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On Signed Edge Domination of Graphs

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Abstract: Let $\gamma_s'(\mathrm{G})$ and $\gamma_l'(\mathrm{G})$ be the numbers of the signed edge and local signed edge domination of a graph G [2], respectively. In this paper we prove mainly that $\gamma_s'(G) \le \lfloor \frac{11}{6}n - 1 \rfloor$ and $\gamma_l'(G) \le 2n - 4$ hold for any graph G of order $n(n \ge 4)$, and pose several open problems and conjectures.

Key words: local signed edge domination function; local signed edge domination number; signed edge domination function; signed edge domination number.
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1. Introduction

We use Bondy and Murty[1] and Xu[2] for terminology and notation not defined here and consider simple graphs only.

Let $G = (V, E)$ be a graph. If $e = uv \in E$, then $N_G[e] = \{u'v' \in E | u' = u \text{ or } v' = v\}$ is called the closed edge-neighbourhood of e in G, and $N_G(e) = N_G[e] \backslash \{e\}$ is the open one. If $v \in V$, then $E_G(v) = \{uv \in E | u \in V\}$. For simplicity, sometimes, $N_G[e]$ and $E_G(v)$ are denoted by $N[e]$ and $E(v)$, respectively. In [2] we introduced the signed edge domination of graphs as follows:

Definition 1[2] *Let $G = (V, E)$ be a nonempty graph. A function $f : E \to \{+1 -1\}$ is called the signed edge domination function (SEDF) of G if $\sum_{e' \in N[e]} f(e') \ge 1$ for every $e \in E(G)$. The signed edge domination number of G is defined as $\gamma_s'(G) = \min\{\sum_{e \in E} f(e) | f$ is an SEDF of $G\}$.*

And define $\gamma_s'(\bar{K}_n) = 0$ for all totally disconnected graphs $\overline{K}_n$.

Next we introduce a new concept of edge domination in graphs:

Definition 2 *Let $G = (V, E)$ be a graph without isolated vertices. A function $f : E \to \{+1 -1\}$ is called the local signed edge domination function (LSEDF) of G if $\sum_{e \in E(v)} f(e) \ge 1$ for every $v \in V(G)$. The local signed edge domination number of G is defined as $\gamma_l'(G) = \min\{\sum_{e \in E} f(e) | f$ is an LSEDF of G }. Obviously, $|\gamma_l'(G)| \le |E(G)|$. It seems natural to define $\gamma_l'(\bar{K}_n) = 0$ for all totally disconnected graphs $\overline{K}_n$.*

Clearly, $\gamma_l'(G_1 \cup G_2) = \gamma_l'(G_1) + \gamma_l'(G_2)$ and $\gamma_s'(G_1 \cup G_2) = \gamma_s'(G_1) + \gamma_s'(G_2)$ for any two disjoint graphs G_1 and G_2. In comparison with the above two definitions, we see that each

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LSEDF of G is an SEDF of G, and hence we have

Lemma 1 *For all graphs G, $\gamma'_s(G) \le \gamma'_l(G)$.*

By Definition 2, we have

Lemma 2 *For all graphs G, $v \in V(G)$, then $\gamma'_l(G) \le \gamma'_l(G-v) + d_G(v)$.*

In recent years, some kinds of domination in graphs have been investigated. Most of those belong to the vertex domination of graphs, such as signed domination[3,4], minus domination[5], majority domination[6], domination[7], etc. A few of results have been obtained about the edge domination of graphs[2]. In this paper we discuss mainly the upper bounds for (local) signed domination numbers of graphs, and pose several open problems and conjectures.

A graph G is said to be a θ-graph if G is a connected graph with degree sequence $d = (2, 2, \cdots, 2, 3, 3)$. That is, a θ-graph consists of a cycle and a path such that two end-vertices of the path are on the cycle.

Lemma 3 *Any θ-graph contains a cycle of even length (even cycle).*

Proof It is obvious.

