Using (2.4), $h_3(T_2(x_{-3}x_{-2}x_{-1}))=f_1(T_1(x_{-2}x_{-1}^0))+f_1(T_1(x_{-2}x_{-1}^1))$, $h_4(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=f_2(T_2(x_{-3}x_{-2}x_{-1}^0))+f_2(T_2(x_{-3}x_{-2}x_{-1}^1))$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0\wedge h_0(T_1(x_{-2}x_{-1}x_0))=1$, by (2.3), $f_1(T_1(x_{-2}x_{-1}x_0))=f(x_{-c+2},\dots,x_0,0,0,\delta_a^2(s_a))$, $\forall x_0\in X$. On the other hand, since $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1\wedge h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$, by (2.3), $f_2(T_2(x_{-3}x_{-2}x_{-1}x_0))=f(x_{-c+3},\dots,x_0,0,0,0,\delta_a^2(s_a))$. Then $\lambda(s_{0,0,0})+\lambda(s_{1,0,0})=\lambda(s_{0,0,0,0})+\lambda(s_{1,0,0,0})+1$. Thus $\lambda(s_{0,0,0})=\lambda(s_{1,0,0})$ iff $\lambda(s_{0,0,0,0})\neq\lambda(s_{1,0,0,0})$. Therefore, if $\lambda(s_0,0)=\lambda(s_1,0)$ then $\lambda(s_{0,0,0})=\lambda(s_{1,0,0})$ and $\lambda(s_{0,0,0,0})\neq\lambda(s_{1,0,0,0})$. Hence s is a 3-step state.

Subcase 2-2. $h_0(T_1(x_{-2}x_{-1}))=1$. Since $h_1(T_2(x_{-3}x_{-2}))=1\rightarrow h_0(T_1(x_{-2}x_{-1}))=1$, we consider $h_1(T_2(x_{-3}x_{-2}))$.

Subcase 2-2-1. $h_1(T_2(x_{-3}x_{-2}))=0$. Since $h_0(T_1(x_{-2}x_{-1}))=1\wedge h_1(T_2(x_{-3}x_{-2}))=0$, by (2.3), $\lambda(s_i,0)=\lambda(s_i,1)$, $\lambda(s_0,0)\neq\lambda(s_1,0)$ ($i=0,1$). Therefore s is a 1-step state.

Subcase 2-2-2. $h_1(T_2(x_{-3}x_{-2}))=1$. By (2.3), $\lambda(s_0,0)=\lambda(s_0,1)=\lambda(s_1,0)=\lambda(s_1,1)$. Since $h_0(T_1(x_{-2}))=1\wedge h_1(T_2(x_{-3}x_{-2}))=1$, by (2.1), $h_2(T_3(x_{-4}x_{-3}x_{-2}))=0$. Since $h_2(T_3(x_{-4}x_{-3}x_{-2}))=1\rightarrow h_1(T_2(x_{-3}x_{-2}x_{-1}))=1$, $h_1(T_2(x_{-3}x_{-2}x_{-1}))=1\rightarrow h_0(T_1(x_{-2}x_{-1}x_0))=1$, $\forall x_{-1},x_0\in X$, we further consider the following cases.

Subcase 2-2-2-1. $h_1(T_2(x_{-3}x_{-2}x_{-1}))=1\wedge h_2(T_3(x_{-4}x_{-3}x_{-2}))=0$. By (2.3), $\lambda(s_{i,j,0})=\lambda(s_{i,j,1})$, $\lambda(s_{i,0,0})=\lambda(s_{i,1,0})$, $\lambda(s_{0,0,0})\neq\lambda(s_{1,0,0})$ ($i,j=0,1$). Therefore s is a 2-step state.

Subcase 2-2-2-2. $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0$. We consider three cases.

Subcase 2-2-2-2-1. $h_0(T_1(x_{-2}x_{-1}x_0))=0\wedge h_0(T_1(x_{-2}x_{-1}^1))=0$. By (2.1), $h_0(T_1(x_{-2}x_{-1}x_0x_1))=1$, $\forall x_0,x_1\in X$. There are two cases to consider.

Subcase 2-2-2-2-1-1. $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$. Since $h_0(T_1(x_{-2}x_{-1}x_0))=0\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$, $\forall x_0\in X$. By (2.1),

$h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1$, by (2.3), $\lambda(s_{i,j,k},0)=\lambda(s_{i,j,k},1)=\lambda(s_{i,0,0},0)$, $\lambda(s_{0,0,0},0)\neq\lambda(s_{1,0,0},0)$ ($i,j,k=0,1$). Thus s is a 3-step state.

