CESÀRO MEANS CONNECTED WITH THE ALLIED SERIES OF A FOURIER SERIES

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1. Introduction.

It is classical that, if the Fourier series of a function $\psi(t)$ of bounded variation is $\Sigma \bar{c}_n \sin nt$, the sequence $n\bar{c}_n$ is bounded. This result cannot be improved. "For, the function $\psi(t) = \frac{1}{2} t (-\pi < t < \pi)$ whose Fourier series is $\Sigma(-1)^{n-1} n^{-1} \sin nt$ is of bounded variation; but we have $n\bar{c}_n = (-1)^{n-1}$". We know, however, that,(1) if $\psi(t)$ is of bounded variation in $(0, 2\pi)$, the sequence $n\bar{c}_n$ converges (c, α) to $2\pi^{-1}\psi(+0)$ for every $\alpha > 0$. In this paper I shall give some results concerning the convergence (C) of the sequence $n\bar{c}_n$ by studying the Cesàro means $\psi_\alpha(t)$ of $\psi(t)$.

We suppose throughout that $f(x)$ is a periodic function, with period 2π, which is integrable in the Lebesgue sence in $(-\pi, \pi)$. We write

$$\psi(t) = \frac{1}{2}\left\{ f(x+t) - f(x-t) \right\}.$$

If we suppose the Fourier series of $f(x)$ to be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.1}$$

the Fourier series of $\psi(t)$ is

$$\sum_{n=1}^{\infty} \bar{c}_n \sin nt,$$

where

(1) A. Zygmund, *Trigonometrical Series*, (1935), p. 62.

$$\sum_{n=1}^{\infty} \bar{c}_n = \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)$$

is the allied series of (1.1). We also write, for $t > 0$,

$$\Psi_0(t) = \psi(t),$$

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \psi(u)\, du \qquad (\alpha > 0),$$

$$\psi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Psi_\alpha(t) \qquad (\alpha \geq 0).$$

We denote by τ_n^α the n-th Cesàro mean of order $\alpha > 0$ of the sequence $n\bar{c}_n$, i.e.

$$\tau_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} \nu \bar{c}_\nu ,$$

where

$$A_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(1+\alpha)} \sim \frac{n^\alpha}{\Gamma(1+\alpha)} .$$

We say that $n\bar{c}_n$ is bounded (c, α) if $\tau_n^\alpha = O(1)$ as $n \to \infty$, and that $n\bar{c}_n$ converges (C, α) to τ if $\tau_n^\alpha \to \tau$ as $n \to \infty$.

2. Lemmas.

In this section I give some lemmas which will be used subsequently.

LEMMA 1(2). *If a sequence converges* (C) *to* τ *and is bounded* (c, α), *then it converges* $(c, \alpha+\delta)$ *to* τ, *for every* $\delta > 0$.

LEMMA 2(3). *If* $\psi_\alpha(t)$ *is of bounded variation, then so is* $\psi_\beta(t)$ *for every* $\beta > \alpha$

LEMMA 3(4). *If* $\psi_\alpha(+0)$ *exists, then so does* $\psi_\beta(+0)$ *for every* $\beta > \alpha$, *and* $\psi_\beta(+0) = \psi_\alpha(+0)$.

(2) A. F. Andersen, *Studier over Cesàro Summabilitetsmetode*, (1921).

(3) L. S. Bosanquet, *Proc. Edinburgh Math. Soc.*, (2), 4 (1933), 12-17.

(4) L. S. Bosanquet, *Proc. London Math. Soc.*, (2) 31 (1930), 144-164.

LEMMA 4([5]). *Let* $\alpha > 0$. *If* $\Psi_\alpha(t)$ *is of bounded variation in an interval* $(0, \eta)$, *where* $\eta > 0$, *and* $\Psi_\alpha(+0) = 0$, *then, for almost all* t *in* $(0, \eta)$ *and for every* $\beta > \alpha - 1$, *we have*

$$\Psi_\beta(t) = \frac{1}{\Gamma(1+\beta-\alpha)} \int_0^t (t-u)^{\beta-\alpha}\, d\,\Psi_\alpha(u).$$

LEMMA 5. *Let* $\alpha > 0$ *and let*

$$g^\alpha(n, t) = \frac{1}{A_n^\alpha}\left(\frac{1}{2} A_n^{\alpha-1} + \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \cos \nu t\right).$$

