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2-CONNECTED k-REGULAR GRAPHS ON AT MOST $3k+3$ VERTICES TO BE HAMILTONIAN (CONTINUED)

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Let X^* and Y^* be generated by $S\cup\{v_0\}$, where $G(S)$ is Hamiltonian connected and $|X^*|=x^*$, $|Y^*|=y^*$ and $S_1^*, S_2^*, \cdots, S_{x^*}^*$ be the sets of vertices contained in the open segments of C between vertices of X^*. Let $S_1^*, x_1, S_2^*, x_2, \cdots, S_{x^*}^*, x_{x^*}$ be the segments and vertices of X^* in order around C. S_i^* is said to be an $X^*(3)$-interval if one of x_{i-1} and x_i belongs to $X_j^*-X_3^*$.

Let $S=S_1^*$, and $S=\{a_1, c_1, c_2, \cdots, c_t, b_1\}$. It is easy to see that the statement in Lemma 2 can be modified as

$$\varepsilon(\{a_1, b_1\}, S_i^*)\leqslant\begin{cases}s_i^*, & \text{if } S_i^* \text{ is an } X^*(3)\text{-intervatl},\\ s_i^*-1, & \text{otherwise}\end{cases}$$

and

$$\varepsilon(\{a_j, b_j\}, S_i^*)\leqslant s_i^*-1,\ j\neq 1.$$

We claim that $x^*=k$. Suppose $x^*=k+t$, $t>0$. If $t=1$, there does not exist an $X^*(3)$-interval S_i^*, and thus in this case the proof of the theorem is the same as that for $x=k+1$.

If $t\geqslant 2$, then $|X^*-X_2^*|\leqslant t-1$, and thus there are at most $2(t-1)$ $X^*(3)$-intervals.

Following Jacksen's argument [1], one can deduce the following inequality

$$(n-2x^*)k\leqslant(n-2x^*)(n-2x^*-1)+4(t-1)$$

or

$$n\geqslant 3k+1+\left(2t-\frac{4(t-1)}{n-2k-2t}\right).$$

As $n=3k+3$, $t\geqslant\dfrac{k+1}{2}$.

On the other hand, it is obvious that $2x^*+1\leqslant n-1$ and thus $t\leqslant\dfrac{k+1}{2}$, which means that $t=\dfrac{k+1}{2}$ and $x^*=k+\dfrac{k+1}{2}$. Thus each interval S_i^*, $i\neq 1$, contains a single vertex, and S_1^* contains exactly two vertices. In this case, we have

$$\varepsilon(V(C)-X^*, X^*)=(k-2)(x^*+1)-2\varepsilon(S, V(C)-X^*).$$

On the other hand

$$\varepsilon(X^*, V(C)-X^*)\leqslant(k-2)x^*-k.$$

Hence

$$\varepsilon(S, V(C)-X^*)\geqslant k-1,$$
$$\varepsilon(S, X^*)\leqslant 2k-4-(k-1)=k-3.$$

Because $D(S)\neq\emptyset$, by Corollary 5 we have $\varepsilon(S, X^*)\geqslant k-2$. The contradiction shows that $D(S)=\emptyset$, i. e. $x^*=k$.

Lemma 8. *Let $S=\{a, c_1, c_2, \cdots, c_t, b\}$, $t\geqslant 2$, be an interval of C between the vertices of X. If $y\leqslant k-3$, then $\varepsilon(S, X)\geqslant k-3$.*

Proof. First of all we will prove two propositions.

Proposition 1. Let c_g be a rAB-vertex of S. If $y\leqslant k-3$, then $\varepsilon(\{c_g\}, X)\geqslant 1$.

