Exploring the Relationship between Eddy-Induced Transport Velocity, Vertical Momentum Transfer, and the Isopycnal Flux of Potential Vorticity

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ABSTRACT

Gent et al. have emphasized the role of the eddy-induced transport (or bolus) velocity as a mechanism for redistributing tracers in the ocean. By writing the momentum equations in terms of the isopycnal flux of potential vorticity, the author shows that any parameterization of the eddy-induced transport velocity must be consistent with the conservation equation for potential vorticity. This places a constraint on possible parameterizations, a constraint that is satisfied by the Gent and McWilliams parameterization only if restrictions are placed on the diffusivity coefficient. A new parameterization is suggested that is the simplest extension of Gent and McWilliams based on the potential vorticity formulation. The new parameterization parameterizes part of the time-mean flow driven by the Reynolds stress terms in addition to the eddy-induced transport velocity. It is also shown that the eddy-induced transport velocity can always be written as the Ekman velocity associated with the vertical derivative of a horizontally directed eddy stress. The author shows how the eddy stress is related to the ''inviscid pressure drag'' or ''form drag'' associated with the eddies, although the correspondence is not exact.

1. Introduction

Gent and McWilliams (1990, hereafter GM90) have suggested a parameterization for mesoscale eddies for use in coarse-resolution ocean models. The parameterization is an almost Fickian diffusion of thickness along isopycnal surfaces. Recently, Gent et al. (1995) have pointed out that the GM90 parameterization can be interpreted as an eddy-induced transport velocity (sometimes referred to as the ''bolus'' velocity). The eddy-induced transport velocity is analogous to the Stokes drift in the theory of surface gravity waves and can play an important role in redistributing tracers even though it is not part of the time-mean flow. For example, Danabasoglu and McWilliams (1995) and Hirst and McDougall (1996) demonstrate that incorporating GM90 in a coarse-resolution global model leads to a dramatic improvement in the ability of the model to represent the distribution of water masses. Gent et al. (1995) point out that when the momentum equations are cast in terms of the tracer transport velocity (the tracer transport velocity is the sum of the time-mean velocity and the eddy-induced transport velocity), the GM90 parameterization is equivalent to a vertical transfer of (geostrophic) momentum. Given that GM90 is designed to mimic the removal of geostrophic shear by baroclinic instability, the correspondence between GM90 and a vertical flux of momentum is not surprising. An approach based on the vertical mixing of momentum was adopted independently by Greatbatch and Lamb (1990, hereafter GL90). GL90 show that at small Ekman number their parameterization is equivalent to GM90.

At the time of writing their paper, GL90 did not appreciate the significance of the eddy-induced transport velocity, or that their momentum equations are really the equations for the tracer transport velocity, as distinct from the time-mean velocity. GL90 were motivated by the correspondence in quasigeostrophic theory between vertical mixing of momentum and horizontal mixing of potential vorticity (Rhines and Young 1982). To show the correspondence, it is necessary to assume that horizontal length scales are large compared to the internal Rossby radius of deformation. GL90 added a vertical eddy viscosity term to the planetary geostrophic momentum equations with the vertical eddy viscosity coefficient assumed to be of the form Af^2/N^2, where f is the Coriolis parameter and N is the local value of the buoyancy frequency. They show that for small Ekman number, and when A satisfies certain conditions, potential vorticity q is homogenized within closed q contours. GL90 show that the GM90 parameterization also leads to homogenization of potential vorticity under analogous conditions.

Homogenization of potential vorticity is a striking feature of the time-mean fields on the subsurface levels of quasigeostrophic models of the wind-driven ocean circulation (see, e.g., Holland et al. 1984). Potential vor-

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ticity homogenization on subsurface isopycnals is also a feature of maps of potential vorticity derived from observations (e.g., McDowell et al. 1982; Keffer 1985; Talley 1988; O'Dywer and Williams 1997), and recent numerical computations using a primitive equation model (Greatbatch et al., in preparation). Indeed, the evidence from both eddy-resolving models and observations suggests that isopycnal mixing of potential vorticity should be a fundamental feature of any parameterization, a point that has also been made by Treguier et al. (1997) and Killworth (1997, submitted to *J. Phys. Oceanogr.*, hereafter K97), and supported by the channel model experiments of Lee et al. (1997). Indeed, the parameterizations suggested by Treguier et al. (1997) and K97 are actually quite similar to that suggested in section 5 of this paper, although the analysis given here is somewhat different and also more general.

In the present paper, we explore the relationship between eddy-induced transport velocity, the vertical transfer of momentum, and isopycnal mixing of potential vorticity. We attempt to link the work of Gent et al. (1995) to the earlier quasigeostrophic theories of Rhines and Holland (1979) and Rhines and Young (1982). In so doing, we show how the potential vorticity equation places a constraint on possible parameterizations for the eddy-induced transport velocity. The key is the link between potential vorticity conservation and the momentum equations, a link that is missing in parameterizations such as GM90 that are derived independently of the momentum equations. We then suggest a new parameterization that is an extension of GM90 but has the potential vorticity equation at its core. We also show how the eddy-induced transport velocity can be related to an "eddy stress," once the momentum equations are cast in terms of the tracer transport velocity, and show the connection between the eddy stress and the eddy "form drag," generalizing a result of Rhines and Holland (1979).

The plan of the paper is as follows. In section 2, we show that the eddy-induced transport velocity can be written as an Ekman velocity associated with the vertical derivative of a horizontally directed eddy stress. In the case of the GM90 parameterization, the horizontal stress is the vertical flux of geostrophic momentum. Section 3 discusses the relationship between the eddy stress and the form drag, while section 4 explores the relationship between the eddy stress and the isopycnal flux of potential vorticity. In section 5 we illustrate the potential vorticity formulation for the particular example of the GM90 parameterization and show how the potential vorticity equation places a constraint on possible parameterizations for the eddy-induced transport velocity. The discussion leads naturally to the suggestion of a new parameterization. A feature of the potential vorticity approach is that the new parameterization parameterizes part of the mean flow driven by the Reynolds stress terms in addition to the eddy-induced transport velocity. Section 6 provides a summary and discussion.

2. The eddy-induced transport velocity and the eddy stress

We develop the formalism within the context of a Boussinesq, hydrostatic, incompressible fluid and use plane Cartesian geometry. We take as our starting point the equation for a tracer τ. We assume that the effect of turbulent mixing, be this by mesoscale eddies or any other process, can be parameterized in terms of the local gradients of the large-scale, averaged tracer field, $\overline{\tau}$, as follows:

$$\frac{D\overline{\tau}}{Dt} = \frac{\partial}{\partial x_j}\left\{A_{i,j}\frac{\partial\overline{\tau}}{\partial x_i}\right\}, \tag{1}$$

where D/Dt is the time derivative following the time-mean flow and $A_{i,j}$ is a general second-order tensor. It should be noted that although the analysis in this section is carried out in Cartesian (x, y, z) coordinates, the averaged large-scale variable need not be the result of averaging at fixed z. In the interpretation given by Gent et al. (1995), the large-scale tracer and velocity fields are unweighted averages on an isopycnal surface. For a detailed analysis of isopycnal averaging (both weighted and unweighted), readers are referred to the paper by de Szoeke and Bennett (1993) where discussion can also be found on how equations averaged in isopycnal coordinates can be transformed to equations in (x, y, z) coordinates.

