Boundary Condition of the Sverdrup Solution from Flow Energetics

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ABSTRACT

The energetics of the wind-driven, quasigeostrophic circulation is used to rule out the vanishing of the Sverdrup solution in the western boundary of the ocean, apart from the total solution near the eastern boundary. The result holds for every wind stress curl.

1. Introduction

The understanding of the physical cause for the intense crowding of streamlines in the proximity of the western boundary of all the oceans is the main question that generated the modern wind-driven ocean circulation theory, beginning from Stommel (1948). Once the variation of the Coriolis parameter with latitude is recognized as the fundamental dynamical ingredient in order to reproduce the westward intensification,

> ''. . . it is a classical problem . . . that shows that it is not possible to add a boundary layer on the eastern boundary of the oceans so that the Sverdrup solution itself must satisfy the boundary condition there. Whether a boundary layer is then possible on the western boundary is less clear.'' (Pedlosky 1994).

It is interesting that if one tries to solve the circulation problem for the western boundary ignoring *tout court* what happens on the eastern, then one is not able to assign to the Sverdrup solution a definite boundary condition on the western coast and the whole procedure becomes problematic. However, a general physical explanation for the preference for westward intensification is given by Pedlosky (1965) on the basis of the anisotropy of the group velocity in the Rossby waves propagation. In this paper we put forward another argument, based on the energetics of the flow field, to show that the vanishing of the Sverdrup solution on the western boundary is not consistent with the steady energy balance of the ocean. In this way, we can assign the correct boundary condition to the Sverdrup solution in the ambit of the whole class of quasigeostrophic models of wind-driven circulation without resorting to the total solution near the eastern boundary.

2. Basic equations

We assume a steady circulation in a rectangular basin on the beta plane, governed by the well-known vorticity equation (Pedlosky 1987a)

$$\left(\frac{\delta_I}{L}\right)^2 J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = \mathbf{k}\cdot\nabla \times \boldsymbol{\tau} - \frac{\delta s}{L}\nabla^2\psi + \left(\frac{\delta_M}{L}\right)^3 \nabla^4\psi \tag{1}$$

with the following boundary conditions

$$\psi|_{\partial D} = 0, \tag{2}$$

$$\nabla^2\psi|_{\partial D} = 0 \quad \text{or} \quad \nabla\psi|_{\partial D} = \mathbf{0} \quad \text{if } \delta_M \neq 0, \tag{3}$$

where

$$D = [x_W \le x \le x_E] \times [y_S \le y \le y_N]$$

is the fluid domain.

Hereafter we put

$$\mathbf{k}\cdot\nabla \times \boldsymbol{\tau} = T(x,y), \tag{4}$$

where T is assumed to be an $O(1)$ given function of x and y, slowly varying with respect to the horizontal dimensions of the basin. Our investigation does not demand specific details on the wind-stress curl, consistent with the fact that we do not have at our disposal any analytical expression of the ''true'' large-scale curl. In fact, in the majority of the models, the forcing field

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mimics one of the nine zonal belts of world $\nabla \times \boldsymbol{\tau}$ fields (Hellerman and Rosenstein 1983) by means of sinusoids or more sophisticated best-fit profiles (i.e., Moro 1988). On the other hand, nothing prevents us from taking into account rather complex wind-stress curl; for instance,

$$\mathbf{k}\cdot\nabla \times \tau = \sin\left[\frac{\pi}{2}(2 - x)y\right],$$

as did Bryan (1963).

The standard way to obtain an energy equation for the system governed by Eqs. (1)–(4) is to multiply Eq. (1) by ψ and then integrate over D with the aid of boundary conditions (2) and (3). The result is

$$\iint_D \psi T\, dx\, dy = -\frac{\delta_S}{L}\iint_D \nabla\psi\cdot\nabla\psi\, dx\, dy - \left(\frac{\delta_M}{L}\right)^3 \iint_D (\nabla^2\psi)^2\, dx\, dy. \quad (5)$$

Equation (5) states an equilibrium condition between the source of mechanical energy due the wind input (first term on the rhs) and the sinks (represented by the last two terms), which holds in the steady motion. In particular, the first term of Eq. (5) is proportional to

$$\iint_D pw_E dx\, dy,$$

where p is the geostrophic perturbation pressure, while w_E is the vertical velocity of the interface between the upper Ekman layer and the geostrophic interior. Thus, this is a pressure-work term arising from the convergence/divergence of the subsurface horizontal transport that, in the steady circulation, is balanced by the energy absorption due to eddy viscosity [last two terms of Eq.(5)].