Then, if k *is a positive integer or zero, we have*([6])

$$(2.1)\qquad \left| \left(\frac{d}{dt}\right)^k g^\alpha(n, t) \right| \begin{cases} \leq A\, n^k & (0 < t \leq \pi), \\ \leq A\, n^{k-\alpha} t^{-\alpha} & (n^{-1} < t \leq \pi; \quad k > \alpha - 1), \\ \leq A\, n^{-1} t^{-k-1} & (n^{-1} < t \leq \pi; \quad k \leq \alpha - 1). \end{cases}$$

This can be proved by an argument used by Zygmund.([7])

LEMMA 6. *Let* $\alpha > 0$ *and* $h = [\alpha]$ *and let*

$$(2.2)\qquad G(n, u; \delta) = \frac{1}{\Gamma(1+h-\alpha)} \int_u^\delta (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n, t)\, dt,$$

where $0 < \delta \leq \pi$. *Then*

$$\left| G(n, u; \delta) \right| \begin{cases} \leq A\, n^\alpha & (0 < u \leq \delta), \\ \leq A\, u^{-\alpha} & (n^{-1} < u \leq \delta), \end{cases}$$

This can be proved by an argument used by Bosanquet.([8])

3. The case $\alpha \geq 1$.

If $\alpha \geq 1$, it is interesting to note that the convergence (C, α) of the sequence $n\bar{c}_n$ depends only on the local property of the function. In fact, we can prove the following theorem.

(5) L. S. Bosanquet, *Quart. J. of Math.*, (Oxford), 6 (1935), 113-123.
(6) A is a constant, not always the same, independent of n, t or u, and ε.
(7) A. Zygmund, *Bull. de l'Acad. Polonaise* (Cracovie), A (1925) 207-217.
(8) L. S. Bosanquet, *Proc. London Math. Soc.*, (2) 41 (1936), 517-528.

THEOREM 3.1. *If $\alpha \geq 1$, the necessary and sufficient condition for the sequence $n\bar{c}_n$ to converge (C, α) to τ is that*

$$-\frac{2}{\pi}\int_0^\delta \psi(t)\frac{d}{dt}g^\alpha(n,t)\,dt \longrightarrow \tau$$

as $n \to \infty$, where δ is an arbitrary positive number.

Proof. Since

$$n\bar{c}_n = \frac{2n}{\pi}\int_0^\pi \psi(t)\sin nt\,dt = -\frac{2}{\pi}\int_0^\pi \psi(t)\frac{d}{dt}\cos nt\,dt,$$

we have

$$\tau_n^\alpha = -\frac{2}{\pi}\int_0^\pi \psi(t)\frac{d}{dt}g^\alpha(n,t)\,dt = -\frac{2}{\pi}\int_0^\delta - \frac{2}{\pi}\int_\delta^\pi$$

and the theorem is proved if we can show that

$$I = \int_\delta^\pi \psi(t)\frac{d}{dt}g^\alpha(n,t)\,dt = o(1)$$

as $n \to \infty$. Suppose first that $\alpha \geq 2$. Then, by the third inequality of (2.1), we have

$$|I| \leq A n^{-1}\int_\delta^\pi |\psi(t)|\, t^{-2}\,dt \leq A n^{-1} = o(1).$$

Next, suppose that $1 < \alpha < 2$. We have, by the second inequality of (2.1),

$$|I| \leq A n^{1-\alpha}\int_\delta^\pi |\psi(t)|\, t^{-\alpha}\,dt \leq A n^{1-\alpha} = o(1)$$

Finally, let $\alpha = 1$. Then, since

$$\frac{d}{dt}g^\alpha(n,t) = \frac{1}{n+1}\frac{d}{dt}\left(\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right),$$

we have

$$I = \frac{n}{n+1}\int_\delta^\pi \frac{\psi(t)}{2\sin\frac{1}{2}t}\cos(n+\tfrac{1}{2})t\,dt - \frac{1}{n+1}\int_\delta^\pi \frac{\psi(t)}{(2\sin\frac{1}{2}t)^2}\sin nt\,dt = o(1)$$

by Riemann-Lebesgue theorem.

THEOREM 3.2. *If* $\alpha \geq 1$ *and if* $\psi_\alpha(t)$ *is of bounded variation in an interval* $(0, \eta)$, *where* $\eta > 0$, *then the sequence* $n\bar{c}_n$ *converges* (c, α) *to* $2\pi^{-1}\psi_\alpha(+0)$.