We now split $A_{i,j}$ into a symmetric and an antisymmetric part. Gent et al. (1995) argue that for mesoscale eddies, the symmetric part corresponds to isopycnal diffusion of tracer. Andrews et al. (1987) note that an analysis based on linearized displacements shows that, in general, the symmetric part of $A_{i,j}$ is tracer dependent, whereas the antisymmetric part is independent of the tracer (see also Plumb 1979). In the following, we shall concentrate on the antisymmetric part, since this is associated with the eddy-induced transport velocity, or Stokes drift (Plumb and Mahlman 1987; Andrews et al. 1987; Middleton and Loder 1989). The general form of the antisymmetric part is

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}. \tag{2}$$

Substituting the antisymmetric part of A into Eq. (1), we see that the contribution from the antisymmetric part can be written as an advection term with eddy-induced transport velocity $\mathbf{u_I} = (u_I, v_I, w_I)$ given by

$$u_I = a_y + b_z \tag{3}$$

$$v_I = -a_x + c_z \tag{4}$$

$$w_I = -b_x + c_y. \tag{5}$$

Here u_I, v_I, and w_I satisfy the continuity equation

$$u_{I_x} + v_{I_y} + w_{I_z} = 0. \tag{6}$$

At the surface, $z = 0$, we have

$$w_I = 0, \tag{7}$$

while at the bottom $z = -H(x, y)$ we have

$$w_I = -(u_I H_x + v_I H_y). \tag{8}$$

Equations (7) and (8) state that there is no eddy flux of tracer normal to the top and bottom boundaries (the same is also true at any side boundaries). For the GM90 parameterization, $a = 0$, and $(b, c) = -\kappa\mathbf{L}$ where $\mathbf{L} = -\boldsymbol{\nabla}\rho/\rho_z$.

We now define

$$\psi = -\int_z^0 a\, dz' + \psi_o(x, y). \tag{9}$$

Then

$$u_I = (b + \psi_y)_z \tag{10}$$

$$v_I = (c - \psi_x)_z \tag{11}$$

$$w_I = -(b + \psi_y)_x - (c - \psi_x)_y. \tag{12}$$

Since it is only u_I, v_I, w_I that matter, a comparison of (10)–(12) with (3)–(5) shows that we can always take $a = 0$ by redefining b and c. Also, because $w_I = 0$ at the surface, $b_x + c_y$ is zero there, allowing us to choose ψ_o so that at the surface

$$\psi_y + b = 0 \tag{13}$$

and

$$-\psi_x + c = 0. \tag{14}$$

It follows that not only can we assume $a = 0$, we can also assume $b = c = 0$ at $z = 0$. It should be noted that for the GM90 parameterization, $a = 0$ everywhere and $\kappa = 0$ at $z = 0$, showing that GM90 already satisfies these conditions.

We now define the tracer transport velocity (U, V, W) to be the sum of the large-scale (time mean) velocity $\overline{\mathbf{u}}$ and the eddy-induced transport velocity $\mathbf{u_I} = (u_I, v_I, w_I)$, that is, $(U, V, W) = (\overline{u} + u_I, \overline{v} + v_I, \overline{w} + w_I)$. The tracer equation (1) can then be written as

$$\frac{\partial\overline{\tau}}{\partial t} + U\frac{\partial\overline{\tau}}{\partial x} + V\frac{\partial\overline{\tau}}{\partial y} + W\frac{\partial\overline{\tau}}{\partial z} = \frac{\partial}{\partial x_j}\left\{A^S_{i,j}\frac{\partial\overline{\tau}}{\partial x_i}\right\}, \tag{15}$$

where $A^S_{i,j}$ is the symmetric part of A.

We now turn to the momentum equations. We shall assume horizontal scales are large compared to the internal radius of deformation, enabling us to make the planetary geostrophic approximation. For convenience, we shall assume the momentum equations reduce to the geostrophic balance

$$-f\overline{v} = -\frac{1}{\rho_o}\frac{\partial\overline{p}}{\partial x} \tag{16}$$

$$f\overline{u} = -\frac{1}{\rho_o}\frac{\partial\overline{p}}{\partial y}, \tag{17}$$

where p is the pressure, overbar denotes averaged values, and ρ_o is a representative density for seawater. We now rewrite these equations in terms of the tracer transport velocity $(U, V, W) = (\overline{u} + u_I, \overline{v} + v_I, \overline{w} + w_I)$. We then get

$$-fV = -\frac{1}{\rho_o}\frac{\partial\overline{p}}{\partial x} + \frac{1}{\rho_o}\frac{\partial X}{\partial z} \tag{18}$$

$$fU = -\frac{1}{\rho_o}\frac{\partial\overline{p}}{\partial y} + \frac{1}{\rho_o}\frac{\partial Y}{\partial z}, \tag{19}$$

where

$$X = -\rho_o f c \tag{20}$$

and

$$Y = \rho_o f b. \tag{21}$$

It follows that the eddy-induced transport velocities u_I, v_I appear as Ekman velocities given by

$$-fv_I = \frac{1}{\rho_o}\frac{\partial X}{\partial z} \tag{22}$$

$$fu_I = \frac{1}{\rho_o}\frac{\partial Y}{\partial z}. \tag{23}$$

The vector (X, Y) appears as a horizontally directed stress acting on the fluid. We shall refer to (X, Y) as the eddy stress. The physical significance of the eddy stress is discussed in section 3 for the case of geostrophic eddies. Equation (23) in Gent et al. (1995) corresponds directly to our Eqs. (18) and (19) for the particular case of the GM90 parameterization. In fact, for the GM90 parameterization

$$(X, Y) = -\rho_o f\kappa\mathbf{k}\times\mathbf{L}, \tag{24}$$

where $\mathbf{k}$ is a unit vector in the upward vertical direction. Use of the thermal wind relation then shows that (X, Y) is actually the vertical flux of geostrophic momentum given by

$$(X, Y) = \rho_o\kappa\frac{f^2}{N^2}\mathbf{u}_{g_z}, \tag{25}$$

where N is the local value of the buoyancy frequency.

In terms of the eddy stress (X, Y), the boundary conditions (7) and (8) take the form

$$\frac{\partial}{\partial x}\left\{\frac{Y}{f}\right\} - \frac{\partial}{\partial y}\left\{\frac{X}{f}\right\} = 0 \tag{26}$$

at $z = 0$ and

$$\frac{\partial}{\partial x}\left\{\frac{Y_b}{f}\right\} - \frac{\partial}{\partial y}\left\{\frac{X_b}{f}\right\} = 0, \tag{27}$$

where (X_b, Y_b) is the stress (X, Y) evaluated at $z = -H(x, y)$ [the equivalence of (27) and (8) can be seen by applying the chain rule for differentiation to (27)]. Since we have chosen $b = 0$ and $c = 0$ at $z = 0$, we actually have zero eddy stress (X, Y) at $z = 0$ (the surface), as is quite reasonable. On the other hand, there is no guarantee that the bottom eddy stress (X_b, Y_b) is zero. For example, the ''Neptune'' effect described by Holloway (1992) is associated with eddy–topography interaction, for which the associated bottom stress need not be zero, an issue to be explored in a later paper.