Lemma 4 *For any graph G, if $\delta(G) \ge 3$, then G contains a θ-graph as subgraph, and hence G contains an even cycle.*

Proof Without loss of generality, we may suppose that G is a connected graph. Let T be a spanning tree of G, and v a pendant-vertex of T. That is, $d_T(v) = 1$. Since $\delta(G) \ge 3$, there exist at least two vertices u and w such that uv, $wv \in E(G) \backslash E(T)$. Define $H = T + \{uv, wv\}$. Then obviously, H contains a θ-graph as subgraph, which is the maximum 2-connected subgraph of H. In view of $H \subseteq G$ and Lemma 3, we have completed the proof of Lemma 4. □

For a graph G, if there exist some subgraphs G_i $(i = 1, 2, \cdots, q)$ of G such that $E(G) = U_{i=1}^{q} E(G_i)$ and $E(G_i) \cap E(G_j) = \phi$ $(1 \le i \ne j \le q)$, then we say that G can be decomposed into $G_1, G_2, \cdots, G_q$.

Lemma 5 *Any forest F can be decomposed into some paths P_{m_i} $(i = 1, 2, \cdots, q; m_i \ge 2)$ such that all end-vertices of all these paths are pairwise distinct.*

Proof We use the induction on $m = |E(F)|$.

It is trivial for $m = 0$. Suppose that the lemma is true for all forests of size $k \le m - 1$. Now we consider a forest F of size m $(m \ge 1)$. In F we choose a path P_t $(t \ge 2)$ whose end-vertices are two pendant-vertices of F.

Let $F_1 = F - E(P_t)$. Clearly, F_1 is a forest of size at most $m - 1$. By the induction hypothesis, F_1 can be decomposed into some paths P_{m_i} $(i = 1, 2, \cdots, q; m_i \ge 2)$ such that all end-vertices of all these paths are pairwise distinct. Thus, F can be decomposed into the paths P_{m_i} $(i = 1, 2, \cdots, q)$ and P_t, all end-vertices of the $q + 1$ paths are pairwise distinct. So, the lemma is true for all forests F of size m. We have completed the proof of Lemma 5. □

For cycles $C_n (n \geq 3)$ and complete graphs $K_n (n \geq 1)$, we have

Lemma 6[8] *$\gamma'_s(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$ and $\gamma'_s(K_n) = \lceil \frac{n-1}{2} \rceil$.*

2. Main results

We first give an upper bound of $\gamma'_l(G)$ for all graphs G.

Theorem 1 *For any graph G of order n $(n \geq 4)$, $\gamma'_l(G) \leq 2n - 4$, and this bound is sharp.*

Proof We use the induction on $m = |E(G)|$. The result is clearly true for $m \leq 3$ (note that $n \geq 4$).

Suppose that the theorem is true for all graphs of size $k (k \leq m - 1)$. Now we consider a graph G with $|E(G)| = m$. By Lemma 2, we may suppose $\delta(G) \geq 1$.

Case 1. $\delta(G) \leq 2$

There exists a vertex $v \in V(G)$ such that $d_G(v) = \delta(G) \leq 2$. Note that $|E(G - v)| \leq m - 1$. By the induction hypothesis, we have $\gamma'_l(G - v) \leq 2(n - 1) - 4 = 2n - 6$. We see from Lemma 2 that $\gamma'_l(G) \leq \gamma'_l(G - v) + d_G(v) \leq 2n - 6 + 2 = 2n - 4$.

Case 2. $\delta(G) \geq 3$

We see from Lemma 4 that G contains an even cycle C. Let $H = G - E(C)$. By the induction hypothesis, H has an LSEDF f with $\sum_{e \in E(H)} f(e) \leq 2n - 4$. Extending f from H by signing $+1$ and -1 alternatively along C, we obtain an LSEDF for G, and hence $\gamma'_l(G) \leq 2n - 4$.