Proof. Since

$$\varepsilon(\{c_g\}, X)=k-\varepsilon\left(\{c_g\}, \bigcup_i S_i\right),$$

and

$$\varepsilon\left(\{c_g\}, \bigcup_i S_i\right)\leqslant 3k+2-1-k-y-2(k-y-1)=y+3,$$

we have

$$\varepsilon(\{c_g\}, X)\geqslant k-3-y.$$

Obviously, if $y\leqslant k-4$, then $\varepsilon(\{c_g\}, X)\geqslant 1$; if $y=k-3$, then $\varepsilon(\{c_g\}, X)\geqslant 0$, and the equality holds only if c_g is joined to every vertex of S and to every inner vertex. In the following we will discuss the case that $y=k-3$.

Let $S_i=\{a_i, c_1, c_2, \cdots, c_{t_i}, b_i\}$, $S_j=\{a_j, d_1, d_2, \cdots, d_{t_j}, b_j\}$ be two other intervals which contains more than one vertex, and let

$$\varepsilon(\{a, b\}, S)=t+\alpha,\ \varepsilon(\{a_i, b_i\}, S_i)=t_i+\alpha_i,$$
$$\varepsilon(\{a_j, b_j\}, S_j)=t_j+\alpha_j.$$

Since c_g is joined to every inner vertex, there is no rA-vertex or rB-vertex in S_i and S_j. Thus if $t_i\geqslant 1$, then $\alpha_i\leqslant -1$; if $t_j\geqslant 1$, then $\alpha_j\leqslant -1$. Since c_g is joined to every vertex in S, by Lemma 4, every inner vertex in S is also a rAB-vertex. Thus $\varepsilon(\{a_i, b_i\}, S)\leqslant 1$, and $\varepsilon(\{a_j, b_j\}, S)\leqslant 1$. Therefore, $\varepsilon(\{a_i, b_i\}, X)\geqslant k-5+t$, and if the equality holds, there must be $t_i=0$; $\varepsilon(\{a_j, b_j\}, X)\geqslant k-5+t$, and if the equality holds, there must be $t_j=0$.

On the other hand, since c_g is joined to every inner vertex, so $\varepsilon(\{a, b\}, S_i)\leqslant 2$ and if the equality holds, we must have $t_i=1$; by the same reason, $\varepsilon(\{a, b\}, S_j)\leqslant 2$ and if the equality holds, we must have $t_j=1$. Hence we have

$$\varepsilon(\{a, b\}, X) \geqslant k - 5 - \alpha + t_i + t_j - 2,$$

and if the equality holds, there must be $t_i = t_j = 1$. Therefore

$$3(k-5) + 2t + t_i + t_j - \alpha - 2 < \varepsilon(\{a_i, b_i\}, X) + \varepsilon(\{a_j, b_j\}, X) + \varepsilon(\{a, b\}, X).$$

Since $t + t_i + t_j = y + 2 = k - 1$, we have

$$y(k-2) + 3(k-5) + k - 1 + t - \alpha - 2 < \varepsilon(V(C) - X, X) = k(k-3).$$

Since $y = k - 3$, $t \geqslant \alpha$, then $k(k-3) + 2(k-6) < k(k-3)$; hence $k < 6$, which is contrary to the assumption that $k \geqslant 6$. This proves that $\varepsilon(\{c_g\}, X) \geqslant 1$.

Proposition 2. Let c_g be a rAB-vertex of S. If $y \leqslant k-3$, then either $\varepsilon(\{c_g\}, X) \geqslant 2$ or c_g is joined to every vertex of S.

Proof. If at least one vertex of S is not joined to c_g, obviously we have $\varepsilon(\{c_g\}, X) \geqslant 2$ if $y \leqslant k-4$. Thus we only consider the case $y = k-3$. By Proposition 1, we know that $\varepsilon(\{c_g\}, X) \geqslant 1$. We will only prove that if $\varepsilon(\{c_g\}, X) = 1$, then c_g must be joined to every vertex of S.