GL90 applied their parameterization directly to the momentum equations by suggesting a particular form for the eddy stress (X, Y) in (18) and (19). As noted in the introduction, their momentum equations should be thought of as the equations for the tracer transport velocity, not the large-scale, time-mean velocity, a point that was not appreciated by GL90. It is straightforward, however, to show that the Ekman velocity associated with their parameterization has the properties of an eddy-induced transport velocity. We can see this by putting $a = 0$, $b = Y/(f\rho_o)$, and $c = X/(f\rho_o)$ in (3)–(5). Although GL90 do not explicitly discuss the surface and bottom boundary conditions for their parameterization, a reasonable choice would be to require (26) and (27) to be satisfied, therefore ensuring that w_I satisfies (7) and (8). In a similar way, instead of writing the tracer equation as in Eq. (15), as implied in GL90, it can be written in terms of the large-scale, time-mean flow and an antisymmetric tensor, as in Eq. (1).

3. Connecting the eddy stress and the form drag

We have shown that the horizontal component of the eddy-induced transport velocity can be expressed in the same form as the Ekman velocity associated the vertical derivative of a horizontally directed stress, as in Eqs. (22) and (23). We next explore the relationship between the eddy stress (X, Y) and the eddy form drag.

We begin by turning to section 1 of Gent et al. (1995). We assume adiabatic flow of a Boussinesq, incompressible fluid and work in isopycnal coordinates. The continuity and density equations can then be combined to give the following equation expressing the conservation of volume

$$\frac{\partial z_\rho}{\partial t} + \boldsymbol{\nabla}_\rho \cdot (z_\rho \mathbf{u}) = 0, \tag{28}$$

where $z(x, y, \rho, t)$ is the physical height of a density surface, z_ρ can be interpreted as the thickness, and $\boldsymbol{\nabla}_\rho$ is the horizontal gradient operator applied at constant ρ. It should be noted that in (28) (and also in what follows), $\mathbf{u}$ is the horizontal component of the velocity (the vertical component is obtained by integrating the continuity equation in z-coordinates with $w = 0$ at the surface). The equation for the conservation of a tracer τ is

$$\frac{\partial (z_\rho \tau)}{\partial t} + \boldsymbol{\nabla}_\rho \cdot (z_\rho \mathbf{u} \tau) = 0. \tag{29}$$

Following Gent et al. (1995) the variables are decomposed into large-scale components denoted by an overbar and eddy components denoted by primes. The large-scale components can be regarded as a time average at fixed (x, y, ρ) (although slow time variations associated with the large-scale flow will be retained). We then obtain

$$\frac{\partial \overline{z_\rho}}{\partial t} + \boldsymbol{\nabla}_\rho \cdot (\overline{z_\rho}\hat{\mathbf{u}}) = \frac{\partial \overline{z_\rho}}{\partial t} + \boldsymbol{\nabla}_\rho \cdot (\overline{z_\rho}\overline{\mathbf{u}}) + \boldsymbol{\nabla}_\rho \cdot (\overline{z_\rho}\mathbf{u}^*) = 0 \tag{30}$$

and

$$\frac{\partial \overline{\tau}}{\partial t} + \frac{1}{\overline{z_\rho}}\frac{\partial \overline{z_\rho' \tau'}}{\partial t} + \hat{\mathbf{u}} \cdot \boldsymbol{\nabla}_\rho \overline{\tau} = -\frac{1}{\overline{z_\rho}} \boldsymbol{\nabla}_\rho \cdot [\overline{(z_\rho \mathbf{u})' \tau'}], \tag{31}$$

where $\hat{\mathbf{u}}$ is the thickness-weighted, isopycnal-averaged velocity given by

$$\hat{\mathbf{u}} = \overline{z_\rho \mathbf{u}}/\overline{z_\rho} = \overline{\mathbf{u}} + \overline{z_\rho' \mathbf{u}'}/\overline{z_\rho} = \overline{\mathbf{u}} + \mathbf{u}^*, \tag{32}$$

$$\mathbf{u}^* = \overline{z_\rho' \mathbf{u}'}/\overline{z_\rho}, \tag{33}$$

and $\overline{\mathbf{u}}$ is the large-scale (time averaged) velocity. Gent et al. (1995) assume that the eddy components of thickness and tracer are uncorrelated so that the second term in Eq. (31) can be neglected. They also assume that the right-hand side of Eq. (31) can be parameterized as a Fickian diffusion along mean isopycnals with coefficient μ so that (31) can be written

$$\frac{\partial \overline{\tau}}{\partial t} + \hat{\mathbf{u}} \cdot \boldsymbol{\nabla}_\rho \overline{\tau} = \frac{1}{\overline{z_\rho}} \boldsymbol{\nabla}_\rho \cdot [\mu \overline{z_\rho} \boldsymbol{\nabla}_\rho \overline{\tau}]. \tag{34}$$

Gent et al. (1995) identify $\hat{\mathbf{u}}$ with the tracer transport velocity and $\mathbf{u}^*$ with the eddy-induced transport velocity, an identification that may not be appropriate, as discussed later in this section [Smith and Dukowicz (1997)]. Despite this word of caution, we shall begin by assuming that the identification made by Gent et al. (1995) is correct and then discuss how the identification might be modified.

We now examine $\mathbf{u}^* = \overline{z_\rho' \mathbf{u}'}/\overline{z_\rho}$ in detail. If $\mathbf{u}^*$ is indeed the eddy-induced transport velocity, then it is possible to associate $\mathbf{u}^*$ with an eddy stress, (X, Y), as in Eqs. (22) and (23). The question arises as to what form is taken by (X, Y)? We shall assume the eddies are geostrophic; that is,

$$-fv' = -\frac{1}{\rho_o}\pi_x'; \qquad fu' = -\frac{1}{\rho_o}\pi_y', \tag{35}$$

where π is the Montgomery potential. Using the hydrostatic equation $\pi_\rho' = gz'$, it can be shown, after some manipulation, that

$$\mathbf{u}^* = \frac{1}{f\rho_o \overline{z_\rho}} \mathbf{k} \times \left\{ -\frac{\partial}{\partial \rho}(\overline{p' \boldsymbol{\nabla}_\rho z'}) + \boldsymbol{\nabla}_\rho (\overline{p' z_\rho'}) \right\}. \tag{36}$$

The first term on the right-hand side corresponds to a vertical flux of momentum. In fact, Rhines and Holland (1979) refer to $\overline{p' \boldsymbol{\nabla}_\rho z'}$ as the ''inviscid pressure drag'' or "form drag." Conversion to (x, y, z) coordinates then shows that the form drag can be naturally associated with an eddy stress (X, Y), as in Eqs. (22) and (23), by putting

$$(X, Y) = \overline{p'\boldsymbol{\nabla}_\rho z'}. \tag{37}$$

The question remains as to the role played by the second term on the right-hand side of equation (36). The second term is a eddy-pressure term that is similar to the form drag and corresponds to a lateral flux of momentum along isopycnals, rather than a vertical flux of momentum. When the horizontal scale of the large-scale flow is large compared to the eddy-scale, the isopycnal flux term can be neglected in comparison with the form drag term, and the assumption expressed by Eq. (37) is valid. Since the eddy-scale is typically the same as the internal radius of deformation (at least for eddies generated by a baroclinic instability process), the isopycnal flux term should also be negligible when the planetary geostrophic approximation is valid for the large-scale flow. Since the GM90 parameterization can be written as a vertical flux of (geostrophic) momentum, the GM90 parameterization can then be regarded as a parameterization for the form drag, enabling us to write

$$\overline{p'\boldsymbol{\nabla}_\rho z'} = -\rho_o f\kappa \mathbf{k} \times \mathbf{L}. \tag{38}$$