3. The energy source

Since the last two terms of Eq. (5) are negative definite [unless ψ is a constant, but this circumstance is not consistent with Eq. (1)], the source term must satisfy the inequality

$$\iint_D \psi(x, y)T(x, y)\, dx\, dy < 0. \quad (6)$$

In order to investigate some consequences of inequality (6) in the presence of the Sverdrup regime in the interior, we consider the total O(1) solution ψ, valid in the interior and in the boundary layer, written as

$$\psi(x, y) = \int_{x_B}^{x} T(\lambda, y)\, d\lambda + \phi_{\tilde{B}}\left(\frac{L}{\delta_F}(x - x_{\tilde{B}}), y\right). \quad (7)$$

In Eq. (7)

$$\psi_I(x, y) = \int_{x_B}^{x} T(\lambda, y)\, d\lambda$$

is the Sverdrup solution, and $\phi_{\tilde{B}}(\xi, y)$ is the boundary layer correction in $x = x_{\tilde{B}}$, where

$$\xi = \frac{L}{\delta_F}(x - x_{\tilde{B}})$$

is the stretched boundary layer coordinated and $\delta_F = \max\{\delta_S, \delta_M\}$ is the boundary layer width.

For the moment, x_B is left indeterminate between the two possible longitudes x_W and x_E. We do not consider the eventuality that x_B may be some interior longitude since it would imply $u_I(x_B, y) = 0$ for the zonal component of the interior current, but this kind of constraint has no physical basis. If $x_B = x_W$, then $\phi_{\tilde{B}} = \phi_E$, that is, the boundary layer correction is on the eastern side of the basin; on the contrary, if $x_B = x_E$, then $\phi_{\tilde{B}}$ is the correction relative to the western side.

We recall that the very assumption of a Sverdrup regime in the interior presupposes

$$\frac{\delta_F}{L} \ll 1. \quad (8)$$

Substitution of Eq. (7) into inequality (6) leads to the following inequality:

$$\int_{y_S}^{y_N}\int_{x_W}^{x_E}\left[T(x, y)\int_{x_B}^{x} T(\lambda, y)\, d\lambda\right] dx\, dy + \int_{y_S}^{y_N}\int_{x_W}^{x_E} \phi_{\tilde{B}}(\xi, y)T(x, y)\, dx\, dy < 0. \quad (9)$$

Consider now the first integral of inequality (9):

$$I_1 = \int_{y_S}^{y_N}\int_{x_W}^{x_E}\left[T(x, y)\int_{x_B}^{x} T(\lambda, y)\, d\lambda\right] dx\, dy.$$

Using the above definition of $\psi_I(x,y)$, we can write

$$I_1 = \frac{1}{2}\int_{y_S}^{y_N}\int_{x_W}^{x_E} \frac{\partial}{\partial x}\psi_I^2(x, y)\, dx\, dy = \frac{1}{2}\int_{y_S}^{y_N}\left[\left(\int_{x_B}^{x_E} T(\lambda, y)\, d\lambda\right)^2 - \left(\int_{x_B}^{x_W} T(\lambda, y)\, d\lambda\right)^2\right] dy.$$

Therefore,

(i) if $x_B = x_E$, then

$$I_1 = -\frac{1}{2}\int_{y_S}^{y_N}\left(\int_{x_E}^{x_W} T(\lambda, y)\, d\lambda\right)^2 dy < 0;$$

while

(ii) if $x_B = x_W$, then

$$I_1 = \frac{1}{2}\int_{y_S}^{y_N}\left(\int_{x_W}^{x_E} T(\lambda, y)\, d\lambda\right)^2 dy > 0.$$

About the second integral of inequality (9), it can be written entirely in terms of ξ and y as follows:

(i) if $x_{\tilde{B}} = x_W$, then $\xi \geq 0$ and

$$I_2 = \int_{y_S}^{y_N}\int_0^{+\infty} \phi_W(\xi, y) T\left(x_W + \frac{\delta_F}{L}\xi, y\right)\frac{\delta_F}{L}\, d\xi\, dy$$
$$\approx \frac{\delta_F}{L}\int_{y_S}^{y_N} T(x_W, y)\int_0^{+\infty}\phi_W(\xi, y)\, d\xi\, dy;$$

while

(ii) if $x_{\tilde{B}} = x_E$, then $\xi \leq 0$ and

$$I_2 = \int_{y_S}^{y_N}\int_{-\infty}^{0} \phi_E(\xi, y) T\left(x_E + \frac{\delta_F}{L}\xi, y\right)\frac{\delta_F}{L}\, d\xi\, dy$$
$$\approx \frac{\delta_F}{L}\int_{y_S}^{y_N} T(x_E, y)\int_{-\infty}^{0}\phi_E(\xi, y)\, d\xi\, dy.$$