Proof. We prove first that we can suppose without loss of generality that $\psi_\alpha(+0) = 0$. In fact, if $\psi_\alpha(+0) \neq 0$, we write

$$\varphi(t) = \frac{1}{2}(\pi - t)$$

so that

$$\varphi_\alpha(t) = \frac{1}{2}\left(\pi - \frac{t}{\alpha+1}\right) \longrightarrow \frac{\pi}{2}$$

as $t \to +0$ Let

$$\psi^*(t) = \psi(t) - \frac{2}{\pi}\psi_\alpha(+0)\varphi(t).$$

Then

$$\psi^*_\alpha(t) = \psi_\alpha(t) - \frac{2}{\pi}\psi_\alpha(+0)\varphi_\alpha(t)$$

and

$$\psi^*_\alpha(+0) = 0.$$

Since the Fourier series of $\psi(t)$ is

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n},$$

if we suppose that the Fourier series of $\psi^*(t)$ is

$$\sum_{n=1}^{\infty} \bar{c}^*_n \sin nt,$$

then

$$\bar{c}_n = \bar{c}^*_n + \frac{1}{n}\frac{2}{\pi}\psi_\alpha(+0)$$

and

$$n \bar{c}_n = n \bar{c}_n^* + \frac{2}{\pi} \psi_\alpha (+0).$$

Then the theorem is proved if we can show that the sequence $n\bar{c}_n^*$ converges (C, α) to 0.

This being so, let us suppose that $\psi_\alpha(t)$ is non-decreasing in $(0, \eta)$. Let ε be an arbitrarily small positive number. Choose δ so small that $\psi_\alpha(t) < \varepsilon$ for $0 < t \leq \delta \leq \eta$. Then

$$\int_0^\delta | d\psi_\alpha(t) | = \int_0^\delta d\psi_\alpha(t) = \psi_\alpha(\delta) < \varepsilon.$$

By Theorem 3.1, it is enough to prove that, as $n \to \infty$,

$$| J | = \left| \int_0^\delta \psi(t) \frac{d}{dt} g^\alpha(n, t)\, dt \right| \leq A\, \varepsilon.$$

Let $h = [\alpha]$. Then, after h times integration by parts, we have

$$J = \left[\sum_{m=1}^{h} (-1)^{m-1} \Psi_m(t) \left(\frac{d}{dt}\right)^m g^\alpha(n, t) \right]_0^\delta + (-1)^h \int_0^\delta \Psi_h(t) \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n, t)\, dt.$$

If α is an integer, then $h = \alpha$ and, letting P be the integrated part of J, we have, for sufficiently large n,

$$| P | = \left| \sum_{m=1}^{h-1} O(n^{-1}) + (-1)^{h-1} \frac{1}{\Gamma(h+1)} \delta^h \psi_h(\delta) \left(\frac{d}{dt}\right)^h g^\alpha(n, \delta) \right| \leq A\, \varepsilon.$$

If α is not an integer, then $h < \alpha$ and

$$P = \sum_{m=1}^{h-1} O(n^{-1}) + O(n^{h-\alpha}) = o(1),$$

as $n \to \infty$. In either case the theorem is proved if we can show that, as $n \to \infty$

$$| J_1 | = \left| \int_0^\delta \Psi_h(t) \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n, t)\, dt \right| \leq A\, \varepsilon.$$

Now, by Lemma 4,

$$J_1 = \frac{1}{\Gamma(1+h-\alpha)} \int_0^\delta \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\, dt \int_0^t (t-u)^{h-\alpha}\, d\Psi_\alpha(u)$$

$$= \frac{1}{\Gamma(1+h-\alpha)} \int_0^\delta d\Psi_\alpha(u) \int_u^\delta (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\, dt$$

$$= \int_0^\delta G(n,u;\delta)\, d\Psi_\alpha(u).$$

Since

$$d\Psi_\alpha(u) = \frac{1}{\Gamma(\alpha)} u^{\alpha-1} \psi_\alpha(u)\, du + \frac{1}{\Gamma(\alpha+1)} u^\alpha\, d\psi_\alpha(u),$$