We now return to the question of whether or not it is appropriate to identify $\mathbf{u}^*$ with the eddy-induced transport velocity $\mathbf{u}_\mathrm{I}$. Smith and Dukowicz have shown that, in general, the eddy-induced transport velocity need only be a part of $\mathbf{u}^*$, the remaining part being associated with a purely rotational eddy thickness flux. To see this, we apply a Helmholtz decomposition to the thickness flux $\overline{z_\rho}\mathbf{u}^*$, enabling us to write $\mathbf{u}^*$ as

$$\mathbf{u}^* = \mathbf{u}^*_D + \mathbf{u}^*_R, \tag{39}$$

where

$$\boldsymbol{\nabla}_\rho \cdot (\overline{z_\rho}\mathbf{u}^*_R) = 0. \tag{40}$$

(It should be noted that the Helmholtz decomposition is applied to the thickness flux $\overline{z_\rho}\mathbf{u}^*$, not to $\mathbf{u}^*$ itself.) Equation (40) says that $\mathbf{u}_R$ makes no contribution to the divergence of the thickness flux, and therefore makes no contribution in Eq. (30). Smith and Dukowicz argue that as a consequence, the eddy-induced transport velocity $\mathbf{u}_\mathrm{I}$ may differ from $\mathbf{u}^*$ by a component $\mathbf{u}_R$, as in (39) and (40), since then (30) is satisfied by the tracer transport velocity $\mathbf{U} = \overline{\mathbf{u}} + \mathbf{u}_\mathrm{I}$, with $\hat{\mathbf{u}}$ replaced by $\mathbf{U}$; that is,

$$\frac{\partial \overline{z_\rho}}{\partial t} + \boldsymbol{\nabla}_\rho \cdot (\overline{z_\rho}\mathbf{U}) = 0, \tag{41}$$

and the volume between mean isopycnals is preserved, as would be the case if $\mathbf{U} = \hat{\mathbf{u}}$. [In fact, Eq. (41) is a consequence of applying Eq. (15) with τ replaced by ρ and using $U_x + V_y + W_z = 0$ in (x, y, z) coordinates]. In fact, in a theoretical justificiation for the GM90 parameterization given by Smith and Dukowicz, the $\mathbf{u}_\mathrm{I}$ that emerges from the analysis differs from $\mathbf{u}^*$ by a nonzero $\mathbf{u}_R$.

Returning to Eq. (36), we note that the thickness flux associated with geostrophic eddies can be written as

$$\overline{z_\rho}\mathbf{u}^* = \frac{1}{f\rho_o}\mathbf{k} \times \left\{-\frac{\partial}{\partial \rho}(\overline{p'\boldsymbol{\nabla}_\rho z'}) + \boldsymbol{\nabla}_\rho(\overline{p'z'_\rho})\right\}. \tag{42}$$

On an f plane, the isopycnal eddy-pressure term $\boldsymbol{\nabla}_\rho(\overline{p'z'_\rho})$ makes no contribution to the divergence of thickness flux, even if the horizontal scale of the large-scale flow is comparable to the eddy scale. It follows from Smith and Dukowicz's argument that the isopycnal eddy-pressure term need not contribute to $\mathbf{u}_\mathrm{I}$ in this case. In general, however, when f varies with latitude, this term will contribute to $\mathbf{u}_\mathrm{I}$ because the associated thickness flux then has nonzero divergence. Also, because eddies are not strictly geostrophic, there will be other contributions to $\overline{z_\rho}\mathbf{u}^*$ in addition to those in Eq. (42), although like the isopycnal eddy pressure term, these contibutions are likely to be small compared to the form drag term. It follows that, in general, it is not possible to completely identify the eddy stress with the form drag, or even with a term that physically corresponds to a vertical flux of momentum, as assumed in the parameterizations of GM90 and GL90. As we shall see in the next section, a more fruitful approach is to consider the link between eddy-induced transport velocity and the isopycnal flux of potential vorticity.

Finally, in this section we note that if the eddy-induced transport velocity $\mathbf{u}_\mathrm{I}$ is not the same as $\mathbf{u}^*$ as defined by (33), then, correspondingly, the tracer transport velocity $\mathbf{U} = (U, V)$ is not the same as $\hat{\mathbf{u}}$. It follows that the simple interpretation presented in Eq. (34) is an oversimplification. In particular, the parameterization for the right-hand side of Eq. (31) needs to be in terms of a mixing tensor, as in Eq. (1) but for isopycnal coordinates. The antisymmetric part of the tensor is associated with an advection by a velocity that is the difference between the tracer transport velocity, (U, V) and $\hat{\mathbf{u}}$. Detailed analysis of output from eddy-resolving models is required for further investigation of the relationship between $\mathbf{u}^*$ and $\mathbf{u}_\mathrm{I}$.

4. The isopycnal flux of potential vorticity

We now explore the relationship between the eddy-induced transport velocity and the isopycnal flux of potential vorticity. We do this by generalizing the argument leading to Eq. (16) of Rhines and Holland (1979).

As in section 3, we assume adiabatic flow of a Boussinesq, incompressible fluid and work in isopycnal coordinates. The momentum equations can be written as

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_o}\pi_x + D(u) \tag{43}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_o}\pi_y + D(v), \tag{44}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \qquad (45)$$

is the material derivative, π is the Montgomery potential, and $D(u)$, $D(v)$ includes all terms arising from turbulent microstructure fluxes [the reader is referred to de Szoeke and Bennett (1993) for a detailed derivation—see their equations (12) and (13)]. For simplicity, we put $D(u) = D(v) = 0$ in the subsequent analysis. Equations (43) and (44) can be written as

$$\frac{\partial u}{\partial t} - (f + \zeta)v = -\frac{1}{\rho_o}B_x \qquad (46)$$

and

$$\frac{\partial v}{\partial t} + (f + \zeta)u = -\frac{1}{\rho_o}B_y, \qquad (47)$$

where $\zeta = v_x - u_y$ is the relative vorticity and B is the Bernoulli function. In terms of the potential vorticity, $q = (f + \zeta)/z_\rho$, (46) and (47) can be written as

$$\frac{\partial u}{\partial t} - q(z_\rho v) = -\frac{1}{\rho_o}B_x \qquad (48)$$

and

$$\frac{\partial v}{\partial t} - q(z_\rho u) = -\frac{1}{\rho_o}B_y. \qquad (49)$$

Averaging these equations now gives

$$\frac{\partial \overline{u}}{\partial t} - \overline{q}\,\overline{z_\rho}\hat{v} = -\frac{1}{\rho_o}\overline{B}_x + \overline{q'(z_\rho v)'} \qquad (50)$$

and

$$\frac{\partial \overline{v}}{\partial t} + \overline{q}\,\overline{z_\rho}\hat{u} = -\frac{1}{\rho_o}\overline{B}_y - \overline{q'(z_\rho u)'}. \qquad (51)$$

Comparison with Eq. (5) of Gent et al. (1995), or Eqs. (31) and (34) in section 3, shows that Gent et al. (1995) identify $(\overline{q'(z_\rho u)'}, \overline{q'(z_\rho v)'})$ with the isopycnal flux of potential vorticity; $\hat{\mathbf{u}} = (\hat{u}, \hat{v})$ is the same as defined by Eq. (32).