Since $\gamma'_l(K_{2,n-2}) = 2n - 4 (n \geq 4)$, the upper bound given in Theorem 1 is sharp. We have completed the proof of Theorem 1. □

For signed edge domination number, by Theorem 1 and Lemma 1, we have

Corollary 1 *For all graphs G of order $n (n \geq 3)$, $\gamma'_s(G) \leq 2n - 4$.*

For the lower bound of $\gamma'_l(G)$, we have

Corollary 2 *For all graphs G of order n, if $\delta(G) \geq 1$, then $\gamma'_l(G) \geq \lceil \frac{n}{2} \rceil$.*

Proof Let f be an LSEDF of G such that $\gamma'_l(G) = \sum_{e \in E(G)} f(e)$. For every edge $e = uv \in E(G)$, $e \in E(u)$ and $e \in E(v)$. Thus, we have

$$\gamma'_l(G) = \sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in E(v)} f(e) \geq \frac{1}{2} \sum_{v \in V(G)} 1 = \frac{n}{2}.$$

Note that $\gamma'_l(G)$ is an integer. The proof is complete. □

We know from Definition 2 that the inequality $\gamma'_l(G) \leq |E(G)|$ holds for all graphs G.

This equality holds for some graphs only.

Theorem 2 *Let G be a graph, $D_3(G) = \{v \in V(G) | d_G(v) \geq 3\}$. Then $\gamma'_l(G) = |E(G)|$ if and only if either $D_3(G) = \phi$ or $D_3(G)$ is an independent set of G.*

Proof It is not difficult to check that the following four statements are equivalent:

(1) $\gamma_l'(G) = |E(G)|$;

(2) For any LSEDF f of G satisfying $\gamma_l'(G) = \sum_{e\in E(G)} f(e)$ and every edge $e \in E(G)$, $f(e) = 1$;

(3) For any two vertices u and v of degree at least 3, $uv \notin E(G)$;

(4) $D_3(G) = \phi$ or $D_3(G)$ is an independent set of G.

We have completed the proof of Theorem 2. □

Next we give an upper bound of $\gamma_s'(G)$ for general graphs G.

Theorem 3 *For any graph G of order n, $\gamma_s'(G) \le \lfloor \frac{11}{6}n - 1\rfloor$.*

Proof Without loss of generality, we may suppose that G is a connected graph and $n \ge 4$.

When G contains a Hamilton cycle C_n, let $T = C_n$.

When G has no Hamilton cycle, we choose a spanning tree T of G such that $|\{v \in V(T)|d_T(v) = 1\}|$ is as small as possible (taken over all spanning tree of G). It is easy to see that any two pendant-vertices of T are not adjacent in G. (Otherwise, there exists a spanning tree T' of G such that T' contains less pendant-vertices than T, which contradicts the choice of T in G.)

Thus, $n - 1 \le |E(T)| \le n$.

For every edge $e \in E(T)$, define $f(e) = +1$.

Let $A = \{v \in V(T)|d_T(v) = 1\}$, note that $A = \phi$ when $T = C_n$.

$$T_0 = T\backslash A, A_0 = \{u \in V(T_0)|d_{T_0}(u) = 1\} \text{ (it is possible that } A_0 = \phi).$$

For each vertex $u_0 \in A_0$, we choose exactly one edge $e_0 \in E(u_0)\backslash E(T)$ when $E(u_0)\backslash E(T) \ne \phi$, where $E(u_0) = \{u_0u \in E(G)|u \in V(G)\}$. Let M be the set of all edges chosen. Clearly, $|M| \le |A_0| \le |A|$ and $A \cap A_0 = \phi$, thus $|M| \le \lfloor \frac{n}{2}\rfloor$.

For every edge $e \in M$, we define $f(e) = +1$.

