On Perfect Neighborhood and Irredundant Sets in Trees

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Abstract: For any tree T, we give an Algorithm (A) with polynomial time complexity to get the perfect neighborhood set in T. Then we prove that S, which is a perfect neighborhood set of T and $|S| = \Theta(T)$, is also a maximal irredundant set in T. We present an Algorithm (B) with polynomial time complexity to form the perfect neighborhood set from the maximal irredundant set in T, and point out that T has an independent perfect neighborhood set U and $|U| \leq |S|$ ($|S|$ the cardinality of a maximal irredundant set of T).

Key words: perfect neighborhood set; irredundant set; independent; tree.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set S, and v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. Let $S \subseteq V$. A vertex v of G is called S-perfect if $|N[v] \cap S| = 1$. The set S is defined as a perfect neighborhood set of G if every vertex of G is S-perfect or adjacent to an S-perfect vertex. Equivalently, S is a perfect neighborhood set of G if for every $u \in V$, there exists a $v \in N[u]$ such that $|N[v] \cap S| = 1$. The lower (upper) perfect neighborhood number $\theta(G)$ ($\Theta(G)$) of G is defined to be the minimum (respectively, maximum) cardinality among all perfect neighborhood sets of G. Let $\varphi(G)$ and $\Phi(G)$ denote the smallest and largest cardinalities of independent perfect neighborhood sets of G, respectively.

For $s \in S$, $pn(s, S) = N[s] - N[S - \{s\}]$ is the S-private neighborhood of s, where for $A \subseteq V$, $N[A]$ denotes the union of closed neighborhoods of elements of A. Elements of $pn(s, S)$ are called S-private neighbors of s. An S-private neighbor of s is either s itself, in which case s is an isolated vertex of $G[S]$, or is a neighbor of s in $V - S$ which is not adjacent to any vertex of $S - \{s\}$. This latter type will be called an external S-private neighbor of s. The set S is irredundant if for all $s \in S, pn(s, S) \neq \varnothing$. Equivalently, S is irredundant if every vertex of S is S-perfect or adjacent with an S-perfect vertex. The parameter $ir(G), IR(G)$ are the smallest and largest cardinalities of maximal irredundant sets of G. Graph theory terminology not presented here can be found in [1].

Perfect neighborhood set was first introduced by Fricke et al.[2]. They defined perfect neighborhood set while studying functions $f : V \to \{0, 1\}$ subject to various edge and neighborhood

conditions. Then the closely relations among perfect neighborhood, dominating set and irre-dundant set were found by Cockaynae et al.[3]. Furthermore, the quantitative relations of the dominating number and perfect neighborhood number were widely researched. Since perfect neighborhood set is deeply related to the other concepts of graph, many complicated characters of graph be revealed by researching perfect neighborhood set. On the other hand, perfect neigh-borhood set was extensively applied to computer network, physiology, transportation, etc.[4,6,7]. So researching perfect neighborhood is very important to develop the theory and application of graph.

By the definitions, perfect neighborhood sets and irredundant sets are clearly related. We begin with the following results.

Proposition 1 $|N[x] \cap S| = 1$ *if and only if* $x \in pn(s, S)$ *for some* $s \in S$.

Proof Since $|N[x] \cap S| = 1$, there exists only one vertex $s \in S$ such that $\{s\} = N[x] \cup S$, then $x \in N[s]$ and $x \notin N[S - \{s\}]$. Clearly, $x \in pn(s, S)$ for $s \in S$.

Conversely, since $x \in N[s]$ and $x \notin N[S - \{s\}]$ for some $s \in S$, we have $s \in N[x]$. Thus $N[S - \{s\}] \cap N[x] = \varnothing$. Hence $|N[x] \cap S| = 1$.   □

Proposition 2 *If* S *is perfect neighborhood set, then* S *is irredundant.*

Proof Suppose S is not irredundant. Then there exists $s \in S$ with $pn(s, S) = \varnothing$, which is equal to neither s nor any neighbor of s is S-perfect. Hence S is not a perfect neighborhood set, which is a contradiction.   □

The following relations involving perfect neighborhood set parameters and irredundant set parameters were obtained in [2,3,4].