Although Eqs. (50) and (51) already demonstrate a link between the $\hat{u}$ and the isopycnal flux of potential vorticity, a more insightful approach is to use thickness-weighted, isopycnal-averaged variables. Putting $\hat{q} = \overline{qz_\rho}/\overline{z_\rho}$, $q = \hat{q} + q''$ and $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}''$, averaging of (48) and (49) gives

$$\frac{\partial \overline{u}}{\partial t} - \hat{q}\overline{z_\rho}\hat{v} = -\frac{1}{\rho_o}\overline{B}_x + \overline{z_\rho q''v''} \qquad (52)$$

and

$$\frac{\partial \overline{v}}{\partial t} - \hat{q}\overline{z_\rho}\hat{u} = -\frac{1}{\rho_o}\overline{B}_y - \overline{z_\rho q''u''}. \qquad (53)$$

Noting that $\hat{q}\overline{z_\rho} = \overline{\zeta} + f$ gives

$$\frac{\partial \overline{u}}{\partial t} - (\overline{\zeta} + f)\hat{v} = -\frac{1}{\rho_o}\overline{B}_x + \overline{z_\rho q''v''} \qquad (54)$$

and

$$\frac{\partial \overline{v}}{\partial t} - (\overline{\zeta} + f)\hat{u} = -\frac{1}{\rho_o}\overline{B}_y - \overline{z_\rho q''u''}. \qquad (55)$$

Taking $\partial(55)/\partial x - \partial(54)/\partial y$ and using (30) leads to the potential vorticity equation

$$\frac{\partial \hat{q}}{\partial t} + \hat{\mathbf{u}}\cdot\boldsymbol{\nabla}_\rho\hat{q} = -\frac{1}{\overline{z_\rho}}\boldsymbol{\nabla}_\rho\cdot(\overline{z_\rho q''\mathbf{u}''}). \qquad (56)$$

It should be noted that the equation corresponding to (56) for the variable $\overline{q}$ is complicated by the fact that $\overline{q} \neq (\overline{\zeta} + f)/\overline{z_\rho}$, and it is for this reason that Eqs. (52)–(55) are more useful than Eqs. (50)–(51). In Eqs. (52)–(55), the isopycnal flux of potential vorticity is given by $\overline{z_\rho q''\mathbf{u}''}$. An analysis similar to the above, also using the thickness-weighted average of potential vorticity, but for the case of zonal averaging, can be found in section 3.9 of Andrews et al. (1987); in particular, compare Eqs. (54) and (55) with Eq. (3.9.9) in Andrews et al. (1987) (see also Tung 1986).

Before proceeding further, we note that the potential vorticity (q-)flux terms on the right-hand side of Eqs. (54) and (55) act as forcing terms in the averaged momentum equations that are analogous to the eddy stress terms in Eqs. (18) and (19). It should be noted, however, that since Eqs. (54) and (55) have been derived by averaging the primitive equations, the q-flux terms include Reynolds stress terms that drive time-mean flow, in addition to the eddy stress terms associated with the eddy-induced transport velocity. The remaining part of the Reynolds stress terms is included in the average of the Bernoulli function $\overline{B}$ through the $(u^2 + v^2)/2$ term. It follows that a parameterization for the q-flux terms is not simply a parameterization for the eddy-induced transport velocity, but also for part of the time-mean flow driven by the Reynolds stresses, an important point we shall return to in section 5.

In order to concentrate attention on the eddy-induced transport velocity, we now simplify the analysis and approximate (46) and (47) by the geostrophic balance

$$-fv = -\frac{1}{\rho_o}\pi_x; \qquad fu = -\frac{1}{\rho_o}\pi_y, \qquad (57)$$

for which $q = f/z_\rho$, $\hat{q} = f/\overline{z_\rho}$, and B is replaced by π. Averaging (57) as before gives

$$-f\hat{v} = -\frac{1}{\rho_o}\overline{\pi}_x + \overline{z_\rho q''v''} \qquad (58)$$

and

$$f\hat{u} = -\frac{1}{\rho_o}\overline{\pi}_y - \overline{z_\rho q''u''}. \qquad (59)$$

Since the nonlinear momentum advection terms are not included in (57), there are no Reynolds stress terms contained in the q-flux terms on the right-hand side of (58) and (59). In fact, since averaging the geostrophic balance gives

$$-f\overline{v} = -\frac{1}{\rho_o}\overline{\pi}_x; \qquad f\overline{u} = -\frac{1}{\rho_o}\overline{\pi}_y \tag{60}$$

and since $\hat{\mathbf{u}} = \overline{\mathbf{u}} + \mathbf{u}^*$, it follows that for geostrophic flow

$$-fv^* = \overline{z_\rho q''v''}; \qquad fu^* = -\overline{z_\rho q''u''}. \tag{61}$$

Equation (61) shows that for geostrophic flow, $\mathbf{u}^*$ can be expressed directly in terms of the isopycnal flux of potential vorticity. We noted at the end of section 3 that the eddy-induced transport velocity $\mathbf{u_I}$ may differ from $\mathbf{u}^*$ by a component $\mathbf{u}_R$ as in Eq. (40) associated with a rotational component of the eddy thickness flux. To take account of this possibility, we write

$$-fv_I = (\overline{z_\rho q''v''})_D; \qquad fu_I = -(\overline{z_\rho q''u''})_D, \tag{62}$$

where the q-flux has been decomposed as

$$(\overline{z_\rho q''\mathbf{u}''}) = (\overline{z_\rho q''\mathbf{u}''})_D + (\overline{z_\rho q''\mathbf{u}''})_R \tag{63}$$

in the sense that

$$\boldsymbol{\nabla}_\rho\cdot\left\{\frac{(\overline{z_\rho q''\mathbf{u}''})_R}{\hat{q}}\right\} = 0. \tag{64}$$

Equation (64) is required by Eq. (40) (note that we have used $\hat{q} = f/\overline{z_\rho}$). The tracer transport velocity can then be expressed in terms of the q-flux as

$$-fV = -\frac{1}{\rho_o}\overline{\pi}_x + (\overline{z_\rho q''v''})_D \tag{65}$$

and

$$fU = -\frac{1}{\rho_o}\overline{\pi}_y + (\overline{z_\rho q''u''})_D. \tag{66}$$

Equations (65) and (66), together with Eq. (41), then give the potential vorticity equation

$$\mathbf{U}\boldsymbol{\nabla}_\rho\cdot\hat{q} = -\frac{1}{\overline{z_\rho}}\boldsymbol{\nabla}_\rho\cdot\{(\overline{z_\rho q''\mathbf{u}''})_D\}. \tag{67}$$

Equation (67) corresponds to (56), with $\hat{\mathbf{u}}$ replaced by the tracer transport velocity $\mathbf{U}$ and the local time derivative dropped because we have assumed geostrophic flow.