Subcase 2-2-2-2-1-2. $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=0$. Since $h_0(T_1(x_{-2}x_{-1}^0))=0\wedge h_0(T_1(x_{-2}x_{-1}^1))=0\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=0$, by (2.1), $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0'))=0$ ($x_0\neq x_0'$). Since $h_0(T_1(x_{-2}x_{-1}x_0x_1))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=0$, $\forall x_0,x_1\in X$, by (2.3), $\lambda(s_{i,j,k},0)=\lambda(s_{i,j,k},1)$, $\lambda(s_{i,j,0},0)\neq\lambda(s_{i,j,1},0)$ ($i,j,k=0,1$). By subcase 2-1-2-2, it suffices to show whether $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$ hold, if $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$. Since $h_0(T_1(x_{-2}x_{-1}^0))=0\wedge h_0(T_1(x_{-2}x_{-1}^1))=0\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=0$, by (2.2), $f_0(x_{-c+2},\dots,x_{-1},0,0)$, $\delta_a^2(s_a)+f_0(x_{-c+2},\dots,x_{-1},1,0)$, $\delta_a^2(s_a)=f_1(T_1(x_{-2}x_{-1}^0)))+f_1(T_1(x_{-2}x_{-1}^1))+1$. Since $h_0(T_1(x_{-2}x_{-1}x_0))=0$, by (2.3), $f_0(x_{-c+2},\dots,x_{-1},x_0,0)$, $\delta_a^2(s_a)=f(x_{-c+2},\dots,x_0,0,0)$, $\delta_a^2(s_a)$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=0\wedge h_0(T_1(x_{-2}x_{-1}x_0x_1))=1$, $\forall x_0,x_1\in X$, by (2.3), $f_1(T_1(x_{-2}x_{-1}x_0))=f(x_{-c+3},\dots,x_0,0,0,0)$, $\delta_a^3(s_a)$. Thus $\lambda(s_{0,0,0},0)+\lambda(s_{1,0,0},0)=\lambda(s_{0,0,0},0)+\lambda(s_{1,0,0},0)+1$. Hence if $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$, then $\lambda(s_{0,0,0},0)\neq\lambda(s_{1,0,0},0)$. Therefore s is a 3-step state.

Subcase 2-2-2-2-2. $h_0(T_1(x_{-2}x_{-1}^0))=1\wedge h_0(T_1(x_{-2}x_{-1}^1))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=0$. Since $h_0(T_1(x_{-2}x_{-1}x_0))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=0$, $\forall x_0\in X$, by (2.3), $\lambda(s_{i,j},0)=\lambda(s_{i,j},1)$, $\lambda(s_{i,0},0)\neq\lambda(s_{i,1},0)$ ($i,j=0,1$). Since $h_0(T_1(x_{-2}x_{-1}))=1\wedge h_0(T_1(x_{-2}x_{-1}x_0))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=0$, by (2.1), $h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$, $\forall x_0\in X$. Since $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1\rightarrow h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$, we consider the following two cases.

Subcase 2-2-2-2-2-1. $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1$. By (2.3), it is easy to see that s is a 3-step state.

Subcase 2-2-2-2-2-2. $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$. Since $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$, by (2.3), $\lambda(s_{i,j,k},0)=\lambda(s_{i,j,k},1)=\lambda(s_{i,j,0},0)$, $\lambda(s_{i,0,0},0)\neq\lambda(s_{i,1,0},0)$ ($i,j,k=0,1$). It suffices to show whether $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$ holds, if $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$. Since $h_0(T_1(x_{-2}x_{-1}x_0))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=0\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}x_0))=1$, $\forall x_0\in X$, by (2.2), $h_3(T_2(x_{-3}x_{-2}x_{-1}))=h_4(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))$. Using (2.4), by the same arguments as above, $\lambda(s_{0,0,0},0)+\lambda(s_{1,0,0},0)=\lambda(s_{0,0,0},0)+\lambda(s_{1,0,0},0)+1$. Thus if $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$, then $\lambda(s_{0,0,0},0)\neq\lambda(s_{1,0,0},0)$. Therefore s is a 3-step state.

Subcase 2-2-2-2-3. $h_0(T_1(x_{-2}x_{-1}x_0))=0\wedge h_0(T_1(x_{-2}x_{-1}x_0'))=1$ ($x_0\neq x_0'$). Without loss of generality, let $h_0(T_1(x_{-2}x_{-1}^0))=0$, $h_0(T_1(x_{-2}x_{-1}^1))=1$. By (2.3), $\lambda(s_{0,j},0)\neq\lambda(s_{0,j},1)$ ($j=0,1$). By (2.1), $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0$, $h_0(T_1(x_{-2}x_{-1}^0x_1))=1$, $\forall x_1\in X$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}))=0\wedge h_0(T_1(x_{-2}x_{-1}^1))=1$, by (2.3), $\lambda(s_{1,j},0)=\lambda(s_{1,j},1)$, $\lambda(s_{1,0},0)\neq\lambda(s_{1,1},0)$ ($j=0,1$). Since $h_0(T_1(x_{-2}x_{-1}))=1\wedge h_0(T_1(x_{-2}x_{-1}^1))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}))=0$, by (2.1), $h_1(T_2(x_{-3}x_{-2}x_{-1}^1))=1$. Since $h_0(T_1(x_{-2}x_{-1}^0x_1))=1$, $\forall x_1\in X$, we further consider $h_1(T_2(x_{-3}x_{-2}x_{-1}^0))$.