Lemma 3[2,4] *For all graphs* G, $IR(G) \geq \Theta(G) \geq \Phi(G)$.

Lemma 4[3] *For any tree* T, $\theta(T) \leq \varphi(T) \leq ir(T)$.

The relations of $ir(G), IR(G), \theta(G)$ and $\Theta(G)$ are well-studied in literature. In this paper, we do not discuss the quantitative relations of parameters anymore. In fact, it is important how to form the maximal irredundant set from the perfect neighborhood set and form the perfect neighborhood set from the maximal irredundant set in trees. In Section 2 we give two main algorithms with polynomial time complexity and theorems to answer the questions.

2. Main results

First, we introduce two lemmas about the maximal irredundant set and perfect neighbor-hood set. An irredundant set S of a graph G induces a partition of the vertex set. Specifically,

$$V = Z_S \cup Y_S \cup F_S \cup E_S \cup Q_S \cup R_S,$$

where $Z_S = \{v \in V | v \text{ is isolated in } G[S]\}$,

$$Y_S = S - Z_S,$$

$$F_S = \{v \in V | v \text{ is an external } S\text{-private neighbour of a vertex of } Z_S\},$$

$$E_S = \{v \in V | v \text{ is external } S\text{-private neighbor of a vertex of } Y_S\},$$

$$R_S = \{v \in V | N[v] \cap S = \varnothing\},$$

$$Q_S = \{v \in V - S | |N(v) \cap S| \geq 2\}.$$

Lemma 5[5] *The irredundant set S of a graph G is maximal if and only if for each $r \in N[R_S]$, there exists $s \in S$ such that $pn(s, S) \subseteq N[r]$.*

If a vertex u is a root of tree, $p(u)$ will denote the parent of u and $C(u)$ the set of children of u. For subsets A, B of V, we say that A dominates B ($A \succ B$) if $B \succ N[A]$.

Lemma 6[3] *The irredundant set S is a perfect neighborhood set if and only if $E_S \cup F_S \cup Z_S \succ V$.*

Let $G = (V, E)$ be a tree and $|v(G)| = n$ ($n > 2$). We state the first main algorithm.

Form S, which is perfect neighborhood set of a tree G and satisfies $|S| = \Theta(G)$.

Algorithm (A) $G \to S$

Step 1. Set $V = \{v_1, v_2, \cdots, v_n\}$ ($n > 2$), $S \leftarrow \varnothing$, $M \leftarrow \varnothing$, $N \leftarrow \varnothing$, $V_0 \leftarrow \varnothing$.

Step 2. If $d(v_i) = 1$, $i = 1, 2, \cdots, n$, then $S \leftarrow S \cup \{v_i\}$, $M \leftarrow M \cup \{v_i\}$.

Step 3. While $d(v_i) > 1$, if $d(v_i, M) = 2$, then $S \leftarrow S \cup \{v_i\}$, $N \leftarrow N \cup \{v_i\}$.

Step 4. Set $V_0 = \{v_i \in V - N[S] | d(v_i) > 1\}$. While $V_0 \neq \varnothing$, if exists $v_i \in V_0$ such that $d(v_i, N) = 2$, then $S \leftarrow S \cup \{v_i\}$, $N \leftarrow N \cup \{v_i\}$, $V_0 = \{v_i \in V - N[S] | d(v_i) > 1\}$.

Step 5. $S(= M \cup N)$ is the perfect neighborhood set of G and satisfies $|S| = \Theta(G)$.

Theorem 7 *The Algorithm (A) can be implemented in $O(n^2)$ time.*

Proof It is easy to check that the Step 2 can be done in $O(n)$ time.