To see the connection between the isopycnal flux of q and the form drag, we now apply thickness-weighted, isopycnal averaging to the geostrophic balance. Following de Szoeke and Bennett (1993) we obtain

$$\overline{z_\rho}\left(-f\hat{v} + \frac{1}{\rho_o}\overline{\pi}_x\right) = -\frac{1}{\rho_o}\overline{z'_\rho\pi'_x} \tag{68}$$

and

$$\overline{z_\rho}\left(f\hat{u} + \frac{1}{\rho_o}\overline{\pi}_y\right) = -\frac{1}{\rho_o}\overline{z'_\rho\pi'_y}. \tag{69}$$

The term on the right-hand side of (68) and (69) is the thickness-pressure gradient covariance term. Using (60) and since $\hat{\mathbf{u}} = \overline{\mathbf{u}} + \mathbf{u}^*$, we immediately obtain

$$-fv^* = -\frac{1}{\overline{z_\rho}}\frac{1}{\rho_o}\overline{z'_\rho\pi'_x}; \qquad fu^* = -\frac{1}{\overline{z_\rho}}\frac{1}{\rho_o}\overline{z'_\rho\pi'_y}. \tag{70}$$

It is then straightforward to decompose the thickness-pressure gradient covariance term to obtain

$$f\mathbf{k}\times\mathbf{u}^* = -\frac{1}{\rho_o}\frac{1}{\overline{z_\rho}}\left\{-\frac{\partial}{\partial\rho}(\overline{p'\boldsymbol{\nabla}_\rho z'}) + \boldsymbol{\nabla}_\rho(\overline{p'z'_\rho})\right\}. \tag{71}$$

Equation (71) is equivalent to Eq. (36). We now equate (71) and (61) to obtain

$$-\mathbf{k}\times(\overline{z_\rho q''\mathbf{u}''}) = -\frac{1}{\rho_o}\frac{1}{\overline{z_\rho}}\left\{-\frac{\partial}{\partial\rho}(\overline{p'\boldsymbol{\nabla}_\rho z'}) + \boldsymbol{\nabla}_\rho(\overline{p'z'_\rho})\right\}. \tag{72}$$

Equation (72) shows the connection between isopycnal mixing of q and the form drag. In fact, (72) formally generalizes the correspondence in quasigeostrophic theory between vertical mixing of momentum and horizontal mixing of potential vorticity. As in section 3, where we found that the eddy-induced transport velocity cannot, in general, be identified exactly with the form drag, then so here, there is not an exact correspondence between the isopycnal flux of potential vorticity and the form drag. Also because $\overline{z_\rho}$, in general, varies along an isopycnal, the $\boldsymbol{\nabla}_\rho(\overline{p'z'_\rho})$ term in general contributes to the divergence of the q-fluxes in Eq. (56).

Finally in this section we note that (68) and (69) can be written

$$\overline{z_\rho}\left(-f\hat{v} + \frac{1}{\rho_o}\overline{\pi}_x\right) = -\frac{1}{\rho_o}\left\{-\frac{\partial}{\partial\rho}\left(\overline{p'\frac{\partial}{\partial x}z'}\right) + \frac{\partial}{\partial x}(\overline{p'z'_\rho})\right\} \tag{73}$$

and

$$\overline{z_\rho}\left(f\hat{u} + \frac{1}{\rho_o}\overline{\pi}_y\right) = -\frac{1}{\rho_o}\left\{-\frac{\partial}{\partial\rho}\left(\overline{p'\frac{\partial}{\partial y}z'}\right) + \frac{\partial}{\partial y}(\overline{p'z'_\rho})\right\}. \tag{74}$$

In this form, the terms on the right-hand side of (71) appear as the divergence of the Eliassen–Palm flux [see, in particular, Lee and Leach (1996), who use time averaging, as in this paper, but also Andrews et al. (1987) and Tung (1986) for the more traditional form of the Eliassen–Palm flux using zonal averaging]. Viewed in this way, the failure to obtain an exact identification between the eddy stress and the form drag can be understood by noting that the Eliassen–Palm flux involves a component along isopycnals as well as a vertical component. Equation (72) can be viewed as a version of the correspondence between the Eliassen–Palm pseudodivergence and the isopycnal flux of Ertel potential vorticity noted by Tung (1986). Another version of (72) using an approximation to the Ertel potential vorticity is discussed by Lee and Leach (1996).

5. The potential vorticity constraint and parameterization of the tracer transport velocity

We now illustrate how the potential vorticity equation [that is, Eq. (56) or (67)] places a constraint on parameterizations for the tracer transport velocity (and hence the eddy-induced transport velocity). We begin by showing how the GM90 parameterization can be written in terms of the isopycnal flux of potential vorticity. Combining Eqs. (18), (19), and (25) from section 2, we know that for the planetary geostrophic system, the GM90 parameterization can be written as

$$-fV = -\frac{1}{\rho_o}\frac{\partial \overline{p}}{\partial x} + \frac{1}{\rho_o}\frac{\partial X}{\partial z} \tag{75}$$

$$fU = -\frac{1}{\rho_o}\frac{\partial \overline{p}}{\partial y} + \frac{1}{\rho_o}\frac{\partial Y}{\partial z}, \tag{76}$$

with

$$(X, Y) = \rho_o \kappa \frac{f^2}{N^2}\mathbf{u}_{g_z}, \tag{77}$$

where N is the local value of the buoyancy frequency, and (U, V, W) is the tracer transport velocity. In isopycnal coordinates, these equations become

$$-fV = -\frac{1}{\rho_o}\frac{\partial \overline{\pi}}{\partial x} - \frac{1}{\overline{z_\rho}}\frac{\partial}{\partial \rho}\left\{\kappa\rho_o \frac{f^2}{g}\frac{\partial \overline{u}_g}{\partial \rho}\right\} \tag{78}$$

$$fU = -\frac{1}{\rho_o}\frac{\partial \overline{\pi}}{\partial y} - \frac{1}{\overline{z_\rho}}\frac{\partial}{\partial \rho}\left\{\kappa\rho_o \frac{f^2}{g}\frac{\partial \overline{v}_g}{\partial \rho}\right\}, \tag{79}$$

where $(\overline{u}_g, \overline{v}_g)$ is the geostrophic velocity. If $\partial\kappa/\partial\rho = 0$, then (78) and (79) can be written in terms of the potential vorticity, $\hat{q} = f/\overline{z_\rho}$, as

$$-fV = -\frac{1}{\rho_o}\frac{\partial \overline{\pi}}{\partial x} - \overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial y} + \kappa\beta \tag{80}$$

and

$$fU = -\frac{1}{\rho_o}\frac{\partial \overline{\pi}}{\partial y} + \overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial x}. \tag{81}$$