In both cases (i) and (ii) $I_1 = O(1)$, while

$$I_2 = O\left(\frac{\delta_F}{L}\right),$$

so because of inequality (8) the sign of inequality (9) is ultimately controlled by that of I_1. This fact allows us to rule out the choice $x_B = x_W$ for the Sverdrup solution, whatever the wind stress curl may be.

4. Concluding remarks

In general, use of energetics in the investigation of rotating fluids is not a powerful tool mainly because the Coriolis force is purely deflecting and its effect can be hardly isolated by resorting to the energy balance of the system. However, in the present paper, due to the assumption of a Sverdrup regime in the interior, we have bypassed this difficulty by introducing into the streamfunction appearing in the energy source the Sverdrup solution that has a typical rotating character; in this way, we have preserved the memory of rotation also in the energetics. The peculiarity of the employed method is based on the direct correlation between the sign of the energy source and the unique correct integration extreme appearing in the Sverdrup solution.

The result that we have obtained in section 3, that is, with reference to Eq. (7), $x_B = x_E$ (and hence $\tilde{x}_{\tilde{B}} = x_W$) relies on Eq. (1), which concerns a homogeneous ocean. A feature of Eq. (1) that is useful in proving the result above is its immediate transformation into the Sverdrup balance (for the geostrophic layer) as far as the sole interior region of the fluid domain is taken into account. These arguments still hold if the homogeneous layer in motion extends down to the thermocline depth only, and the fluid in the deeper layer remains at rest. In this case the vanishing ageostrophic vertical velocity (w_1, see below) at the bottom level is substituted by the corresponding velocity at the depth of the flat interface between the thermocline and the quiescent abyss. In any case the same Eq. (1) is obtained by the vertical integration of the vorticity equation

$$J(\psi, \nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = \frac{\partial w_1}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^4\psi \qquad (10)$$

across the geostrophic interior. In Eq. (10) $\beta = (L/\delta_I)^2$ and $\mathrm{Re} = L\delta_I^2/\delta_M^3$ is the Reynolds number. From this last point of view, the term

$$-\frac{\delta s}{L}\nabla^2\psi$$

appearing in Eq. (1) is perhaps better understood as a formal eddy viscosity parameterization, rather than the effect of Ekman pumping at the interface depth.

The vanishing of the Sverdrup solution along the eastern coast of the basins and the subsequent westward intensification seem to be intimately linked to some ''key inequalities'' arising from the dynamics of the circulation. This can be seen starting from the time-dependent version of Eq. (1); that is,

$$\left(\frac{\delta_I}{L}\right)^2\left[\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi\right)\right] + \frac{\partial\psi}{\partial x}$$
$$= F(\psi) + T(x, y), \qquad (11)$$

where $F(\psi)$ represents any frictional mechanism.

With reference to Eq. (11), Pedlosky (1965) considers what happens *in a meridional boundary layer,* where the main balance is

$$\left(\frac{\delta_I}{L}\right)^2\left[\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi\right)\right] + \frac{\partial\psi}{\partial x} = F(\psi). \qquad (12)$$

The time-dependent nature of Eq. (12) implies that the small-scale decaying Rossby wave solution is characterized by a positive group velocity

$$C_{gx} > 0, \qquad (13)$$

so, since the dissipation allows only short travel distances, the only source of small-scale motion is represented by the reflection of the large-scale motion on the western boundary. Thus, assumption $\partial/\partial t \neq 0$ leads to inequality (13).

In a complementary way, we have taken into account the flow behavior *in the interior* where, unlike Eq. (12), the dominant balance involves the wind stress forcing and is typically steady. As we have already seen, the related equation

$$\left(\frac{\delta_I}{L}\right)^2 J\left(\psi, \nabla^2\psi\right) + \frac{\partial\psi}{\partial x} = F(\psi) + T(x, y)$$

leads to inequality (6).