For the augment S in the Step 3, there exist n vertices at most. Since every vertex need to compute n times at most, this step can then be performed in $O(n^2)$ time.

Since $|V_0| < n$ and $|V| < n$, the overall time needed to update S can be done in $O(n^2)$ time.

It follows that the running time of the algorithm is bounded by $O(n^2)$. □

From the Algorithm (A), we can get the first main theorem for every tree G.

Theorem 8 *If S is a perfect neighborhood set of G and $|S| = \Theta(G)$, then S is a maximal irredundant set of G.*

Proof If $|V(G)| \leq 2$, then $S = \{v\}$ and $\Theta(G) = 1$. The conclusion is obviously.

So we assume $|V(G)| > 2$. Since S is a perfect neighborhood set and $|S| = \Theta(G)$, the set S can be obtained by the above-mentioned algorithm. Consider the following two supposes.

Suppose $R_S = \varnothing$. It is easy to verify that S is a maximal irredundant set of G by the Lemma 5. Suppose $R_S \neq \varnothing$. Then there at least exists one vertex $v_i \in R_S$. It means $N[v_i] \cap S = \varnothing$, which is equal to $d(v_i, S) \geq 2$. By the Algorithm (A), we have $d(v_i, M \cup N) \geq 2$. If $d(v_i, M \cup N) = 2$, then $d(v_i, M) = 2$ or $d(v_i, N) = 2$, and $S_0 = S \cup \{v_i\}$ is the perfect neighborhood set of G and satisfies $|S_0| = \Theta(G)$, which is a contradiction. If $d(v_i, M \cup N) > 2$, we must continue to augment

S, then $V_0 \neq \varphi$, which is also a contradiction. Hence $R_S = \varnothing$, which satisfies the conditions of Lemma 5. □

It is easy to get a maximal irredundant set of a tree through the Algorithm (A) and Theorem 8. So we can give the following algorithm, which maximize a irredundant set in a tree. This algorithm is similar to Algorithm (A) and also of polynomial time complexity.

Let G be a tree and $|V(G)| = n(n > 2)$, and S_0 be an irredundant set of G.

Algorithm (A') $S_0 \rightarrow S$

Step 1. Set $V = \{v_1, v_2, \cdots, v_n\}$, $M \leftarrow \varnothing$.

Step 2. For $i = 1, 2, \cdots, n$, if there exists $v_i \in V - S_0$ such that $d(v_i, S_0) = 2$, then

$$S_0 \leftarrow S_0 \cup \{v_i\},$$
$$M \leftarrow M \cup \{v_i\}.$$

Step 3. While $R_S \neq \varnothing$, if there exists $v_i \in R_S$ such that $d(v_i, M) = 2$, then

$$S_0 \leftarrow S_0 \cup \{v_i\},$$
$$M \leftarrow M \cup \{v_i\},$$
$$R_S = \{v \in V | N[s] \cap S_0 = \varnothing\}.$$

Step 4. Set $S \leftarrow S_0$. S is a maximal irredundant set of G.

We now state the second main algorithm as follows.

Let T be a tree. Suppose $Y_S \neq \varnothing$ and let $Y_1, Y_2, \cdots, Y_n$ be the vertex sets of the components of $T[Y_S]$. Root T at a vertex r and for $i = 1, 2, \cdots, n$, let $y_i \in Y_i$ be the (unique) vertex in Y_i such that $d(r, y_i) = d(r, Y_i)$. We may assume the ordering of the Y_i to be such that $d(r, y_i) \leq d(r, Y_i)$ whenever $i \leq j$. Let $E_i = \cup_{y \in Y_i} pn(y, S)$ and observe that $E_S = \cup_{i=1}^n E_i$. For any subset $Y = \{x_1, x_2, \cdots, x_t\}$ of Y_S, let $W(Y)$ denote any set $\{w_1, w_2, \cdots, w_t\}$ where $w_i \in pn(x_i, S)$.

