

Nonlinear Energy Transfer from Semidiurnal Barotropic Motion to Near-Inertial Baroclinic Motion

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ABSTRACT

A theoretical investigation into the subharmonic parametric instability of internal gravity waves to progressive surface waves in a continuously stratified medium is presented. A perturbation and multiple scales analysis is utilized and is carried out to the third order. Recent parallel efforts for a two-layer model have shown that surface waves can resonate, exponentially in time, two oblique internal wave trains whose frequencies are nearly subharmonic to that of the surface wave. This analysis is generalized in the current paper to allow for continuous variation of density with depth, as well as planetary rotation. Specific attention is focused on the energy transfer from the semidiurnal barotropic tide to near-inertial baroclinic motion. At low latitudes, linear analysis demonstrates that such a resonant triad indeed exists, suggesting that this mechanism may play a role in the development of the inertial peak seen in many internal wave spectra. At the second order, evolution equations are obtained that demonstrate that the internal waves initially grow exponentially in time. Finally, when the analysis is carried to the third order, it is shown that an equilibrium exists. Nonlinear detuning of the internal wave frequencies results in the internal wave amplitudes reaching a steady-state value, which is readily calculated.

1. Introduction

Numerous mechanisms have been proposed for the generation of internal waves in the ocean. Reviews of the main energy sources are given by Thorpe (1975), Phillips (1966), and LeBlond and Mysak (1978). Conspicuously absent in many of these reviews is the role of the ever-present tide as a possible source in the open ocean. It is true that the generation of baroclinic modes by barotropic flow over sharp topography, such as the continental shelf break, is a mechanism that is now well understood (e.g., Baines 1971; Cox and Sandstrom 1962). However, in regions far removed from coastal boundaries, the tide has not been viewed as a viable source.

The present paper proposes a generation mechanism that provides for the direct transfer of energy, without the aid of topography, from the barotropic tide to a group of baroclinic modes in the near-inertial frequency band. It is within this near-inertial band that most oceanic internal wave energy has been documented to reside. In the Garret and Munk (1975) model spectrum, the frequency band between f and $2f$, where f denotes the inertial frequency, accounts for more than 30% of the total internal wave energy. In spite of numerous attempts, the mechanisms responsible for this clustering

of energy near the inertial limit have not been conclusively demonstrated. In particular, some of the main physical processes responsible for *maintaining* the observed energy distributions are still missing. The nature, and indeed the very existence, of such a balance are still open questions.

McComas and Bretherton (1977) proposed a subharmonic parametric instability, which was shown to lead to the cascading of energy from higher-frequency internal waves toward the inertial limit. The instability involved a resonant triad of internal waves, where energy was transferred from a primary long-scale internal wave to a pair of much shorter perturbation internal waves with half the frequency. The authors acknowledged, however, that their instability mechanism, which fed energy into the inertial range, needed to be balanced by an energy sink in that range in order to achieve equilibrium. Indeed, long-term averages of month-long open-ocean records certainly show little variation around what appears to be some sort of equilibrium level (Fofonoff and Webster 1971). Without such a sink, no steady-state saturation would be possible and a full explanation of the inertial peak observed in many internal wave spectra would still be lacking. Unfortunately, no appropriate mechanism was identified by the authors as a candidate for the necessary sink.

The current paper is concerned more with the *generation* than the redistribution of internal wave energy. The proposed mechanism is intended to offer a self-contained interaction between barotropic and baroclinic modes, leading, at maturity, to a balance between a clear

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source and a clear sink. As such, the analysis will lead to a definite equilibrium amplitude. To motivate this, note that the full interaction will be described by a nonlinear Schroedinger equation having the following form:

$$\frac{\partial B}{\partial t} = i\beta AB^* + i\gamma|B|^2B.$$

In this equation, B is the amplitude of the internal wave mode, A is the amplitude of the generating surface wave, and α and β are constant interaction coefficients. Initially, B is extremely small so that the last term in the above equation is negligible. As such, there will be an exponential growth of the internal waves at the expense of the surface wave. As B grows, the cubic term gains prominence, representing a feedback effect. In other words, energy is returned to the surface wave and this path provides the required sink for the internal waves. Invoking balance between the last two terms implies that, at equilibrium, $|B| \sim |A|^{1/2}$. This suggests that if $|A| \sim O(\epsilon)$, where ϵ is a small parameter, then the steady-state internal wave amplitude reaches the much higher value of $|B| \sim O(\epsilon^{1/2})$.

It is important to note that this work is a contribution to a wide body of literature on the general interaction problem between surface and internal waves. Ball (1964) initiated this area of study, and important advancements were contributed by Thorpe (1966), Joyce (1974), and Watson et al. (1976), among others. The interactions in these studies were fundamentally different from the current investigation, however, in that the resonant triad consisted of two surface waves and a single internal wave of much longer scale.

Significant additional contributions were made by Dysthe and Das (1981), who considered the nonlinear coupling of internal waves to modulations in the surface wave spectrum, and Olbers and Herterich (1979), who investigated the generation of internal waves by surface waves in the framework of spectral scattering theory. Both analyses were pursued without the usual assumption of coherent wave trains possessing deterministic phase relations, rendering them more realistic models of observed oceanic processes.

In a series of papers, Watson (1985, 1990, 1994) has recently contributed a great deal to the understanding of the various sources and fluxes of internal wave energy. The foremost of these works considered the interactions between internal waves and mesoscale currents in an effort to describe the transferral of near-inertial internal wave energy to higher vertical wavenumbers. In the lattermost, a substantial transfer of energy from the internal wave field to the surface wave field was demonstrated.

Hill and Foda (1998) recently identified the additional path of interaction consisting of a single finite-amplitude surface wave and two perturbation internal waves. It should be noted that Smith (1972) considered this triad but from the point of view of having a single finite-

amplitude internal wave, with perturbation surface and internal waves completing the triad. His motivation was the study of how a single finite-amplitude internal wave could modify the surface wave field.

The present investigation formulates this interaction problem for the case of a fluid layer of constant buoyancy frequency. It is important to note that the goal of this work is not to introduce any new techniques or offer any improvements to the works of the authors mentioned above. Rather, the aim is to highlight, using a simplistic approach, one path of interaction that has not been specifically considered before and that may be responsible for some observed characteristics of internal wave spectra. A full understanding of the internal wave spectrum in the ocean, however, will require careful consideration of all of the physical processes present.

A perturbation and multiple timescale analysis is used to solve the boundary value problem at successive orders. At the leading order, the linear solutions are obtained and the conditions for resonance are determined. It is demonstrated that at latitudes below 28.8° in the Northern Hemisphere, a resonant triad consisting of the semidiurnal barotropic tide and two near-inertial baroclinic waves exists. Observational spectra that lend support to this hypothesis are considered.

At the second order, evolution equations for the internal wave amplitudes are derived. It is shown that, initially, the internal wave amplitudes grow exponentially in time. Finally, the equilibrium problem is considered. As mentioned above, when the internal wave amplitudes grow to be sufficiently large, a balance exists between quadratic and cubic terms, allowing for determination of the steady-state internal wave amplitude. This equilibrium amplitude is readily calculated and is shown to be a function of latitude, stratification, and internal wave vertical mode number.

2. Theoretical formulation

The origin of a three-dimensional Cartesian coordinate system is placed on the undisturbed free surface of an ocean of constant depth H and density profile $\rho_0(z)$. The z coordinate is defined as pointing vertically upward and the x and y coordinates define the horizontal. The density is assumed to vary exponentially with depth so that the buoyancy frequency is constant. As noted by Cox and Sandstrom (1962), this is not the most realistic model of stratification in the deep ocean, but serves as a useful and tractable first approximation.

The wave field is assumed to be made up of a single surface wave train and two internal wave trains. The surface wave has amplitude A , wavenumber vector \mathbf{k} , which is aligned with the positive x axis, and frequency ω . The internal waves have amplitudes a_1 and a_2 , wavenumber vectors $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$, and frequencies σ_1 and σ_2 . The phase functions of the three linear waves are therefore given by the following:

$$\begin{aligned} \theta_1 &= i(\boldsymbol{\lambda}_1 \cdot \mathbf{x} - \sigma_1 t) \\ \theta_2 &= i(\boldsymbol{\lambda}_2 \cdot \mathbf{x} - \sigma_2 t) \\ \theta_3 &= i(kx - \omega t). \end{aligned}$$

The three amplitudes a_1 , a_2 , and A are taken to be complex and functions of a slow timescale t_1 .

For resonant interactions to occur, it is necessary for the following resonance conditions to be satisfied:

$$\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \mathbf{k} \tag{1}$$

$$\sigma_1 + \sigma_2 = \omega. \tag{2}$$

The first of these conditions simply dictates that the wavenumber vectors of the wave triad form the sides and diagonal of a parallelogram in the x - y plane. The second requires that the individual dispersion relationships be satisfied.

The governing equations are given, to the Boussinesq approximation, by the following statements of continuity and conservation of momentum:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \rho'_t + w\rho'_{0z} &= -\mathbf{u} \cdot \nabla \rho' \\ u_t - fv + \mathbf{u} \cdot \nabla u &= \frac{-p'_x}{\rho_0} \\ v_t - fu + \mathbf{u} \cdot \nabla v &= \frac{-p'_y}{\rho_0} \\ w_t + \mathbf{u} \cdot \nabla w &= \frac{-p'_z}{\rho_0} - \rho'g. \end{aligned}$$

In the above, $\mathbf{u} = (u, v, w)$ is the fluid velocity vector, ρ' and p' are the perturbation density and pressure, ρ_0 is the equilibrium density, f is the Coriolis parameter, and g is the gravitational acceleration. Note that the x , y , z , and t subscripts are used to denote differentiation.

The boundary value problem is to be solved subject to the following boundary conditions:

$$\begin{aligned} w &= \eta_t + \mathbf{u} \cdot \nabla \eta, & z &= \eta \\ p_0 + p' &= 0, & z &= \eta \\ w &= 0, & z &= -1. \end{aligned}$$

In the above, η is the free surface displacement from equilibrium and p_0 is the equilibrium fluid pressure. It is convenient to recast the problem nondimensionally, so the following scalings are adopted:

$$\begin{aligned} (x^*, y^*) &= k(x, y) & z^* &= \frac{z}{H} \\ t^* &= k\sqrt{gH}t & (\omega^*, \sigma_1^*, \sigma_2^*, f^*) &= \frac{(\omega, \sigma_1, \sigma_2, f)}{k(gH)^{1/2}} \\ \eta^* &= \frac{\eta}{A} & p^{*'} &= \frac{p'}{\rho_0 g A} & w^* &= \frac{w}{Ak(gH)^{1/2}} \\ (u^*, v^*) &= \frac{(u, v)H}{A(gH)^{1/2}} & \rho^{*'} &= \frac{\rho'H}{\rho_0 A} & N^* &= \sqrt{\frac{H}{g}}N \\ (\boldsymbol{\lambda}_1^*, \boldsymbol{\lambda}_2^*) &= \frac{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)}{k}. \end{aligned}$$

Note that N is the buoyancy frequency and that the asterisks denote dimensionless quantities and are hereafter dropped for convenience. Given the recast equations, if Airy theory is adopted, which is appropriate for tidal frequencies in the deep ocean, the vertical momentum equation is replaced by the hydrostatic approximation.

Manipulation of the equations of motion yields the following nonlinear equation for the vertical velocity:

$$\begin{aligned} w_{tzz} + \nabla_h^2(N^2w) + f^2w_{zz} &= \epsilon \{ \nabla_h^2(\mathbf{u} \cdot \nabla \rho') + (\mathbf{u} \cdot \nabla u)_{xzt} + (\mathbf{u} \cdot \nabla v)_{yzt} \\ &\quad + f(\mathbf{u} \cdot \nabla v)_{xz} - f(\mathbf{u} \cdot \nabla u)_{yz} \}. \end{aligned} \tag{3}$$

Note the presence of ϵ , which is the longwave nonlinearity parameter defined by A/H . Furthermore, note that ∇_h^2 denotes the horizontal Laplacian operator and that the equation exhibits quadratic nonlinearity.

Revisiting the boundary conditions, the bottom boundary condition remains unchanged and is linear in w . The free surface boundary conditions may be combined, in order to eliminate η , and Taylor expanded around $z = 0$. As such, the following two conditions are obtained:

$$w = 0, \quad z = -1 \tag{4}$$

$$\begin{aligned} w_{tzz} + f^2w_z - \nabla_h^2w &= \epsilon \{ \nabla_h^2(w_z p' - N^2 p' p'_t - (p' p'_z)_t - \mathbf{u} \cdot \nabla p') + (\mathbf{u} \cdot \nabla u)_{xt} + (\mathbf{u} \cdot \nabla v)_{yt} + f(\mathbf{u} \cdot \nabla v)_x - f(\mathbf{u} \cdot \nabla u)_y \} \\ &\quad + \epsilon^2 \left\{ \nabla_h^2 \left(\frac{1}{2} N^2 w_z \eta^2 + w_z \eta p'_z + \frac{1}{2} p' p' w_{zz} \right) - \nabla_h^2 \left(\left(\frac{1}{2} \eta^2 N^2 + \eta p'_z \right) (N^2 p' + p'_z) - \frac{1}{6} \eta^3 N^4 - \frac{1}{2} \eta^2 p'_{zz} \right) \right. \\ &\quad \left. - \nabla_h^2 \mathbf{u} \cdot \nabla \left(\frac{1}{2} \eta^2 N^2 + \eta p'_z \right) - p' (\mathbf{u} \cdot \nabla \eta)_z \right\}, \quad z = 0. \end{aligned} \tag{5}$$

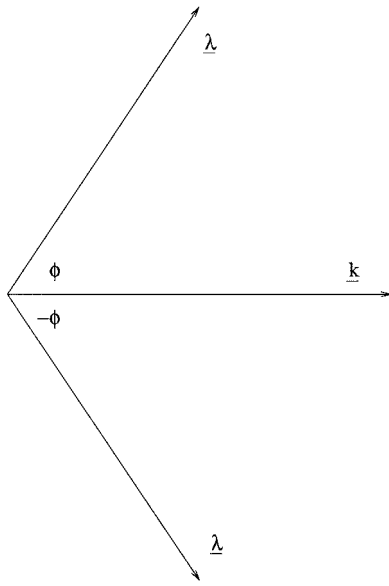


FIG. 1. Resonant triad wavenumber vector orientation for case of exact subharmonic resonance.

In the free surface boundary condition, terms up to $O(\epsilon^2)$ have been retained, so that both quadratic and cubic nonlinearity are present.

3. Initial instability

To address first the instability problem, the internal waves are assumed to be very small in comparison to the surface wave. The internal wave amplitudes may therefore be scaled as follows:

$$(a_1^*, a_2^*) = \frac{(a_1, a_2)}{\epsilon A}.$$

Again, the asterisks denote nondimensional values and are subsequently dropped. At this point in the analysis, restriction is made to the case of exact subharmonic resonance only. In other words, $\sigma_1 = \sigma_2 = \omega/2$, $\lambda_{1x} = \lambda_{2x} = 1/2$, and $\lambda_{1y} = -\lambda_{2y} \equiv \lambda_y$. The internal waves therefore propagate at angles of $\pm\phi$ relative to the surface wave, as demonstrated in Fig. 1. The motivation for restricting the analysis to this special case is twofold. Note first that, in general, a continuous spectrum of pairs of internal waves may be resonated by a single surface wave. The only restriction is that (1), (2) be satisfied. As demonstrated by Hill and Foda (1998) for the two-layer model, the pair of internal waves that grew at the fastest rate was the one that was exactly subharmonic to the surface wave. This same result can be shown for the current case of continuous stratification, and it is reasonable to consider here only this most unstable pair. Second, and of no less importance, the algebraic simplifications that result are enormous and result in the analysis being highly more compact.

To proceed, the variables of the problem are expanded

in power series of ϵ . For example, the vertical velocity w is expanded as follows:

$$w = w_3(z)e^{i\theta_3} + \epsilon w_1(z)e^{i\theta_1} + \epsilon w_2(z)e^{i\theta_2} + \epsilon^2 w_1'(z)e^{i\theta_1} + \epsilon^2 w_2'(z)e^{i\theta_2} + \dots + c.c.$$

The first three terms in the expansion are the linear harmonics describing the interacting wave triad. The internal wave harmonics appear at $O(\epsilon)$ since, for the moment, attention is focused on initial instability only. The slow timescale, of which the wave amplitudes are functions, is formalized as $t_1 = \epsilon t$. The remaining terms in the expansion are forced internal harmonics in phase with the linear internal harmonics. To clarify this, note that the product of $w_3(z)e^{i\theta_3}$ and $w_1(z)e^{i\theta_1}$, with the help of (1), (2), is in phase with $w_2'(z)e^{i\theta_2}$. As such, the quadratic terms in the governing equation and free surface boundary condition will provide forcing in the inhomogeneous boundary value problem for $w_2'(z)$.

a. $O(1)$

At the leading order, the homogeneous boundary value problem for the linear surface wave is obtained. Solution for the vertical velocity component and, in turn, the other linear variables is quite straightforward (see Gill 1982) and is not presented here in its entirety. In summary, normal mode analysis reveals that the eigenvalues of the linear problem are given by

$$NC_e = \tan\left(\frac{N}{C_e}\right). \tag{6}$$

The first eigenvalue of (6) is that of the surface wave and is given by $C_0 \approx 1$, with the accompanying dispersion relationship

$$\omega^2 = 1 + f^2.$$

b. $O(\epsilon)$

At the next order, homogeneous boundary value problems for the linear internal waves arise. At this point, the additional restriction that the internal waves be of the same vertical mode number is imposed. Again, this is done to simplify the analytic results. The restriction represents little loss of generality, fortunately. For it can be readily shown that interactions do exist for triads containing internal waves of different vertical mode numbers. However, the interactions are found to be orders of magnitude weaker than those interactions where the internal waves are of the same vertical mode number.

The dispersion relationship of the internal waves is given by the following:

$$\frac{\omega^2}{4} = C_n^2 |\boldsymbol{\lambda}|^2 + f^2, \tag{7}$$

where C_n is any eigenvalue of (6) subsequent to C_0 and is, upon invocation of the rigid-lid approximation, given

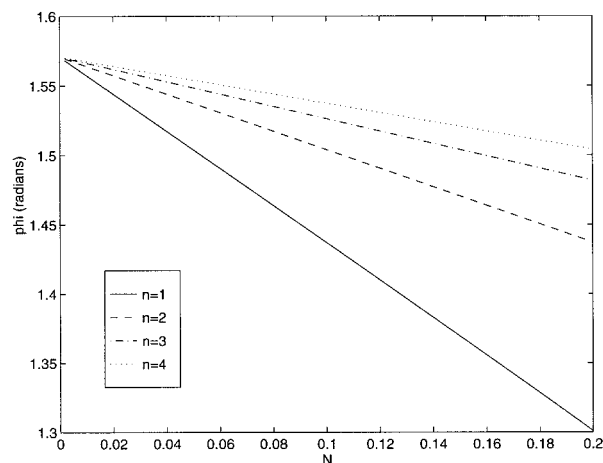


FIG. 2. Angle of propagation of internal waves as a function of buoyancy frequency and vertical mode number: $\omega = 1.146$, $f = 0.5606$

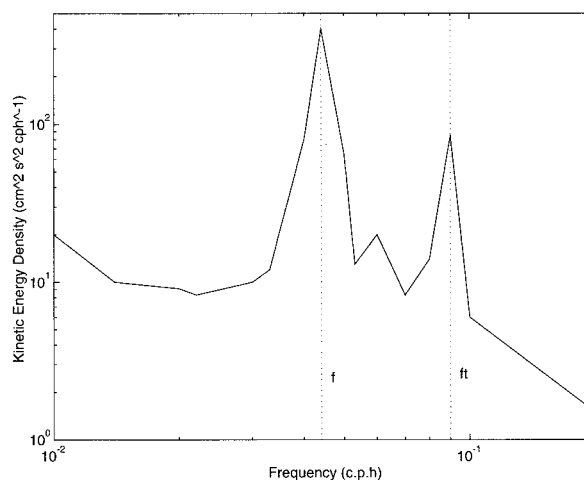


FIG. 3. Internal wave frequency spectrum from the western Sargasso Sea. Note the presence of peaks at the semidiurnal (f_i) and inertial (f) frequencies. Additionally note that $f_i \approx 2f$. [After Gould et. al. (1974)].

by $C_n \approx N/(n\pi)$. The vertical mode number of the internal waves is given by the integer n . Invoking this approximation, in the framework of a perturbation analysis, is *only* valid so long as the order of the error introduced is less than $O(\epsilon)$.

The linear analysis to this point is sufficient for determining the frequencies and wavenumbers of the internal waves. The frequencies and x components of the wavenumber vectors are given by the condition of exact subharmonic resonance. The y components of the wavenumber vectors are found by first solving (7) for $|\lambda|$.

For a numerical example, ω and f are set to values of 1.146 and 0.5606 respectively, corresponding, in an ocean depth of 4 km, to the semidiurnal tidal frequency f_i and a latitude of 28°N. The internal waves then have a frequency of 0.573, which is indeed very near to the inertial limit. Note that resonant triads in the Northern Hemisphere are possible only at latitudes below 28.8°N. At higher latitudes, the semidiurnal frequency will be less than twice the inertial frequency, with the result that subharmonic internal waves are not permissible. Therefore, at low latitudes only, there exists a possibility for energy to be transferred from the barotropic tide to subharmonic, near-inertial internal waves. The angle of propagation of the internal waves is shown in Fig. 2 as a function of both buoyancy frequency and vertical mode number. Note that the internal waves are highly oblique to the surface wave.

To provide evidence that this period doubling phenomenon exists, thereby establishing this mechanism as a possible candidate for the generation of internal wave energy, observational internal wave spectra demonstrating peaks at both f and f_i were sought out. A convincing example of this can be found in the observations of Gould et al. (1974). The authors reported on a series of observations from the western Sargasso Sea, located at 28°N. As can be seen from Fig. 3, strong peaks in the

internal wave spectrum are indeed seen at both inertial and semidiurnal frequencies.

c. $O(\epsilon^2)$

With the linear harmonics determined, solution of the second-order inhomogeneous boundary value problem for the internal waves can be pursued. Note that at this order forcing that is in phase with one of the internal waves, in both the governing equation and the free surface boundary condition, arises from quadratic interactions between the surface wave and the other internal wave. As a result of the homogeneous problem having a nontrivial solution, the inhomogeneous problem has a solution only if this forcing is orthogonal to the homogeneous solution. This introduces a solvability condition, known as the Fredholm alternative, which is obtained by applying Green's theorem.

For example, applying Green's theorem to $w'_2(z)$ and $w_2(z)$ across the depth of the fluid layer yields

$$\int_{-1}^0 [w_2(z)w'_{2z}(z) - w'_2(z)w_{2z}(z)] dz = [w_2(z)w'_{2z}(z) - w'_2(z)w_{2z}(z)]_{-1}^0. \quad (8)$$

Through substitution of the bottom boundary conditions, the free surface boundary conditions, and the governing equations for $w_2(z)$ and $w'_2(z)$, lengthy but straightforward manipulation leads to the following evolution equation:

$$\frac{da_2}{dt_1} = i\alpha a_1^*. \quad (9)$$

The asterisk in this instance denotes the complex conjugate. Repeating this process for the other internal

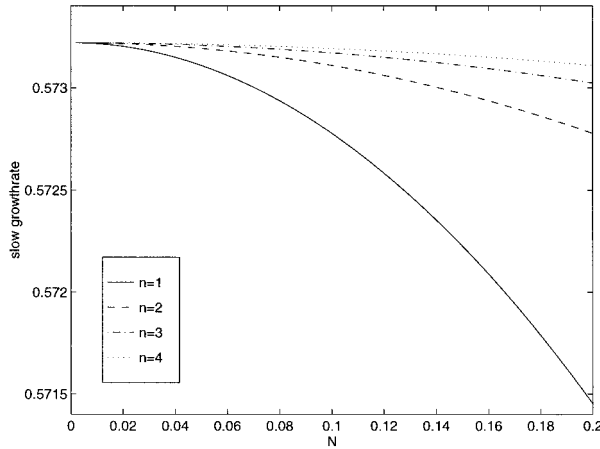


FIG. 4. Internal wave growth rate as a function of buoyancy frequency and vertical mode number: $\omega = 1.146$, $f = 0.5606$

forced harmonic, w'_1 , yields the second evolution equation:

$$\frac{da_1}{dt_1} = i\alpha a_2^*. \quad (10)$$

The interaction coefficient α that appears is a purely real coefficient, given by the simple expression:

$$\alpha = \frac{N^2}{2\omega n^2 \pi^2} - \frac{\omega}{2}.$$

Cross differentiation of (9) and (10) reveals that the internal wave amplitudes grow exponentially in time,

$$a_1, a_2 \propto \exp(\pm \alpha t_1).$$

Therefore, the absolute value of α is the growth rate, on the slow timescale, of the internal waves. This growth rate is shown in Fig. 4, again as a function of both stratification and vertical mode number. Of interest is that higher vertical modes are found to have larger growth rates and that the effect of rotation is not felt.

4. Equilibrium analysis

The result that the internal waves grow exponentially in time is clearly one that is not uniformly valid. For, as $t \rightarrow \infty$, the amplitudes grow to be infinitely large, violating the assumption of weak nonlinearity implicit in a perturbation analysis such as this. The possibility of a steady state, however, arises when the cubic interactions heretofore ignored are retained. As discussed in the introduction, this equilibrium requires that the relative scaling between the internal wave amplitudes and the surface wave amplitude be given by

$$(a_1^*, a_2^*) = \frac{\epsilon^{1/2}(a_1, a_2)}{A},$$

where, once again, the asterisks are temporarily used to denote the scaled amplitudes. Note that these scaling

arguments, and those of the previous section, are identical to those made by Minzoni and Whitham (1977) for the highly analogous case of edge waves on a sloping boundary.

The variables of the problem are rescaled correspondingly. For example, the expansion for the vertical velocity takes the revised form

$$w = \epsilon^{-1/2} w_1(z) e^{i\theta_1} + \epsilon^{-1/2} w_2(z) e^{i\theta_2} + w_3(z) e^{i\theta_3} \\ + \epsilon^{1/2} w'_1(z) e^{i\theta_1} + \epsilon^{1/2} w'_2(z) e^{i\theta_2} + \dots + \text{c.c.}$$

In this expansion, the surface waves appear at $O(1)$ as before. The linear internal waves appear at $O(\epsilon^{-1/2})$ however, since they are, at equilibrium, much larger than the surface wave. The final two terms in the expansion, as before, represent the forced internal harmonics.

a. $O(\epsilon^{-1/2})$

No forcing exists at this order, yielding homogeneous boundary value problems for the two internal waves. The linear solutions and dispersion relationships obtained are identical to those discussed in the previous section.

b. $O(1)$

At this order, an *inhomogeneous* boundary value problem for the surface wave is obtained. This is because quadratic products between the two linear internal waves will yield terms of the same phase and order as the surface wave. There will be both a homogeneous and a particular solution so that, for example, the vertical velocity can be expressed as follows:

$$w_3(z) = w_{3_h}(z) + w_{3_p}(z).$$

The homogeneous part of the solution may be taken to be the linear solution discussed previously. The particular solution is readily found from (3) to be as follows:

$$w_{3_p} = \frac{ia_1 a_2 6\omega n \pi N^2 (\lambda_y^2 / |\lambda|^2)}{4n^2 \pi^2 - N^2} \\ \times \cos[n\pi(z+1)] \sin[n\pi(z+1)].$$

Particular solutions for the other variables of the problem are found in a similar fashion.

c. $O(\epsilon^{1/2})$

Finally, the forced internal wave harmonics are considered. As before, there is forcing in both the governing equation and the free surface boundary condition. Now, however, cubic interactions are considered in addition to the quadratic interactions considered previously. Note that terms cubic in internal wave amplitude arise both from third-order self-interactions of the internal waves and second-order interactions between the internal waves and the particular solution of the surface wave.

The process of solvability is identical to before. A solution to the inhomogeneous problem is ensured only if the forcing is orthogonal to the eigenfunction of the homogeneous problem. Revisiting (8), manipulation leads to the following:

$$\frac{da_2}{dt_1} \propto i\beta a_1^* + i\gamma|a_1|^2 a_2. \tag{11}$$

Clearly, a steady state exists when the two terms on the

right-hand side balance. The companion equation for the other internal wave is found to be

$$\frac{da_1}{dt_1} \propto i\beta a_2^* - i\gamma^*|a_2|^2 a_1. \tag{12}$$

The interaction coefficients β and γ are real and complex, respectively, and are given by the following:

$$\beta = \frac{1}{8}\omega(N^2 - \omega^2 n^2 \pi^2)$$

$$\gamma = \frac{6\omega n \pi N^2 \lambda_y^2}{|\lambda|^2 (4n^2 \pi^2 - N^2)} \left\{ -\frac{3}{8} n \pi N^2 \lambda_y^2 - i f \lambda_y n \pi \left(\frac{1}{4} \omega n^2 \pi^2 + \frac{N^2}{8\omega} \right) \right\} - \frac{n^2 \pi^2 N^2 \lambda_y^3}{4|\lambda|^2} (\omega \lambda_y + i f).$$

The equilibrium physical amplitude a_e that the two internal waves will attain is therefore given simply as follows:

$$a_e = \sqrt{\frac{|\beta|}{|\gamma|}}.$$

The dependence of this result upon the strength of the stratification is shown in Fig. 5. Note that the equilibrium amplitude decreases with increasing stratification, a result consistent with intuition. For, if a given amount of energy is supplied to the internal wave field and the stratification is strong, the resulting internal wave amplitude will be less than if the stratification had been weak. To give some physical significance to the results, note that, if 40 cm is taken to be the surface wave amplitude, then a scaled value of $a_e = 1$ corresponds to a dimensional value of 40 m.

The effect of the inertial frequency on the equilibrium amplitude is detailed in Fig. 6. The dependence is seen

to be significant, with a_e becoming very large as the latitude approaches the limit of 28.8°N. As such, it is expected that internal wave spectra taken at latitudes near this limit will be the likeliest to demonstrate the presence of this mechanism.

Finally, Fig. 7 demonstrates the dependence of the equilibrium amplitude on the vertical mode number of the internal waves. While the second-order analysis indicated that internal waves of higher vertical mode number would be resonated on a faster timescale, the third-order analysis reveals that the lowest vertical mode will achieve the largest amplitude.