It is easy to check the following statements:

For every nonpendant-edge e of T, $N_G[e]$ contains at least three edges of T. For any pendant-edge e of T, $e = uv \in E(T)$ with $d_T(u) = 1$, when $d_G(v) \ge 3$; $N_G[e]$ has at least three edges in $E(T) \cup M$, when $d_G(v) = 2$ (note that $d_G(v) \ne 1$); $N_G[e]$ contains two edges of T. For every edge $e \in E(G)\backslash E(T)$, since any two vertices of A are not adjacent in G, $N_G[e]$ contains at least three edges of T.

Write $G_0 = G - (E(T) \cup M)$.

If there exist even circuits in G_0, then we choose some pairwise edge-disjoint even circuits, denoted by H_i $(1 \le i \le t)$, so that the graph $G_1 = G_0 - \cup_{i=1}^{t} E(H_i)$ contains no even circuit. If there is no even circuits in G_0, then $G_1 = G_0$.

For each even circuit H_i, we define f by signing $+1$ and -1 alternatively along H_i $(1 \le i \le t)$.

Since G_1 does not contain any even circuit, any two odd cycles in G_1 are vertex-disjoint. (Otherwise, there exists an even circuit in G_1, which is impossible.)

Let $C_{r_i} (1 \le i \le s)$ be all odd cycles of G_1, where $r_i \ge 3$ is odd for each i. Noting that $V(C_{r_i}) \cap V(C_{r_j}) = \phi (1 \le i \ne j \le s)$, we have $s \le \lfloor \frac{n}{3}\rfloor$.

For every C_{r_i}, let M_i be a maximum matching of C_{r_i}, and define f as follows:

$$f(e)=\begin{cases} -1, & \text{when } \ e\in M_i \\ +1, & \text{when } \ e\in E(C_{r_i})\backslash M_i \end{cases}$$

Clearly, $\sum_{e\in E(C_{r_i})} f(e)=1$ for each i $(1\le i\le s)$.

Let $F=G_1-\cup_{i=1}^{s}E(C_{r_i})$. Obviously, F is a forest. By Lemma 5, F can be decomposed into some paths such that all end-vertices of these paths are pairwise distinct. These paths are written as P_{m_i} $(m_i\ge 2, 1\le i\le q)$, namely, $E(F)=\cup_{i=1}^{q}E(P_{m_i})$ and $E(P_{m_i})\cap E(P_{m_j})=\phi$ $(1\le i\ne j\le q)$.

For every path $P_{m_i}(1\le i\le q)$, $m_i\ge 2$, let N_i be a maximum matching of P_{m_i}. When $e\in N_i$, define $f(e)=-1$; when $e\in E(P_{m_i})\backslash N_i$, define $f(e)=+1$. Note that $|N_i|=\lceil\frac{m_i}{2}\rceil\ge |E(P_{m_i})\backslash N_i|$, we have $-1\le\sum_{e\in E(P_{m_i})} f(e)\le 0$, $i=1,2,\cdots,q$.

We have completed the definition of f on $E(G)$.

Next we check that f is an SEDF of G.

(1) For any edge $e=uv\in E(G)\backslash E(T)$;

Since any two vertices of A are not adjacent in G, thus, $N_G[e]$ contains at least three edges of T. Note that u (also, v) is an end-vertex of at most one path defined before, thus $N_G[e]$ contains at most two pendant-edges of all paths $P_{m_i}(1\le i\le q)$. So, we have $\sum_{e'\in N[e]} f(e')\ge 1$.

(2) For any edge $e=uv\in E(T)$;

When e is not any pendant-edge of T, obviously, $N_G[e]$ contains at least three edges of T. Similarly to (1), we have $\sum_{e'\in N[e]} f(e')\ge 1$.