we have

$$J_1 = \frac{1}{\Gamma(\alpha)} \int_0^\delta u^{\alpha-1} \psi_\alpha(u)\, G(n,u;\delta)\, du + \frac{1}{\Gamma(\alpha+1)} \int_0^\delta u^\alpha\, G(n,u;\delta)\, d\psi_\alpha(u)$$

$$= \frac{1}{\Gamma(\alpha)} \left(\int_0^{n^{-1}} + \int_{n^{-1}}^\delta\right) + \frac{1}{\Gamma(\alpha+1)} \left(\int_0^{n^{-1}} + \int_{n^{-1}}^\delta\right)$$

$$= \frac{1}{\Gamma(\alpha)} (J_{11} + J_{12}) + \frac{1}{\Gamma(\alpha+1)} (J_{13} + J_{14}).$$

By (2.2), we have

$$|J_{11}| \leqslant A n^\alpha \int_0^{n^{-1}} u^{\alpha-1} \psi_\alpha(u)\, du \leqslant A\varepsilon n^\alpha \int_0^{n^{-1}} u^{\alpha-1}\, du \leqslant A\varepsilon,$$

$$|J_{13}| \leqslant A n^\alpha \int_0^{n^{-1}} u^\alpha\, d\psi_\alpha(u) \leqslant A \int_0^{n^{-1}} d\psi_\alpha(u) \leqslant A\varepsilon,$$

$$|J_{14}| \leqslant A \int_{n^{-1}}^\delta d\psi_\alpha(u) \leqslant A\varepsilon.$$

It remains to show that $|J_{12}| \leq A\varepsilon$ as $n \to \infty$.

By the second mean-value theorem, we have

$$J_{12} = \psi_\alpha(\delta) \int_{\delta'}^\delta u^{\alpha-1} G(n,u;\delta)\, du \quad (n^{-1} < \delta' < \delta)$$

$$= \frac{\psi_\alpha(\delta)}{\Gamma(1+h-\alpha)} \int_{\delta'}^\delta u^{\alpha-1}\, du \int_u^\delta (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\, dt$$

$$= \frac{\psi_\alpha(\delta)}{\Gamma(1+h-\alpha)} \int_{\delta'}^\delta \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t) \int_{\delta'}^t (t-u)^{h-\alpha} u^{\alpha-1}\, du$$

$$= \frac{\psi_\alpha(\delta)}{\Gamma(1+h-\alpha)} \int_{\delta'}^{\delta} t^h \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t) \int_{\delta'/t}^{1} (1-v)^{h-\alpha} v^{\alpha-1}\, dv$$

$$= \frac{\psi_\alpha(\delta)}{\Gamma(1+h-\alpha)} \int_{\delta'}^{\delta} t^h \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t) F(t)\, dt,$$

where

$$F(t) = \int_{\delta'/t}^{1} (1-v)^{h-\alpha} v^{\alpha-1}\, dv$$

is an increasing function of t. Applying the second mean-value theorem again, we have

$$J_{12} = \frac{\psi_\alpha(\delta) F(\delta)}{\Gamma(1+h-\alpha)} \int_{\delta''}^{\delta} t^h \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\, dt \quad (n^{-1} < \delta' < \delta'' < \delta).$$

Since, by (2.1),

$$\int_{\delta''}^{\delta} t^h \left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\, dt = \left[\sum_{m=0}^{h} (-1)^{h-m} \frac{h!}{m!} t^m \left(\frac{d}{dt}\right)^m g^\alpha(n,t) \right]_{\delta''}^{\delta} = O(1)$$

as $n \to \infty$, it follows that $|J_{12}| \leq A\varepsilon$. This completes the proof.

4. The case $0 < \alpha < 1$.

If $0 < \alpha < 1$, Theorem 3.1 is no longer true. In this case, we can prove the following theorem.

THEOREM 4.1. *If $0 < \alpha < 1$ and if $\psi_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the sequence $n\bar{c}_n$ is bounded (C, α).*