From this viewpoint, the east–west asymmetry of the large-scale circulation patterns can be ascribed to two complementary inequalities: the first (13) implied by the wavelike mesoscale dynamics and the second (6) coming from the basin-scale energetics.

Strictly speaking, to account for negative values of the planetary vorticity gradient, say β_0, the term $\partial\psi/\partial x$ in Eq.(1) should be substituted by

$$\frac{\beta}{|\beta|}\frac{\partial\psi}{\partial x},$$

where

$$\beta = \left(\frac{L}{\delta_I}\right)^2 = \frac{\beta_0 L^2}{U}.$$

As a consequence, the Sverdrup solution should take the form

$$\psi_I(x, y) = \frac{|\beta|}{\beta}\int_{x_B}^{x} T(\lambda, y)\, d\lambda,$$

and, finally, the integral I_1 becomes

$$I_1 = \frac{|\beta|}{2\beta}\int_{y_S}^{y_N}\left[\left(\int_{x_B}^{x_E} T(\lambda, y)\, d\lambda\right)^2 - \left(\int_{x_B}^{x_W} T(\lambda, y)\, d\lambda\right)^2\right] dy. \quad (14)$$

From Eq. (14), recalling inequality (6), we conclude that, if $\beta < 0$, we must rule out the choice $x_B = x_E$ in order that $I_I < 0$, so an eastern boundary current arises. Obviously, in this case the (x,y) coordinates are referred to the beta plane associated to the observer that detects a negative planetary vorticity gradient.

We underline that the followed method is not only independent from the explicit form of the wind stress curl, but it is also largely independent from the details of the parameterization of turbulence.

For instance, the same conclusion holds even if we consider a dissipation including also a biharmonic term of the kind $-|A|\nabla^6\psi$. In this case, instead of Eq. (5), we have

$$\iint_D \psi T\, dx\, dy = -|A|\iint_D \nabla(\nabla^2\psi)\cdot\nabla(\nabla^2\psi)\, dx\, dy - \frac{\delta_2}{L}\iint_D \nabla\psi\cdot\nabla\psi\, dx\, dy - \left(\frac{\delta_M}{L}\right)^3\iint_D (\nabla^2\psi)^2\, dx\, dy,$$

but again inequality (6) follows, provided that at least one of the dissipative terms be nonvanishing.

The fluid domain D, defined in section 2, has not necessarily a rectangular shape. In fact, we can easily see that the same conclusions hold if we allow x_W and x_E to vary with y, provided that $x_W < x_E$. Therefore, the correct boundary condition of the Sverdrup solution is quite independent from the details of the meridional shorelines.

There is an analogy between the problem of finding the correct boundary condition of the Sverdrup solution and that related to the outcropping of the isopycnals considered by Pedlosky (1987b) within a wind-driven two-layered model of the subtropical–subpolar circulation. The governing equations of this last model for the upper warm water layer of thickness D are the geostrophic equilibrium

$$\mathbf{u} = \frac{\gamma}{f}\mathbf{k} \times \nabla D, \quad (15)$$

(γ is the reduced gravity and k is the vertical unit vector) and the Sverdrup balance

$$\beta v_I D_I = f w_e(y) \quad (16)$$

(the subscript I refers to the geostrophic interior and w_e is the latitude-dependent Ekman pumping velocity).

Well south of the subtropical gyre

$$w_e < 0, \quad (17)$$

while the outcrop can only exist if w_e is positive. This property of the forcing field determinates the correct integration interval of the differential equation for D_I that follows from Eqs. (15) and (16). In fact, from these equations we obtain

$$\frac{\partial}{\partial x} D_I^2 = \frac{2f^2}{\beta\gamma}w_e(y); \quad (18)$$

however, the integration of Eq. (18) can be performed, a priori, both from x_W to x and from x to x_E, x_W and x_E being the longitudes of the meridional boundaries. In the first case

$$D_I^2(x, y) = D_I^2(x_W, y) + \frac{2f^2}{\beta\gamma}w_e(y)(x - x_W), \quad (19)$$

but because of inequality (17) the rhs of solution (19) is not positive definite in the subtropical gyre; so solution (19) itself must be rejected. On the contrary, in the second case

$$D_I^2(x, y) = D_I^2(x_E, y) - \frac{2f^2}{\beta\gamma}w_e(y)(x_E - x), \quad (20)$$

where

$$-\frac{2f^2}{\beta\gamma}w_e(y)(x_E - x) > 0$$

in the subtropical gyre, so the outcropping is consistently forbidden in this region. