Article ID: 1000-341X(2007)01-0047-06　　**Document code**: A

On Sufficient Conditions for p-Valently Starlikeness and Strong Starlikness

XU Neng

(Department of Mathematics, Changshu Institute of Technology, Jiangsu 215500, China)
(E-mail: xuneng11@pub.sz.jsinfo.net)

Abstract: Let $A_n(p)$ $(p, n \in N = \{1, 2, 3, \cdots\})$ denote the class of functions of the form $f(z) = z^p + a_{p+n}z^{p+n} + \cdots$ that are analytic in the unit disk $E = \{z : |z| < 1\}$. By using the method of differential subordinations we give some sufficient conditions for a function $f(z) \in A_n(p)$ to be p-valently starlike or strong starlike.

Key words: analytic function; starlikeness; strongly starlikeness; subordination.
MSC(2000): 30C45
CLC number: O174.51

1. Introduction

Let $A_n(p)(p, n \in N = \{1, 2, 3, \cdots\})$ be the class of functions of the form

$$f(z) = z^p + \sum_{m=n}^{\infty} a_{p+m}z^{p+m}$$

that are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z) \in A_n(p)$ is said to be p-valently starlike of order α in E if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > p\alpha \quad (z \in E)$$

for some α $(0 \le \alpha < 1)$. We denote by $S_n^*(p, \alpha)$ $(0 \le \alpha < 1)$ the subclass of $A_n(p)$ consisting of functions $f(z)$ which are p-valently starlike of order α in E. Clearly, $S_n^*(p, \alpha) \subset S_n^*(p, 0)$ for $0 \le \alpha < 1$. Also, we write $A_1(p) = A(p), A(1) = A, S_1^*(p, \alpha) = S^*(p, \alpha)$ and $S^*(1, \alpha) = S^*(\alpha)$.

A function $f(z)$ in $A(p)$ is said to be p-valently strong starlike of order α in E if it satisfies

$$\frac{zf'(z)}{f(z)} \prec p(\frac{1+z}{1-z})^{\alpha}$$

for some α $(0 < \alpha \le 1)$, where the symbol $\prec$ denotes subordination. We denote by $\widetilde{S}^*(p, \alpha)$ $(0 < \alpha \le 1)$ the subclass of $A(p)$ consisting of all functions which are p-valently strong starlike of order α in E. It is clear that $\widetilde{S}^*(p, 1) = S^*(p, 0)$[1].

Recently, Owa et al.[2,3], Yang[4], Silverman[5], Ponnusamy and Singh[6] and others have obtained various sufficient conditions for a function $f(z)$ to be in $S_n^*(p, \alpha)$ $(0 \le \alpha < 1)$ and

Received date: 2004-12-06; **Accepted date**: 2005-09-01

$\widetilde{S}^*(p,\alpha)$ $(0<\alpha\le 1)$. In the present paper, using the method of differential subordinations, we give new criteria for $f(z)$ to be in the classes $S_n^*(p,\alpha)$ $(0\le\alpha<1)$ and $\widetilde{S}^*(p,\alpha)$ $(0<\alpha\le 1)$.

To derive our results, we need the following lemmas.

Lemma 1[7] *Let $g(z)$ be analytic and univalent in E, and $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(E)$, with $\varphi(w)\neq 0$ when $w\in g(E)$. Set*

$$Q(z)=zg'(z)\varphi(g(z)),\quad h(z)=\theta(g(z))+Q(z),$$

and suppose that

(i) $Q(z)$ is univalent and starlike in E, and

(ii) $\mathrm{Re}\frac{zh'(z)}{Q(z)}=\mathrm{Re}\{\frac{\theta'(g(z))}{\varphi(g(z))}+\frac{zQ'(z)}{Q(z)}\}>0$ $(z\in E)$.

If $p(z)$ is analytic in E, with $p(0)=g(0)$, $p(E)\subset D$ and

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(g(z))+zg'(z)\varphi(g(z))=h(z),$$

then $p(z)\prec g(z)$ and $g(z)$ is the best dominant of the subordination.

Lemma 2[4] *Let $g(z)=b_0+b_nz^n+b_{n+1}z^{n+1}+\cdots$ $(n\in N)$ be analytic in E and $h(z)$ be analytic and starlike (with respect to the origin) univalent in E with $h(0)=0$. If $zg'(z)\prec h(z)$, then*

$$g(z)\prec b_0+\frac{1}{n}\int_0^z\frac{h(t)}{t}\mathrm{d}t.$$

2. Main results

Theorem 1 *Let $0<\alpha\le 1$, μ be an integer, and $-1\le\mu\alpha\le 1$. If $f(z)\in A(p)$ satisfies $f(z)\neq 0$ in $0<|z|<1$, $f'(z)\neq 0$ when $\mu\ge 0$, and*

$$(\frac{f(z)}{zf'(z)})^\mu(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)})\prec\frac{2\alpha z}{p^\mu(1+z)^{1+\mu\alpha}(1-z)^{1-\mu\alpha}}=h(z),\tag{1}$$

then $f(z)\in\widetilde{S}^*(p,\alpha)$ and the order α is sharp.

Proof Let us define the function $p(z)$ in E by

$$p(z)=\frac{zf'(z)}{pf(z)}.$$

Then $p(z)$ is analytic in E and

$$(\frac{f(z)}{zf'(z)})^\mu(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)})=\frac{1}{p^\mu}\frac{zp'(z)}{p^{1+\mu}(z)}.\tag{2}$$

From (1) and (2) we have

$$\frac{1}{p^\mu}\frac{zp'(z)}{p^{1+\mu}(z)}\prec h(z).\tag{3}$$

Let $0 < \alpha \le 1$, μ be an integer, $-1 \le \mu\alpha \le 1$,

$$D = \begin{cases} C & (\mu \le -1) \\ C\backslash\{0\} & (\mu > -1), \end{cases}$$

and choose

$$g(z) = (\frac{1+z}{1-z})^{\alpha}, \quad \theta(w) = 0, \quad \varphi(w) = \frac{1}{p^{\mu}}\frac{1}{w^{1+\mu}}. \tag{4}$$

Then $g(z)$ is analytic and univalent in E, $g(0) = p(0) = 1, p(E) \subset D, \theta(w)$ and $\varphi(w)$ satisfy the conditions of Lemma 1. The function

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{2\alpha z}{p^{\mu}(1+z)^{1+\mu\alpha}(1-z)^{1-\mu\alpha}} \tag{5}$$

is univalent and starlike in E because

$$\mathrm{Re}\frac{zQ'(z)}{Q(z)} = 1 + (1+\mu\alpha)\mathrm{Re}(-\frac{z}{1+z}) + (1-\mu\alpha)\mathrm{Re}\frac{z}{1-z} > 0 \quad (z \in E).$$

Furthermore, we have

$$\theta(g(z)) + Q(z) = \frac{2\alpha z}{p^{\mu}(1+z)^{1+\mu\alpha}(1-z)^{1-\mu\alpha}} = h(z)$$

and

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} = \mathrm{Re}\frac{zQ'(z)}{Q(z)} > 0 \tag{6}$$

for $z \in E$. The Inequality (6) shows that the function $h(z)$ is close-to-convex and univalent in E. Now it follows from (2)–(6) that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z).$$

Therefore, by virtue of Lemma 1, we conclude that $p(z) \prec g(z)$, that is, $f(z) \in \widetilde{S}^*(p,\alpha)$.

For the function

$$f(z) = z^p \exp\int_0^z \frac{p}{t}((\frac{1+t}{1-t})^{\alpha} - 1)\mathrm{d}t \in A(p),$$

it is easy to verify that

$$(\frac{f(z)}{zf'(z)})^{\mu}(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}) = h(z)$$

and

$$|\arg\frac{zf'(z)}{pf(z)}| = \alpha|\arg\frac{1+z}{1-z}| \to \frac{\alpha\pi}{2} \quad \text{as} \quad z \to i.$$

This completes the proof of the theorem.

Letting $\mu = \alpha = 1$ in Theorem 1, we obtain

Corollary 1 *If $f(z) \in A(p)$ satisfies $f(z)f'(z) \ne 0$ $(0 < |z| < 1)$ and*

$$\frac{f(z)}{zf'(z)}(1 + \frac{zf''(z)}{f'(z)}) \prec 1 + \frac{2z}{p(1+z)^2} = h_1(z),$$

then $f(z) \in S^*(p,0)$ and the order 0 is sharp.

Remark 1 Because

$$h_1(E)=\{w:|\arg(w-(1+\frac{1}{2p}))|>0\},$$

hence Corollary 1 coincides with the result of Owa et al.[2]. Furthermore we see that the result of [2] is sharp.

Letting $\mu=0$, $0<\alpha\le 1$ in Theorem 1, and noting

$$h(z)=\frac{2\alpha z}{1-z^2},\quad h(e^{i\theta})=\frac{2\alpha e^{i\theta}}{1-e^{2i\theta}}=\frac{\alpha i}{\sin\theta},$$

we have $\mathrm{Re}h(e^{i\theta})=0$ and $|\mathrm{Im}h(e^{i\theta})|\ge\alpha$, and

Corollary 2 *If $f(z)\in A(p)$ satisfies $f(z)f'(z)\ne 0$ $(0<|z|<1)$ and*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\ne ib,$$

where b is a real number and $|b|\ge\alpha$ $(0<\alpha\le 1)$, then $f(z)\in\widetilde{S}^*(p,\alpha)$ and the order α is sharp.

Letting $\mu=-1$ in Theorem 1, we have

Corollary 3 *If $f(z)\in A(p)$ satisfies $f(z)\ne 0(0<|z|<1)$ and*

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec\frac{2p\alpha z}{(1+z)^{1-\alpha}(1-z)^{1+\alpha}},$$

where $0<\alpha\le 1$, then $f(z)\in\widetilde{S}^*(p,\alpha)$ and the order α is sharp.

Letting $p=\alpha=1$ in Corollary 3, we have

Corollary 4 *If $f(z)\in A$ satisfies $f(z)\ne 0(0<|z|<1)$ and*

$$\frac{zf'(z)}{f(z)}(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)})\prec\frac{2z}{(1-z)^2}=h_2(z),$$

then $f(z)\in S^*(0)$ and the order 0 is sharp.

Remark 2 Owa and Obradovic[3] have proved that if $f(z)\in A$ satisfies

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)})\}>-\frac{1}{2}\quad(z\in E),$$

then $f(z)\in S^*(0)$. Since

$$h_2(E)=\{w:|\arg(w+\frac{1}{2})|<\pi\},$$

we see that Corollary 4 improves the result in [3].

Theorem 2 *If $f(z)\in A_n(p)$ satisfies $f(z)f'(z)\ne 0(0<|z|<1)$ and*

$$(\frac{f(z)}{zf'(z)})^{\mu}(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)})\prec az,\tag{7}$$

where $0<\mu\le 1, 0<a\le\frac{n}{p^\mu}$. Then $f(z)\in S_n^*(p,1/(1+\frac{\mu p^\mu a}{n})^{\frac{1}{\mu}})$ and the order $1/(1+\frac{\mu p^\mu a}{n})^{\frac{1}{\mu}}$ is sharp.

Proof Let

$$g(z)=\frac{zf'(z)}{pf(z)}. \tag{8}$$

Then $g(z)=1+b_nz^n+\cdots$ is analytic in E and

$$z(\frac{1}{g^{\mu}(z)})'=-\mu p^{\mu}(\frac{f(z)}{zf'(z)})^{\mu}(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}).$$

From (7) and (8), we have

$$z(\frac{1}{g^{\mu}(z)})'\prec -\mu p^{\mu}az. \tag{9}$$

For $0<\mu\leq 1$ and $0<a\leq \frac{n}{p^{\mu}}$, applying Lemma 2 to (9) we have

$$\frac{1}{g^{\mu}(z)}\prec 1-\frac{\mu p^{\mu}a}{n}z,$$

which implies that

$$g(z)\prec (\frac{1}{1-\frac{\mu p^{\mu}a}{n}z})^{\frac{1}{\mu}}=h_3(z). \tag{10}$$

The region $h_3(E)$ is symmetric with respect to the real axis and $h_3(z)$ is convex univalent in E because

$$\text{Re}\{1+\frac{zh_3''(z)}{h_3'(z)}\}=\text{Re}\{\frac{1+\frac{p^{\mu}a}{n}z}{1-\frac{\mu p^{\mu}a}{n}z}\}>\frac{1-\frac{p^{\mu}a}{n}}{1+\frac{\mu p^{\mu}a}{n}}\geq 0 \quad (z\in E).$$

Hence $\text{Re}h_3(z)\geq h_3(-1)\geq 0$ for $z\in E$ and it follows from (8) and (10)

$$\text{Re}\frac{zf'(z)}{pf(z)}>(\frac{1}{1+\frac{\mu p^{\mu}a}{n}})^{\frac{1}{\mu}} \quad (z\in E).$$

This shows that $f(z)\in S_n^*(p,1/(1+\frac{\mu p^{\mu}a}{n})^{\frac{1}{\mu}})$.

If we take

$$f(z)=z^p\exp\int_0^z\frac{p}{t}((\frac{1}{1-\frac{\mu p^{\mu}a}{n}t^n})^{\frac{1}{\mu}}-1)\mathrm{d}t,$$

then it is easy to see $f(z)\in A_n(p)$ satisfies (7) and

$$\text{Re}\frac{zf'(z)}{pf(z)}\to(\frac{1}{1+\frac{\mu p^{\mu}a}{n}})^{\frac{1}{\mu}} \quad \text{as} \quad z\to e^{i\pi/n}.$$

Thus the order $1/(1+\frac{\mu p^{\mu}a}{n})^{\frac{1}{\mu}}$ is sharp.

Letting $\mu=n=p=1$ in the Theorem 2 yields

Corollary 5 *Let $f(z)\in A$ satisfy $f(z)f'(z)\neq 0$ $(0<|z|<1)$ and*

$$\frac{f(z)}{zf'(z)}(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)})\prec az \quad (z\in E),$$

where $0<a\leq 1$. Then $f(z)\in S^*(\frac{1}{1+a})$ and the order $\frac{1}{1+a}$ is sharp.

Remark 3 Silverman[5, Theorem 1] has proved if $f(z) \in A, 0 < a \leq 1$, and

$$G_a = \{f : |(\frac{\frac{1+zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}) - 1| < a,\ z \in E\},$$

then $G_a \subset S^*(2/(1+\sqrt{1+8a}))$. Because $2/(1+\sqrt{1+8a}) < 1/(1+a)(0 < a < 1)$, we see that Corollary 5 improves the result in [5].

Acknowledgement The author would like to express deep appreciation to Professor YANG Ding-gong for enlightening discussions.

References:

[1] DUREN P L. *Univalent Functions* [M]. Springer-Verlag, New York, 1983.
[2] OWA S, NONOKAWA M, SAITON H. *Sufficient conditions for multivalent starlikeness* [J]. Ann. Polon. Math., 1995, **62**(1): 75–78.
[3] OWA S, OBRADOVIC M. *An application of differential subordinations and some criteria for univalency* [J]. Bull. Austral. Math. Soc., 1990, **41**(3): 487–494.
[4] YANG Ding-gong. *On sufficient conditions for multivalent starlikeness* [J]. Bull. Korean Math. Soc., 2000, **37**(4): 659–668.
[5] SILVERMAN H. *Convex and starlike criteria* [J]. Int. J. Math. Math. Sci., 1999, **22**(1): 75–79.
[6] PONNUSAMY S, SINGH V. *Criteria for strongly starlike functions* [J]. Complex Variables Theory Appl., 1997, **34**(3): 267–291.
[7] MILLER S S, MOCANU P T. *On some classes of first order differential subordinations* [J]. Michigan Math. J., 1985, **32**: 185–195.

关于 p- 叶星形性和强星形性的充分条件

徐 能
(常熟理工学院数学系，江苏 常熟 215500)

摘要: 本文利用微分从属的方法得到了单位圆盘内 p- 叶星形函数和强星形函数的某些充分条件.

关键词: 解析函数；星形函数；强星形函数；从属.