Proof. Here we have

$$\tau_n^\alpha = -\frac{2}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} g^\alpha(n,t) dt = -\frac{2}{\pi} \int_0^\pi \frac{d}{dt} g^\alpha(n,t) dt \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha}\, d\Psi_\alpha(u)$$

$$= -\frac{2}{\pi} \frac{1}{\Gamma(1-\alpha)} \int_0^\pi d\Psi_\alpha(u) \int_u^\pi (t-u)^{-\alpha} \frac{d}{dt} g^\alpha(n,t)\, dt$$

$$= -\frac{2}{\pi} \int_0^\pi G(n,u;\pi)\, d\Psi_\alpha(u)$$

$$= -\frac{2}{\pi} \frac{1}{\Gamma(\alpha)} \int_0^\pi \psi_\alpha(u) u^{\alpha-1} G(n,u;\pi) du - \frac{2}{\pi} \frac{1}{\Gamma(1-\alpha)} \int_0^\pi u^\alpha G(n,u;\pi)\, d\Psi_\alpha(u)$$

$$= -\frac{2}{\pi} \frac{1}{\Gamma(\alpha)} \left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\pi} \right) - \frac{2}{\pi} \frac{1}{\Gamma(1-\alpha)} \left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\pi} \right)$$

$$= -\frac{2}{\pi}\frac{1}{\Gamma(\alpha)}(I_1 + I_2) - \frac{2}{\pi}\frac{1}{\Gamma(1-\alpha)}(I_3 + I_4).$$

Since $\psi_\alpha(t)$ is bounded in $(0, \pi)$, we have, by (2.2)

$$| I_1 | \leqslant An^\alpha \int_0^{n^{-1}} u^{\alpha-1}\, du \leqslant A,$$

$$| I_3 | \leqslant An^\alpha \int_0^{n^{-1}} u^\alpha \,| d\psi_\alpha(u) | \leqslant A \int_0^{n^{-1}} | d\psi_\alpha(u) | \leqslant A,$$

$$| I_4 | \leqslant A \int_{n^{-1}}^{\pi} | d\psi_\alpha(u) | \leqslant A.$$

Now, suppose that $\psi_\alpha(t)$ is non-decreasing in $(0, \pi)$. By the second mean-value theorem, we have

$$I_2 = \int_{n^{-1}}^{\pi} \psi_\alpha(u) u^{\alpha-1} G(n, u\,; \pi)\, du = \psi_\alpha(\pi-0) \int_\delta^\pi u^{\alpha-1} G(n, u\,; \pi)\, du$$

$$(n^{-1} < \delta < \pi)$$

$$= \frac{\psi_\alpha(\pi-0)}{\Gamma(1-\alpha)} \int_\delta^\pi u^{\alpha-1}\, du \int_u^\pi (t-u)^{-\alpha} \frac{d}{dt} g^\alpha(n, t)\, dt$$

$$= \frac{\psi_\alpha(\pi-0)}{\Gamma(1-\alpha)} \int_\delta^\pi \frac{d}{dt} g^\alpha(n, t)\, dt \int_\delta^t u^{\alpha-1}(t-u)^{-\alpha}\, du$$

$$= \frac{\psi_\alpha(\pi-0)}{\Gamma(1-\alpha)} \int_\delta^\pi F_1(t) \frac{d}{dt} g^\alpha(n, t)\, dt = \frac{\psi_\alpha(\pi-0) F_1(\pi)}{\Gamma(1-\alpha)} \int_{\delta'}^\pi \frac{d}{dt} g^\alpha(n, t)\, dt$$

$$(\delta < \delta' < \pi)$$

$$= \frac{\psi_\alpha(\pi-0) F_1(\pi)}{\Gamma(1-\alpha)} \left\{ g^\alpha(n, \pi) - g^\alpha(n, \delta') \right\} = O(1),$$

for

$$F_1(t) = \int_\delta^t u^{\alpha-1}(t-u)^{-\alpha}\, du = \int_{\delta/t}^1 v^{\alpha-1}(1-v)^{-\alpha}\, dv$$

is an increasing function of t and $F_1(\pi)$ is bounded. Hence $\tau_n^\alpha = O(1)$ and the theorem is therefore proved.

THEOREM 4.2. *If $0 < \alpha < 1$, and if $\psi_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the sequence $n\bar{c}_n$ converges (C, β) to $2\pi^{-1}\psi_\alpha(+0)$, for every $\beta > \alpha$.*

Proof. Under the given conditions, $\psi_1(t)$ is, by Lemma 2, of bounded variation, and $\psi_1(+0)$ is, by Lemma 3, equal to $\psi_\alpha(+0)$. Hence, by Theorem 3.2, the sequence $n\bar{c}_n$ converges $(C, 1)$, and, by Lemma 1, converges (C, β), for every $\beta>\alpha$, to $2\pi^{-1}\psi_\alpha(+0)$.

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