In fact, if $\varepsilon(\{c_g\}, X) = 1$ and c_g is not joined to every vertex of S, then c_g must be joined to every inner vertex outside S. Let other two intervals of φ be $S_i = \{a_i, c_1, c_2, \cdots, c_{t_i}, b_i\}$ and $S_j = \{a_j, d_1, d_2, \cdots, d_{t_j}, b_j\}$. If $s_i \leqslant 3$ and $s_j \leqslant 3$, by Lemma 7, we have $\varepsilon(S_i, X) \geqslant k-3$, $\varepsilon(S_j, X) \geqslant k-3$. Furthermore, by Lemma 4 and Proposition 1, we obtain $\varepsilon(S, X) \geqslant k-5$. Hence we have

$$(k-3)(k-2) + 2(k-3) + (k-5) \leqslant \varepsilon(V(C) - X, X) \leqslant k(k-3),$$
$$k(k-3) + k - 5 \leqslant k(k-3),$$

that is $k \leqslant 5$, which is contrary to $k \geqslant 6$.

Therefore we may assume that $s_i \geqslant 4$, $s_j \geqslant 2$. Since c_g is joined to every inner vertex outside S, there is no rAB-vertex in S_i or S_j.

If $t_i \geqslant 2$, $t_j \geqslant 2$, we have $\varepsilon(\{a, b\}, S_i) = 0$, $\varepsilon(\{a, b\}, S_j) = 0$. By Lemma 4 and Proposition 1, we obtain

$$\varepsilon(S, X) \geqslant k - 5 + t_i + 1 + t_j + 1 \geqslant k + 1.$$

Since there is no rAB-vertex in S_i or S_j, we have

$$\varepsilon(S_i, X) \geqslant k-5, \ \varepsilon(S_j, X) \geqslant k-5.$$

Hence, we have

$$k + 1 + 2(k-5) + (k-3)(k-2) \leqslant k(k-3),$$
$$k(k-3) + k - 3 \leqslant k(k-3),$$

which is contrary to the assumption that $k \geqslant 6$.

If $t_i \geqslant 2$, $t_j \leqslant 1$, by Lemma 7 we know that $\varepsilon(S_j, X) \geqslant k-3$; by Lemma 4 and Proposition 1 we obtain

$$\varepsilon(S, X) \geqslant k-5+t_i+1 \geqslant k-2,$$
$$\varepsilon(S_i, X) \geqslant k-5.$$

Hence, we have

$$(k-2)(k-2)+(k-3)+k-5 \leqslant k(k-3),$$
$$k(k-3)+k-4 \leqslant k(k-3),$$

which is also contrary to $k \geqslant 6$.

Now we prove this lemma by the following cases.

1. There are $f>0$ rAB-vertices in S.

(a) $0<f<t$.

By Lemma 4 we know that no rAB-vertex of S can be joined to all vertices of S. Hence by Proposition 2 we know that $\varepsilon(\{c_g\}, X) \geqslant 2$ for every rAB-vertex c_g of S. So we obtain

$$\varepsilon(S, X) \geqslant k-5-\alpha+2f \geqslant k-5+f \geqslant k-4,$$

and if the equality holds, there must be $\alpha=f=1$. By Lemma 5, $(a, c_{2i+1}) \in E$, $(b, c_{2i}) \in E$, $i=1, 2, \cdots, \frac{t}{2}$, and either $(a, c_2) \in E$ or $(b, c_{t-1}) \in E$. We may suppose that $(a, c_2) \in E$. In this case c_1 is the unique rAB-vertex of S. Put $S'=\{c_1, c_2, \cdots, c_t\}$. There is no rAB-vertex in S'. Otherwise there are at least two rAB-vertices in S, which is contrary to $f=1$. Since $(c_1, b)\bar{\in}E$, $(c_t, a)\bar{\in}E$, we can conclude that

$$\varepsilon(\{c_1, c_t\}, S)=\varepsilon(\{c_1, c_t\}, S') \leqslant t-2.$$

Let $\varepsilon(\{c_1\}, S)=r_1$, $\varepsilon(\{c_t\}, S)=r_2$, $\varepsilon(\{a\}, S)=\beta_1$, $\varepsilon(\{b\}, S)=\beta_2$. Since $(a, b) \in E$, there exists a Hamiltonian chain from a to c_1 in $G(S)$; there also exists a Hamiltonian chain from b to c_t. Hence we have

$$\varepsilon(\{a, c_1\}, X) \geqslant k-5-r_1-\beta_1+t,$$
$$\varepsilon(\{b, c_t\}, X) \geqslant k-5-r_2-\beta_2+t.$$

This reduces to

$$\begin{aligned}\varepsilon(S, X) &\geqslant 2(k-5)+2t-(r_1+r_2)-(\beta_1+\beta_2)\\ &=2(k-5)+2t-t+2-t-1\\ &=2(k-5)+1=k-3+k-6 \geqslant k-3.\end{aligned}$$

(b) $f=t$.

If $\alpha \leqslant t-2$, then

$$\varepsilon(S, X) \geqslant k-5-\alpha+f \geqslant k-3.$$

If $\alpha \geqslant t-1$, then there is at least one vertex of a and b, which is joined to every vertex of S. We may assume that it is b. If there also exists a rAB-vertex c_g in S such that c_g is joined to every vertex of S, then $G(S)$ must be Hamiltonian connected.

In fact, Since c_g is a rAB-vertex, there exists a Hamiltonian chain from c_g to b in $G(S)$. Let this chain be

$$Q=(c_g=d_0, d_1, \cdots, d_t, d_{t+1}=b).$$

Since c_g and b are joined to every vertex of Q respectively, for any two vertices d_i and d_j $(i<j)$ in this chain we can obtain a Hamiltonian chain

$$(d_i, d_{i-1}, \cdots, d_0, d_{j+1}, \cdots, d_{j-1}, d_{t+1}, d_t, \cdots, d_j).$$

By the arbitrariness of d_i and d_j, $G(S)$ is Hamiltonian connected.

Since c_g is joined to every vertex of S, by Lemma 4 we know that every inner vertex of S is a rAB-vertex. In addition to $D(S)=\emptyset$, we can deduce

$$\varepsilon(S, X)\geqslant k-3+t(k-2)-t^2-2.$$

If $t<k-2$, clearly, $\varepsilon(S, X)\geqslant k-3$.

If $t\geqslant k-2$, then $\varepsilon(S, X)\geqslant f-t\geqslant k-2$ by proposition 1.

If $\alpha\geqslant t-1$, and any rAB-vertex of S can not be joined to all vertices of S, by Proposition 2 we obtain

$$\varepsilon(S, X)\geqslant k-5-\alpha+2f\geqslant k-5+f\geqslant k-3.$$

2. There is no rAB-vertex in S.

Since $\varepsilon(S, X)\geqslant k-5-\alpha$, clearly, the lemma holds if $\alpha\leqslant-2$, Thus we only consider the following two cases.

(a) $\alpha=-1$.

If $(a, b)\bar{\in}E$, by Lemma 5 we know that either $(a, c_2)\in E$ or $(c_{t-1}, b)\in E$. We may suppose that $(a, c_2)\in E$. In this case we can conclude from Lemma 4 Corollary 4 that $\varepsilon(\{c_1, b\}, S)\leqslant t-2$. Since $(c_1, a, c_2, c_3, \cdots, b)$ is a Hamiltonian chain of $G(S)$, we obtain

$$\varepsilon(S, X)\geqslant\varepsilon(\{c_1, b\}, X)\geqslant k-3.$$

If $(a, b)\in E$, by Lemma 4 Corollary 3 we know that $\varepsilon(\{c_1, c_t\}, S)\leqslant t-2$. In a way similar to 1(a) we can obtain

$$\varepsilon(S, X)\geqslant\varepsilon(\{a, c_1\}, X)+\varepsilon(\{b, c_t\}, X)\geqslant k-3.$$

(b) $\alpha=0$.

In this case by Lemma 4 we know that $(a, b)\in E$. Again by Lemma 4 Corollary 3 we know that $\varepsilon(\{c_1, c_t\}, S)\leqslant t-2$. Thus following the preceding case we can conclude that

$$\varepsilon(S, X)\geqslant k-3.$$

The proof of this lemma is complete.

In the following we will prove the theorem by the value of y.

1. $y=0$.

Let $S_1, S_2, \cdots, S_k$ be k intervals of C in order around C. Let $S=S_1=\{a, b\}$ be the interval which contains two vertices. Let $C=\{S_1, x_1, S_2, x_2, \cdots, S_k, x_k\}$ be the arrangement of C in a right order. By Lemma 7 and Lemma 8 we know that $\varepsilon(S_i, X)\geqslant k-$

3 for every i. Since the strict inequality cannot hold, we can conclude that $\varepsilon\left(S_i, \bigcup_{j\neq i} S_j\right) = k-1$ for every i.

(a) There exists an interval S_l which contains two vertices such that

$$\varepsilon(S_l, X^+\cup X^-)\geqslant k-2.$$

We may assume without loss of generality that S satisfies this condition. When $\varepsilon(\{a\}, X^-)>0$ and $\varepsilon(\{b\}, X^+)>0$, if $(a, b_j)\in E$ and $(b, a_i)\in E$, then these two chords must "cross" each other, that is $i>j$. Thus $(b, a_2)\bar{\in}E$, $(b_k, a)\bar{\in}E$. Since for arbitrary b_j, if $(a, b_j)\in E$, then $(a, x_j)\bar{\in}E$, $(b, x_j)\bar{\in}E$ and $(b, x_k)\bar{\in}E$, similarly, if $(b, a_i)\in E$, then $(a, x_{i-1})\bar{\in}E$, $(b, x_{i-1})\bar{\in}E$ and $(a, x_1)\bar{\in}E$.

Since $\varepsilon(S_l, X^+\cup X^-)\geqslant k-2$, there exists at most one vertex in X which can be joined to a and b. Hence $\varepsilon(\{a, b\}, X)\leqslant 2$, which is contrary to $\varepsilon(\{a, b\}, X)=k-3\geqslant 3$.

When $\varepsilon(\{a\}, X^-)=0$ and $\varepsilon(\{b\}, X^+)\geqslant k-2$, obviously, if $(a_i, b)\in E$, then $(x_{i-1}, a)\bar{\in}E$ and $(a, x_1)\bar{\in}E$. Thus there exists at most one vertex in X which can be joined to a, that is $\varepsilon(\{a\}, X)\leqslant 1$. Hence

$$\varepsilon(\{a, b\}, X)\leqslant 1$$

whcih is contrary to $\varepsilon(\{a, b\}, X)=k-3\geqslant 3$.

(b) $\varepsilon(S_l, X^+\cup X^-)\leqslant k-3$ for all intervals S_l which contains two vertices.

Since $\varepsilon\left(S_l, \bigcup_{j\neq l} S_j\right)=k-1$, we have

$$\varepsilon\left(S_l, \bigcup_{j\neq l} S_j-(X^+\cup X^-)\right)\geqslant 2.$$

Because there are only two inner vertices in C and $D(S_l)=\emptyset$, these two inner vertices must be joined to S_l.

First of all, these two inner vertices cannot be in the same interval, otherwise they must be joined to a_l and b_l respectively. Thus there exists a longer cycle. Suppose

$$S_i=\{a_i, c, b_i\},\ S_j=\{a_j, d, b_j\}$$

and other intervals all contain two vertices.

Since for all $l\neq i, j$, we have $\varepsilon(\{a_l, b_l\}, \{c, d\})=2$ and $D(S_l)=\emptyset$, we claim that $\varepsilon(\{a_i, b_i, a_j, b_j\}, S_l)\leqslant 1$.

If there exists $l\neq i, j$, such that $\varepsilon(\{a_i, b_i, a_j, b_j\}, S_l)=2$, there must be $\varepsilon(\{a_i, b_i\}, S_l)=1$, $\varepsilon(\{a_j, b_j\}, S_l)=1$. Since $\varepsilon(\{a_l, b_l\}, \{c, d\})=2$, there are two ways to connect:

(i) $(a_l, c)\in E$, $(b_l, d)\in E$.

In this case, $(b_i, a_l)\in E$, $(a_j, b_l)\in E$. Hence we can get a longer cycle.

(ii) $(a_l, c)\in E$, $(a_l, d)\in E$.

In this case there must be $(b_i, a_l)\in E$, $(b_j, a_l)\in E$; thus $\varepsilon(\{b_l\}, X^+)>0$. Otherwise $\varepsilon(S_l, X) \geqslant \varepsilon(\{b_l\}, X) = k-2$, which is contrary to $\varepsilon(S_l, X) = k-3$. Suppose $(a_l, b_l) \in E$. In this case, it is contrary to Lemma 1. These prove that $\varepsilon(\{a_i, b_i, a_j, b_j\}, S_l) \leqslant 1$.

Since c and d cannot be rAB-vertices, thus $(a_i, b_i) \bar{\in} E$, $(a_j, b_j) \bar{\in} E$, and we have

$$\varepsilon(\{a_i, b_i\}, X) + \varepsilon(\{a_j, b_j\}, X) \geqslant 2(2k-4) - (k-2) - 4$$
$$= 3k - 10 > 2k - 6 = 2(k-3),$$

which is contrary to $\varepsilon(S_i, X) = k-3$ and $\varepsilon(S_j, X) = k-3$.

We conclude that $y = 0$ cannot occur.

2. $1 \leqslant y \leqslant k-3$.

By Lemmas 7 and 8, we can deduce

$$k(k-3) \geqslant \varepsilon(V(C) - X, X) \geqslant y(k-2) + (k-y)(k-3)$$
$$= k(k-3) + y,$$

which is contrary to the assumption $y \geqslant 1$.

3. $y = k-2$.

Let $S_j = \{a_j, c_1, c_2, \cdots, c_{t_j}, b_j\}$ and $S_l = \{a_l, d_1, d_2, \cdots, d_{t_l}, b_l\}$ be two intervals which contain more than one vertex. Put

$$\varepsilon(\{a_i, b_i\}, S_i) = t_i + \alpha_i, \quad i = j, l.$$

Then

(a) S_j and S_l cannot be consecutive intervals on C. Otherwise, let x_j be the vertex of X between S_l and S_j. Hence the degree of x_j is at least $k+1$ since x_j is joined to every vertex of Y, which is contrary to the assumption of k-regularity.

(b) $(a_j, b_l) \bar{\in} E$, $(a_l, b_j) \bar{\in} E$.

If they are not true, suppose $(a_j, b_l) \in E$ (see Fig.2*). We can get a longer cycle since the vertex y_i of Y between S_l and S_j is joined to every vertex of X. Hence $(a_j, b_l) \bar{\in} E$. Similarly, we have $(a_l, b_j) \bar{\in} E$.

(c) By Lemma 1 we can get the following inequalities

$$\varepsilon(\{a_j\}, S_j) + \varepsilon(\{a_l\}, S_j) \leqslant s_j - 1,$$
$$\varepsilon(\{b_j\}, S_j) + \varepsilon(\{b_l\}, S_j) \leqslant s_j - 1,$$
$$\varepsilon(\{a_l\}, S_l) + \varepsilon(\{a_j\}, S_l) \leqslant s_l - 1,$$
$$\varepsilon(\{b_l\}, S_l) + \varepsilon(\{b_j\}, S_l) \leqslant s_l - 1.$$

These four inequalities must be strict. Otherwise, suppose $\varepsilon(\{a_j\}, S_j) + \varepsilon(\{a_l\}, S_j) = s_j - 1$. Then $(a_l, b_j) \in E$, which is contrary to (b). Similarly we can prove that the others are strict too. So we have

$$\varepsilon(\{a_j, b_j, a_l, b_l\}, S_j \cup S_l) \leqslant 2(s_j + s_l - 2) - 4,$$

* Figure 2 in the first part of this paper at page 47 of the last issue of this Journal.

$$\varepsilon(S_i \cup S_l, X) \geqslant 4(k-2) - 2(s_i + s_l - 2) + 4 \geqslant 2k - 8.$$

Hence

$$k(k-3) \geqslant \varepsilon(V - X, X) \geqslant (k-2)(k-2) + 2k - 8 > k(k-3),$$

which is in contradiction.

4. $y = k - 1$.

$y = k - 1$ cannot hold since

$$k(k-3) \geqslant \varepsilon(V(C) - X, X) \geqslant (k-1)(k-2) > k(k-3).$$

As we have said the proof of the theorem is for the case that $r = 1$. Now we prove briefly that the theorem also holds when $r = 2$ or $r = 3$.

Assume that there exists another vertex u_0 besides v_0 in R. Then the number of the inner vertices of C is reduced. Thus for any interval S_i, $\varepsilon\left(\{a_i, b_i\}, \bigcup_{j \neq l} S_j\right)$ must be reduced and the number which is reduced is exactly $r - 1$. If $s_i = 2$, since at most one of a_i and b_i can be adjacent to u_0, we still have $\varepsilon(S, X) \geqslant k - 3$. If $s_i = 3$ and $(a_i, b_i) \in E$, by the proof of Lemma 7, we shall have $\varepsilon(S_i, X) \geqslant k - 3$, if $(a_i, b_i) \bar{\in} E$ and $(a_i, u_0) \in E$, $(b_i, u_0) \in E$, then we may take the chain $\{a_i, u_0, b_i\}$ as edge (a_i, b_i). In this case which reduces to the preceding case and we still have $\varepsilon(S_i, X) \geqslant k - 3$. Therefore the result of Lemma 7 still holds if there exist more than one isolated vertex in R.

If $s_i \geqslant 4$, let $S_i = \{a_i, c_1, c_2, \cdots, c_t, b_i\}$. We have used the following types of chords in the proof of Lemma 8:

(a) $\varepsilon(\{a_i, b_i\}, X)$ and $\varepsilon(\{c_g\}, X)$, where c_g is the rAB-vertex of S_i;

(b) $\varepsilon(\{c_1, b_i\}, X)$ or $\varepsilon(\{a_i, c_t\}, X)$;

(c) $\varepsilon(\{c_1, c_t\}, X)$.

They all use an A-vertex (or a rA-vertex) and a B-vertex (or a rB-vertex). Besides, it may use a rAB-vertex.

For rAB-vertex c_g whether $r = 1$ or $r > 1$, we have $\varepsilon(\{c_g\}, X) \geqslant 1$. Thus propositions 1 and 2 in Lemma 8 still hold.

When using an A-vertex (or a rA-vertex) and a B-vertex (or a rB-vertex) of S_i, even if u_0 is not joined to these two vertices at the same time, we still have $\varepsilon(S_i, X) \geqslant k - 3$ since the number of the inner vertices is reduced. Otherwise, if u_0 is joined to these two vertices at the same time, then u_0 cannot be joined to the A-vertex or B-vertex or rA-vertex of rB-vertex of other interval. So for any S_j, $j \neq i$, we have $\varepsilon(S_j, X) \geqslant k - 2$. This is contrary to $(k-2)(k-1) \leqslant k(k-3)$. This proves that Lemma 8 holds if $r > 1$. Hence it guarantees that the technique of the preceding proof is efficient for $r > 1$.

Part II. R contains no isolated vertices.

Let $Q = \{q_1, q_2, \cdots, q_g\}$ be a chain in R, $t(Q)$ be the number of occurrences of ordered pairs (c_i, c_j) of the vertices of C, such that c_i is joined to one of q_1 and q_g, c_j is joined to the other, and

$$\varepsilon(\{q_1, q_g\}, \{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\}) = 0.$$

$\{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\}$ is said to be an exchange-interval of C relative to Q.

Lemma 9 *Let $Q = \{q_1, q_2, \cdots, q_g\}$ be a chain with maximal length in R such that $t(Q) = t \geq 2$, and $S_1, S_2, \cdots, S_t$ denote the exchange-intervals of C relative to Q. For any S_i and S_j, $i \neq j$, $S_i = \{c_l, c_{l+1}, \cdots, c_m\}$, $S_j = \{c_w, c_{w+1}, \cdots, c_z\}$. Then*

$$\varepsilon(\{c_l, c_m\}, S_j) \leq s_j - g.$$

Proof. Obviously

$$|N(c_l) \cap S_j| \leq s_j - g.$$

Let $A = N(c_l) \cap S_j$, $B = N(c_m) \cap S_j$. Put

$$A^+ = \{c_f \in S_j,\ h+1 \leq f \leq h+g \mid c_h \in A\},$$

$$A^- = \{c_f \in S_j,\ h-g \leq f \leq h-1 \mid c_h \in A\}.$$

Then

$$B \subseteq S_j - \{A^+ \cup A^- \cup \{c_z, c_{z-1}, \cdots, c_{z-g+1}\}\}.$$

Since

$$|\{A^+ \cup A^- \cup \{c_z, c_{z-1}, \cdots, c_{z-g+1}\}\}| \geq |A| + g,$$

so

$$|B| \leq s_j - |A| - g.$$

That is

$$\varepsilon(\{c_l, c_m\}, S_j) = |A| + |B| \leq s_j - g.$$

Now we return to the proof of the theorem for part II.

We know that $|V(C)| \geq 3k$ if $n \leq 3k + 3$[3]; thus $|R| \leq 3$. Since R contains no isolated vertices, there is only one component. Let the chain in R be $Q = \{q_1, q_2, \cdots, q_g\}$; thus $g = 2$ or $g = 3$.

Case A. $g = 2$.

If $|(N(q_1) \cap V(C)) \cup (N(q_2) \cap V(C))| \geq k$, the vertices q_1 and q_2 are contracted to one vertex q. So this case can be proved in the same way as part I. In the following we only consider the case that

$$|(N(q_1) \cap V(C)) \cup (N(q_2) \cap V(C))| = k - 1.$$

We have that $N(q_1) \cap V(C) = N(q_2) \cap V(C)$ and $|N(q_1) \cap V(C)| = |N(q_2) \cap V(C)| = k - 1$. Put $N(q_1) \cap V(C) = X$. The cycle C is divided into $k-1$ open intervals $S_1, S_2, \cdots, S_{k-1}$ by the vertices of X. Let $S_i = \{c_l, c_{l+1}, \cdots, c_m\}$. By Lemma 9, we have

$$\varepsilon(\{c_l, c_m\}, S_j) \leq s_j - 2 \quad (j \neq i),$$

$$\varepsilon\left(\{c_l, c_m\}, \bigcup_{j\neq i} S_j\right) \leqslant \sum_{j\neq i} (s_j - 2)$$
$$\leqslant 3k + 1 - (k-1) - s_i - 2(k-2) \leqslant 4.$$
$$\varepsilon(S_i, X) \geqslant \varepsilon(\{c_l, c_m\}, X)$$
$$\geqslant 2k - 2(s_i - 1) - 4 = 2(k - s_i - 1).$$
$$\varepsilon\left(\bigcup_{i=1}^{k-1} S_i, X\right) \geqslant \sum_{i=1}^{k-1} 2(k - s_i - 1)$$
$$= k^2 - 3k + 2 + (k-6)(k+1) + 2.$$

If $k \geqslant 6$, we have

$$\varepsilon\left(\bigcup_{i=1}^{k-1} S_i, X\right) > k^2 - 3k + 2.$$

On the other hand,

$$\varepsilon\left(X, \bigcup_{i=1}^{k-1} S_i\right) \leqslant (k-2)(k-1) = k^2 - 3k + 2,$$

which is in contradiction.

Case B. $g = 3$.

The proof of this case is analogous to that of $g = 2$.

The proof of the theorem is now completed.