NOTE ON THE DIOPHANTINE EQUATION

$x^x y^y = z^z$

BY CHAO KO (柯召)

Dr. Erdös conjectured that the Diophantine equation

$$(1) \qquad x^x y^y = z^z$$

has no integer solution, if $x>1$, $y>1$, $z>1$. In the present note, I shall prove that *his conjecture is correct only when* $(x, y)=1$ *and* (1) *has infinitely many solutions when* $(x, y)>1$.

Without loss of generality, we can suppose that

$$z > y \geq x > 0.$$

If $(x, y)=1$, from (1), we have obviously

$$x = x't, \qquad y = y's, \qquad z = st,$$

where x', y', s, t are positive integers and $(s, x't)=(t, y's)=1$, and, therefore, (1) becomes

$$x'^{x't} y'^{y's} = t^{(s-x')t} s^{(t-y')s},$$

where $s>x'$ and $t>y'$. Since $(x', s)=(y', t)=1$,

$$x'^{x'} = t^{s-x'}, \qquad y'^{y'} = s^{t-y'}.$$

Since $(x', s-x')=(y', t-y')=1$, it is evident that

$$(2) \qquad x' = m^{s-x'}, \quad t = m^{x'}; \quad y' = n^{t-y'}, \quad s = n^{y'};$$

where m and n are positive integers.

Suppose $y' \geq x'$. If

$$s - x' < x',$$

from (2), we have

$$2x' > s > x',$$

which contradicts to that x' is a positive integer, since $2x'>n^{y'}>x'$ leads to $s=1$ and, therefore, $1>x'>0$. Hence $s-x'\geqq x'$ and $t=m^{x'}$ divides x', which leads to $m=1$ and so $t=1$, contrary to that $t>y'>0$. Similarly, we can prove that (2) has no integer solution when $y'<x'$. This proves that Erdös' conjecture is correct when $(x, y)=1$.

If we omit the restriction $(x, y)=1$ and put

$$x = kx', \quad y = ky', \quad z = kz',$$

equation (1) becomes

$$k^{x'+y'-z'}x'^{x'}y'^{y'} = z'^{z'}. \tag{3}$$

Let $x'=2^4$, $y'=3^2$, $z'=2^3.3$, then $x'+y'-z'=1$ and from (3), $k=2^8.3^6$. Hence

$$x = 2^{12}.\,3^6, \quad y = 2^8.\,3^8, \quad z = 2^{11}.\,3^7$$

is a solution of (1).

Mr. W. H. Chu communicated to me that

$$x' = 2^6, \quad y' = 7^2, \quad z' = 2^4.\,7$$

give another solution

$$x = 2^{70}.\,7^{14}, \quad y = 2^{64}.\,7^{16}, \quad z = 2^{68}.\,7^{15}$$

of (1).

The above two solutions suggest that if we write

$$x' = 2^{2n}, \qquad y' = (2^n - 1)^2, \qquad z' = 2^{n+1}(2^n - 1),$$

then from (3)

$$k = 2^{2^{n+1}(2^n-n-1)}\,(2^n-1)^{2(2^n-1)},$$

which is an integer for any positive integer n, since then $2^n-n-1\geqq 0$. Hence (1) has infinitely many integer solutions

$$x = 2^{2n+1(2^n-n-1)+2n}(2^n-1)^{2(2^n-1)},$$

$$y = 2^{2n+1(2^n-n-1)}(2^n-1)^{2(2^n-1)+2},$$

and

$$z = 2^{2n+1(2^n-n-1)+n+1}(2^n-1)^{2(2^n-1)+1}.$$

National Szechuan University, Chengtu.

(Received 28 March, 1939).