Equations (80) and (81) can be identified with equations (65) and (66) by putting

$$(\overline{z_\rho q'' v''})_D = -\overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial y} + \kappa\beta \tag{82}$$

and

$$-(\overline{z_\rho q'' u''})_D = \overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial x}. \tag{83}$$

The appearance of the $\kappa\beta$ on the right-hand side of (82) should be noted. This term ensures that when the isopycnals are horizontal, implying no vertical shear of the geostrophic velocity, the eddy-induced transport velocity $(U - \overline{u}, V - \overline{v})$ is zero, as one might expect given that GM90 is designed to mimic the effect of baroclinic instability. For the β-plane geometry we are using, and as long as κ is independent of y, $\kappa\beta$ plays no role in the potential vorticity equation because it is part of the rotational component of the q-flux. We can then obtain the potential vorticity equation

$$U\frac{\partial \hat{q}}{\partial x} + V\frac{\partial \hat{q}}{\partial y} = \frac{1}{\overline{z_\rho}}\nabla_\rho\cdot(\overline{z_\rho}\kappa\nabla_\rho \hat{q}). \tag{84}$$

It should be noted that to derive (84), it was necessary to assume that both $\partial\kappa/\partial\rho = 0$ and $\partial\kappa/\partial y = 0$. It is not a coincidence that GL90 made these same two assumptions in order to obtain homogenization of potential vorticity within closed q contours using the GM90 and GL90 parameterizations. That Eq. (84) is satisfied by the GM90 parameterization only when restrictions are placed on κ is a weakness of GM90. Indeed, the analysis in section 4 shows that the potential vorticity equation [that is, Eqs. (56) or (67) or their equivalent] should be satisfied by any parameterization for (U, V).

We next note that the simplest parameterization in terms of q-fluxes would be to use (80) and (81) to compute (U, V) with a spatially variable, time-dependent κ and the $\kappa\beta$ term dropped; that is,

$$-fV = -\frac{1}{\rho_o}\frac{\partial \overline{\pi}}{\partial x} - \overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial y} \tag{85}$$

and

$$fU = -\frac{1}{\rho_o}\frac{\partial \overline{\pi}}{\partial y} + \overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial x}. \tag{86}$$

Equation (84) is then automatically satisfied irrespective of the form of κ. In the context of geostrophic flow [see the discussion following Eq. (57)], the eddy-induced transport velocity is then given by

$$-fv_I = -\overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial y}; \qquad fu_I = \overline{z_\rho}\kappa\frac{\partial \hat{q}}{\partial x}. \tag{87}$$

Writing these equations as

$$-fv_I = -\kappa\beta + \kappa f\frac{1}{\overline{z_\rho}}\frac{\partial}{\partial \rho}\left\{\frac{\partial z}{\partial y}\right\};$$

$$-fu_I = -\kappa f\frac{1}{\overline{z_\rho}}\frac{\partial}{\partial \rho}\left\{\frac{\partial z}{\partial x}\right\} \tag{88}$$

makes clear the connection with Treguier et al. (1997) and K97 [compare Eq. (88) with Eq. (39) in Treguier et al. (1997) and Eq. (35) in K97]. The analysis given in section 4, however, shows that the approach taken here is actually more general than that in Treguier et al. (1997) or K97, a point we now explore further.

We first observe, by analogy with (54) and (55), that we can generalize (85) and (86) to the primitive equations by using the following equations to compute the tracer transport velocity (U, V)

$$\frac{\partial \overline{u}}{\partial t} - (\overline{\zeta} + f)V = -\frac{1}{\rho_o}\overline{B}_x - \overline{z_\rho}\,\kappa\frac{\partial \hat{q}}{\partial y} \qquad (89)$$

and

$$\frac{\partial \overline{v}}{\partial t} - (\overline{\zeta} + f)U = -\frac{1}{\rho_o}\overline{B}_y - \overline{z_\rho}\,\kappa\frac{\partial \hat{q}}{\partial x}, \qquad (90)$$

where $\hat{q} = (\overline{\zeta} + f)/\overline{z_\rho}$ is the (unapproximated) thickness-weighted, isopycnal averaged potential vorticity, and the q-flux terms on the right-hand side of (89) and (90) are parameterizations for the q-flux terms on the right-hand side of (54) and (55). A similar parameterization, but for the case of zonal averaging (rather than the time-averaging used here), has been proposed for the stratosphere by Tung (1986) [see his equation (5.10) and note that Tung includes additional terms that arise from the effect of diabatic heating, an issue to be explored in a later paper]. Then $\partial(90)/\partial x - \partial(89)/\partial y$ gives the potential vorticity equation

$$\frac{\partial \hat{q}}{\partial t} + U\frac{\partial \hat{q}}{\partial x} + V\frac{\partial \hat{q}}{\partial y} = \frac{1}{\overline{z_\rho}}\boldsymbol{\nabla}_\rho\cdot(\overline{z_\rho}\,\kappa\boldsymbol{\nabla}_\rho\hat{q}), \qquad (91)$$

where use has been made of (41). Equation (91) ensures that potential vorticity is homogenized inside closed $\hat{q}$ contours in the absense of forcing and dissipation of the large-scale $\hat{q}$-field. It should be noted that in (91), and also (84), the term on the right-hand side refers only to mixing along isopycnal surfaces; in particular, diapycnal mixing of potential vorticity is not implied (Haynes and McIntyre 1987).

We next note that in Eqs. (89) and (90) the q-flux terms parameterize not only the eddy-induced transport velocity, but also part of the Reynolds stresses that drive time-mean flow [that the q-fluxes include a part of the Reynolds stress terms was noted in the discussion following Eq. (56)]. Killworth argues that restrictions are necessary on the diffusion coefficient κ in order to ensure that the q-fluxes have the properties of the eddy-induced transport velocity, as would be the case if equation (87) were valid [in particular, it would be necessary to satisfy the boundary condition (27) at the ocean bottom, together with $(X, Y) = 0$ at $z = 0$]. In general, however, these restrictions may not be necessary since Eq. (87) does not account for the contribution of the Reynolds stress terms to the q-fluxes, an issue to be discussed further in a later paper. It should also be noted that even when the planetary geostrophic approximation is valid for the large-scale flow, in which case (89) and (90) can be approximated by (85) and (86), the q-flux terms still contain a contribution from the Reynolds stress terms, implying that (87) may still not be valid. The validity of (87) required the assumption of geostrophic flow, as in (57), before the application of averaging.

As written, Eqs. (89) and (90) do not provide a complete closure. For example, part of the Reynolds stress terms is included in the time-averaged Bernoulli function and requires parameterization. Noting that

$$\overline{B} = \rho_o\frac{(\overline{u}^2 + \overline{v}^2)}{2} + \overline{p} + g\rho\overline{z} + \rho_o\frac{\overline{(u'^2 + v'^2)}}{2}, \qquad (92)$$

we see that developing a closure for the missing Reynolds stress term requires parameterizing the eddy kinetic energy. One way to do this might be in terms of a Richardson number for the large-scale flow (see, e.g., Treguier et al. 1997; Visbeck et al. 1997). Substituting (92) into (89) and (90) shows that for both the mean and eddy kinetic energy terms, it is only their gradient that appears in the governing equations, which explains why these terms play no role in the potential vorticity equation (91). [It should be noted that a more complete parameterization would likely include a rotational potential vorticity flux that would also appear as the gradient of a scalar in (89) and (90) and would need to be included in the closure.] Another problem with Eqs. (89), (90), and (92) is that they contain a mixture of the time-mean velocity $\overline{\mathbf{u}}$ and the tracer transport velocity $\mathbf{U}$. Since the tracer transport velocity is the fundamental velocity variable, we need to express $\overline{\mathbf{u}}$ in terms of $\mathbf{U}$. In general, the difference between $\overline{\mathbf{u}}$ and $\mathbf{U}$ is likely to be order the Rossby number (Smith and Dukowicz), so it should be possible to simply replace $\overline{\mathbf{u}}$ by $\mathbf{U}$, including for the calculation of $\overline{\zeta}$. [It should be noted that the terms involving $\overline{\mathbf{u}}$ are themselves of order the Rossby number in (89), (90), and (92)]. There is also a need to specify κ. One approach is to try and use linear stability theory, along the lines suggested by K97. Another would be to base the coefficient on a Richardson number for the large-scale flow, as in Visbeck et al. (1997).

Finally, we note that as in Eq. (88), the q-flux terms on the right-hand side of (89) and (90) can be written as

$$-\overline{z_\rho}\,\kappa\frac{\partial \hat{q}}{\partial y} = -\kappa\beta + \kappa f\frac{1}{\overline{z_\rho}}\frac{\partial}{\partial \rho}\left\{\frac{\partial z}{\partial y}\right\};$$

$$\overline{z_\rho}\,\kappa\frac{\partial \hat{q}}{\partial x} = -\kappa f\frac{1}{\overline{z_\rho}}\frac{\partial}{\partial \rho}\left\{\frac{\partial z}{\partial x}\right\}. \qquad (93)$$

In a later paper, we shall describe implementation of the new parameterization in a model using the form for the q-fluxes given in (93). Detailed discussion of the implementation and, in particular, the top and bottom boundary conditions, is given there. For now we note that at the surface and the bottom, the slope $\boldsymbol{\nabla}_\rho z$ is required to be parallel to the bounding surface; that is, zero at the surface and parallel to the bottom slope at the bottom, as required by considering the role in the potential vorticity budget played by the top and bottom boundaries in the stretching and squashing of a vertical column of fluid. In cases where the isopycnals intersect the top and bottom boundaries without passing through a well-mixed layer (that is, a layer, with $\rho_z = 0$), these boundary conditions correspond to the delta functions introduced by K97. When there is a mixed layer or a well-mixed region due to deep convection, the isopycnal slope is linearly interpolated across the region where $\rho_z = 0$ in a manner analogous to the suggestion of Treguier et al. (1997). Detailed discussion of this and other related issues is deferred until a later paper.

6. Summary and discussion

We began in section 2 by showing that the eddy-induced transport velocity can always be written as an Ekman velocity associated with a horizontally directed stress term in the momentum equations for the tracer transport velocity. In the case of the GM90 parameterization, the stress is provided by the vertical flux of geostrophic momentum. We then went on to show that the eddy stress has a natural interpretation as the ''inviscid pressure drag'' or ''form drag'' associated with the eddies, although the correspondence is not exact. We have also shown the link between the eddy-induced transport velocity, the eddy form drag, and the isopycnal flux of potential vorticity, generalizing the quasigeostrophic result of Rhines and Holland (1979) and Rhines and Young (1982) that equates vertical mixing of momentum with horizontal mixing of potential vorticity. The link to the isopycnal flux of potential vorticity provides a basis for developing parameterizations in terms of potential vorticity mixing. We noted that the GM90 parameterization is compatible with the potential vorticity equation only when restrictions are placed on the diffusivity coefficient. We regard the need for these restrictions to be a weakness of GM90 since the potential vorticity equation should be satisfied by any parameterization. We suggest that the development of improved parameterizations should be based on specifying forcing terms in the momentum equations for the tracer transport velocity and that the forcing terms should be expressed in terms of the isopycnal flux of potential vorticity, as illustrated by Eqs. (89) and (90). A new parameterization is suggested that is the simplest parameterization with the required form. We have shown that the new parameterization not only parameterizes the eddy-induced transport velocity, but also part of the time-mean flow driven by the Reynolds stress terms in the momentum equations. By working with the momentum equations, as in this paper, it is the tracer transport velocity that emerges as the fundamental variable. Indeed, the analysis of sections 4 and 5 suggests that effort should be directed at parameterizing the tracer transport velocity directly, rather than the eddy-induced transport velocity on its own, thereby including the Reynolds stress driven flow directly in the parameterization.

The approach taken in this paper has some similarity to that taken by Tung (1986). For the case of zonal averaging applied to the atmosphere, Tung relates the isentropic flux of Ertel's potential vorticity to the Eliassen–Palm pseudodivergence, a result analogous to that expressed by Eq. (72). Tung also advocates parameterizing the isentropic flux of Ertel potential vorticity as a forcing term in the averaged momentum equations, and in fact proposes a parameterization similar to that proposed here. Tung's parameterization includes additional terms [see his Eq. (5.10)] that arise from considering the effect of diabatic heating of the atmosphere, corresponding to diapycnal mixing in the ocean. Future work will address the possible role of these extra terms in parameterizations applicable to the ocean, generalizing the parameterization proposed here.

A somewhat different approach to parameterizing the momentum equation can be found in Gent and McWilliams (1996). These authors advocate parameterizing the eddy-induced transport velocity separately from the momentum equations and do not take advantage, as is done in this paper or in Tung (1986), of the close correspondence between the pseudodivergence of the Eliassen–Palm flux and the isopycnal flux of potential vorticity.

A feature of the proposed parameterization is that for horizontally flat isopycnals, the q-flux terms in (93) are nonzero if κ is nonzero, on account of the $\kappa\beta$ term. We do not regard this as a weakness of the proposed parameterization. In fact, as pointed out by Holloway (1992), although baroclinic instability processes act to remove horizontal density gradients, the effect of eddies is to drive the large-scale ocean circulation towards a state of motion, not one of rest. The possibility of a connection between the parameterization proposed here and the ''Neptune'' effect of Holloway (1992) is discussed in detail in a later manuscript. The $\kappa\beta$ term also appears in the work of Welander (1973) and Tung (1986). Indeed, Tung (1986) notes that without the $\kappa\beta$ term the winter stratospheric jet would reach unrealistically large velocities.

A complication throughout our analysis has been that whereas Gent et al. (1995) assumed the tracer transport velocity and the thickness-weighted, isopycnal-averaged velocity to be synonymous, recent theoretical work by Smith and Dukowicz (1997) suggests that this may not be true in general. Guidance from eddy-resolving model output is required to clarify this issue.

A feature of our analysis, and other related papers, for example, Gent et al. (1995), McDougall et al. (1996, submitted to *J. Phys. Oceanogr.*), and McDougall and McIntosh (1996) is the application of different kinds of averaging to the equations of motion. Indeed, our manuscript illustrates the use of both weighted and unweighted averaging on isopycnals. It follows that care is required when interpreting the variables carried by models, a particularly important issue when assimilating observations into a model.

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