5. Concluding remarks

A third-order multiple-scales analysis has been presented in an effort to describe a path of energy transfer from the surface wave field to the internal wave field that has been heretofore largely ignored. Specific atten-

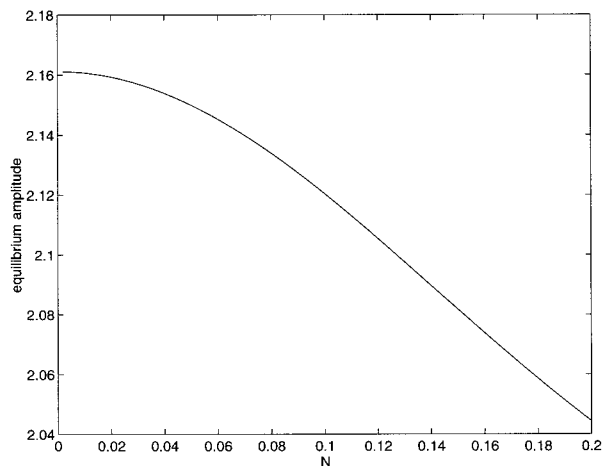


FIG. 5. Equilibrium internal wave amplitude as a function of buoyancy frequency: $\omega = 1.146$, $f = 0.5606$, $n = 1$

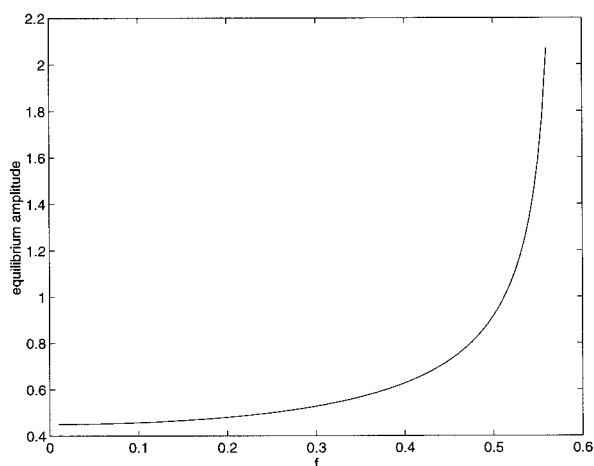


FIG. 6. Equilibrium internal wave amplitude as a function of Coriolis parameter: $\omega = 1.146$, $N = 0.1$, $n = 1$

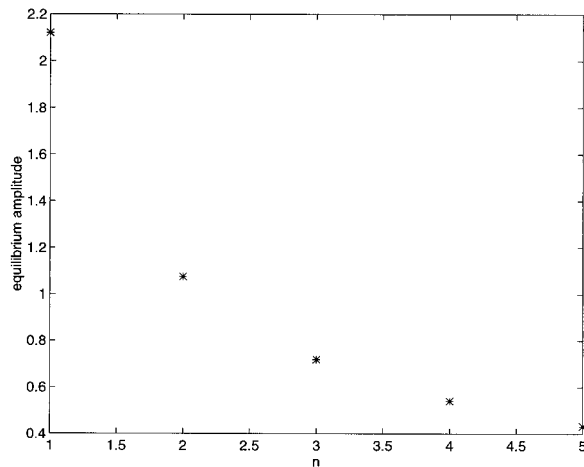


FIG. 7. Equilibrium internal wave amplitude as a function of vertical mode number: $\omega = 1.146$, $f = .5606$, $N = 0.1$

tion has been directed to the transfer, at low latitudes, from the semidiurnal barotropic tide to near-inertial baroclinic modes. The motivation for pursuing this was to demonstrate that the ever-present barotropic tide may be in part responsible for the development of the inertial peak seen in many internal wave energy spectra.

The initial instability problem, in which the internal waves were assumed to be small perturbations, was solved first. Solution of the inhomogeneous boundary value problem for the internal waves revealed that the amplitudes grew exponentially in time.

To describe the equilibrium, or saturation, of the internal waves, however, it was necessary to consider third-order effects. It was shown that a balance existed when the internal waves had grown to be much larger in amplitude than the forcing surface wave. This balance between second- and third-order terms allow for the ready calculation of the equilibrium amplitudes of the internal waves. The dependence of these steady-state amplitudes on stratification strength, latitude, and internal wave vertical mode number was discussed.

In closing, the goal of this work was to offer a compact and straightforward analysis of a mechanism of substantial significance to the budget of internal wave energy in the ocean. The emphasis was on the generation rather than the redistribution of energy and the identification of the barotropic tide as a viable source was of paramount importance.

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