Subcase 2-2-2-2-3-1. $h_1(T_2(x_{-3}x_{-2}x_{-1}^0))=1$. By (2.1), $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=1$, by (2.3), $\lambda(s_{i,j,k},0)=\lambda(s_{i,j,k},1)=\lambda(s_{i,0,0},0)$, $\lambda(s_{0,0,0},0)\neq\lambda(s_{1,0,0},0)$ ($i,j,k=0,1$). Therefore s is a 3-step state.

Subcase 2-2-2-2-3-2. $h_1(T_2(x_{-3}x_{-2}x_{-1}^0))=0$. Since $h_0(T_1(x_{-2}x_{-1}^0x_1))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}^0))=0$, $\forall x_1\in X$, by (2.3), $\lambda(s_{0,j,k},0)=\lambda(s_{0,j,k},1)$, $\lambda(s_{0,j,0},0)\neq\lambda(s_{0,j,1},0)$ ($j,k=0,1$). Since $h_1(T_2(x_{-3}x_{-2}x_{-1}^0))=0$, by (2.1), $h_2(T_3(x_{-4}x_{-3}x_{-2}x_{-1}))=0$. Since $h_1(T_2(x_{-3}x_{-2}x_{-1}^1))=1$, by (2.3), $\lambda(s_{1,j,k},0)=\lambda(s_{1,j,k},1)=\lambda(s_{1,j,0},0)$, $\lambda(s_{1,0,0},0)\neq\lambda(s_{1,1,0},0)$ ($j,k=0,1$). Using subcase 2-1-2-2, $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$ if $\lambda(s_{0,0},0)=\lambda(s_{0,1},0)$. It suffices to show whether $\lambda(s_{0,0,0},0)=\lambda(s_{1,0,0},0)$ holds, if $\lambda(s_{0,0},0)=\lambda(s_{1,0},0)$. Since $h_0(T_1(x_{-2}x_{-1}^0))=0\wedge h_0(T_1(x_{-2}x_{-1}^1))=1\wedge h_1(T_2(x_{-3}x_{-2}x_{-1}^0))=0$, by (2.2), $f_0(x_{-c+2},\dots,x_{-1},0,0)$, $\delta_a^2(s_a)+f_1(x_{-c+2},\dots,x_{-1},1)$, $\delta_a^2(s_a)=f_1(x_{-c+3},\dots,x_{-1},0,0)$, $\delta_a^3(s_a)+f_2(x_{-c+3},\dots,x_{-1},1)$, $\delta_a^3(s_a)+1$. By the same arguments as above, we can conclude that s is a 3-step state.

To sum up, if condition (c) holds, then any state s of M is a t -step state ($0\leq t\leq 3$). Therefore M is weakly invertible with delay 3.

5 Binary Feedforward Inverse Finite Automata

Theorem 2. Let M be a c -order SIMFA $C(M_a, f)$, $M_a=\langle S_a, Y_a, \delta_a, \lambda_a \rangle$ be cyclic, $X=Y=\{0,1\}$, $c\geq 3$. Then if one of the following conditions holds, M is a feedforward inverse with delay 3.

- (a) There exists a mapping f_0 from $X^c \times S_a$ to Y such that $f(x_{-c}, \dots, x_0, s_a) = f_0(x_{-c}, \dots, x_{-1}, s_a) + x_0$;
- (b) There exists a mapping f_3 from $X^{c-3} \times S_a$ to Y such that $f(x_{-c}, \dots, x_0, s_a) = f_3(x_{-c}, \dots, x_{-4}, s_a) + x_{-3}$;
- (c) There exist mappings h_0 from $X^{c-1} \times S_a$ to $\{0,1\}$, h_1 from $X^{c-2} \times S_a$ to $\{0,1\}$, h_2 from $X^{c-3} \times S_a$ to $\{0,1\}$, f_0 from $X^c \times S_a$ to Y , f_1 from $X^{c-1} \times S_a$ to Y , f_2 from $X^{c-2} \times S_a$ to Y , f_3 from $X^{c-3} \times S_a$ to Y , such that (2.1), (2.2) and (2.3) hold, where h_3 and h_4 are defined by (2.4).

Proof: Since M_a is cyclic, M is strongly connected. By Theorem 2 in Ref.[6], M is a feedforward inverse with delay 3 iff M is weakly invertible with delay 3. Therefore, by Theorem 1, M is a feedforward with delay 3 if one of conditions (a), (b) and (c) holds.

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