When $e=uv$ is a pendant-edge of T, where $u\in A$ and $v\in A_0$. If $d_G(v)\ge 3$, then $N_G[e]$ contains at least three edges in $E(T)\cup M$. Similarly to (1), we have $\sum_{e'\in N[e]} f(e')\ge 1$. If $d_G(v)$=2 (note that $d_G(v)\ne 1$), $N_G[e]$ contains two edges of T, and v is not end-vertex of any path P_{m_i} $(1\le i\le q)$. Thus $N_G[e]$ contains at most one pendant-edge in $\cup_{i=1}^{q}E(P_{m_i})$, and we have $\sum_{e'\in N[e]} f(e')\ge 1$.

So, f is an SEDF of G. Note $n-1\le |E(T)|\le n$. When $T=C_n, A_0=\phi$ and hence $M=\phi$; when T is a spanning tree of G, $|M|\le\lfloor\frac{n}{2}\rfloor$. These imply $|E(T)|+|M|\le n-1+\lfloor\frac{n}{2}\rfloor$.

Note that $s\le\lfloor\frac{n}{3}\rfloor$, we have

$$\begin{aligned}\sum_{e\in E(G)} f(e) &= |E(T)|+|M|+\sum_{i=1}^{t}\sum_{e\in E(H_i)} f(e)+\sum_{i=1}^{s}\sum_{e\in E(C_{r_i})} f(e)+\sum_{i=1}^{q}\sum_{e\in E(P_{m_i})} f(e) \\ &\le n-1+\lfloor\frac{n}{2}\rfloor+0+s+0\le\lfloor\frac{11}{6}n-1\rfloor.\end{aligned}$$

Therefore, $\gamma_s'(G)\le\sum_{e\in E(G)} f(e)\le\lfloor\frac{11}{6}n-1\rfloor$. We have completed the proof of Theorem 3. □

In particular, if G is a bipartite graph, then in the proof of Theorem 3, $s=0$. So we have

Corollary 3 *For any bipartite graph G of order n, $\gamma_s'(G)\le\lfloor\frac{3}{2}n-1\rfloor$.*

If a graph G has a 2-regular spanning subgraph H, then in the proof of Theorem 3, let $T=H$, and hence $M=\phi$. Analogously, we have $\gamma_s'(G)\le\sum_{e\in E(G)} f(e)\le |E(H)|+s\le n+\lfloor\frac{n}{3}\rfloor$, where $n=|V(G)|$. Namely, we have

Corollary 4 *Let G be a graph of order n. If G has a 2-regular spanning subgraph, then*

$$\gamma_s'(G) \le \lfloor \frac{4}{3}n \rfloor.$$

3. Some open problems and conjectures

We know from Lemma 1 that $\gamma_s'(G) \le \gamma_L'(G)$. A natural problem is

Problem 1 Characterize the graphs which satisfy the equality $\gamma_s'(G) = \gamma_L'(G)$.

Although in [2] we have determined the exact value of $\psi(m) = \min\{\gamma_s'(G)|G$ is a graph of size $m\}$ for all positive integers m, it seems more difficult to solve the following

Problem 2[2] Determine the exact value of $g(n) = \min\{\gamma_s'(G)|G$ is a graph of order $n\}$ for every positive integer n.

Conjecture 1 For any graph G of order $n(n \ge 1)$, $\gamma_s'(G) \le n-1$.

If it is true, the super bound is the best possible for odd n. For example, let G be the subdivision of the star $K_{1,\frac{n-1}{2}}$, then clearly, $\gamma_s'(G) = |E(G)| = n-1$. (The subdivision of a graph G is the graph obtained from G by subdividing each edge of G exactly once.)

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关于图符号的边控制

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摘要: 设 $\gamma_s'(G)$ 和 $\gamma_l'(G)$ 分别表示图 G 的符号边和局部符号边控制数，本文主要证明了：对任何 n 阶图 $G(n \ge 4)$, 均有 $\gamma_s'(G) \le \lfloor \frac{11}{6}n - 1 \rfloor$ 和 $\gamma_l'(G) \le 2n-4$ 成立，并提出了若干问题和猜想.

关键词: 局部符号边控制函数；局部符号边控制数；符号边控制函数；符号边控制数.