Algorithm (B) $S \rightarrow U$

Step 1. $W_1 \leftarrow W(Y)$, $U \leftarrow Z_S \cup W_1$, $A, B, D, M \leftarrow \varnothing$.

Step 2. For $i = 1, 2, \cdots, n$, do

(1) If $p(p(y_i)) \notin W_1 \cup \cdots \cup W_{i-1} \cup A \cup B \cup D$ or $p(y_i) \notin pn(y_i, S)$, then

$$W_i \leftarrow W(Y_i);$$

If $p(p(y_i)) \in B$ and $p(y_i) \in pn(y_i, S)$, then

$$W_i \leftarrow W(Y_i - \{y_i\}) \cup \{p(y_i)\},$$
$$B \leftarrow B - \{p(p(y_i))\},$$
$$D \leftarrow D \cup \{p(p(y_i))\};$$

If $p(p(y_i)) \in W_1 \cup \cdots \cup W_{i-1} \cup A \cup U$ and $\{p(y_i)\} \subset pn(y_i, S)$, then choose any $w \in pn(y_i, S) - \{p(y_i)\}$, $W_i \leftarrow \{w\} \cup W(Y_i - \{y_i\})$;

If $p(p(y_i)) \in W_1 \cup \cdots \cup W_{i-1}$ and $\{p(y_i)\} = pn(y_i, S)$, then

$$W_i \leftarrow W(Y_i - \{y_i\}),$$
$$A \leftarrow A \cup \{p(y_i)\};$$

If $p(p(y_i)) \in A \cup D$ and $\{p(y_i)\} = pn(y_i, S)$, then

$$W_i \leftarrow W(Y_i - \{y_i\}),$$
$$B \leftarrow B \cup \{p(y_i)\}.$$

(2) Set $U \leftarrow U \cup W_i$.

Step 3. While $B \neq \varnothing$, for $b \in B$, if each vertex in $C(b)$ is dominated by $U \cup F_U$, then $B \leftarrow B - \{b\}$; If there exists $v \in C(b)$ which is not dominated by $U \cup F_U$, then

$$M \leftarrow M \cup \{v\},$$
$$B \leftarrow B - \{b\}.$$

Step 4. Set $U \leftarrow U \cup M$. U is the perfect neighborhood of T.

We omit the proof of the following result since it is similar to that of the Theorem 7.

Theorem 9 *The Algorithm (B) can be implemented in $O(n^2 \log n)$ time.*

From the Algorithm (B), we can get the second main theorem for every tree T.

Theorem 10 *If S is a maximal irredundant set of T, then T has an independent perfect neighborhood set U with $|U| \leq |S|$.*

Proof Suppose $Y_S = \varnothing$. Then $E_S = \varnothing$ and $R_S = \varnothing$. In this case $V = Z_S \cup F_S \cup Q_S$ and is dominated by $Z_S \cup F_S$. The result follows from Lemma 6 since $S = Z_S$ is independent.

Suppose $Y_S \neq \varnothing$. The reader is referred to [3], for a similar proof.   □

3. Conjecture

Observe that perfect (independent perfect) neighborhood sets and maximal irredundant sets are clearly related. Based on Lemmas 3 and 4, and Theorems 8 and 10, we state the following conjecture.

Conjecture 11 For all trees T, $\Phi(T) = \Theta(T) = IR(T)$.

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关于树的完美邻域集与无冗余集

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摘要: 本文首先给出了求树图 T 的完美邻域的多项式时间复杂度算法 (A), 并在此基础上证明了当 S 是 T 的任一完美邻域且 $|S| = \Theta(T)$, 则 S 是 T 的一极大无冗余集. 然后给出了由 T 的一极大无冗余集生成完美邻域集的多项式时间复杂度算法 (B), 并依此算法证明了若 S 为 T 的任一极大无冗余集, 则 T 存在一独立完美邻域集 U 且 $|U| \leq |S|$.

关键词: 完美邻域集; 无冗余集; 独立; 树.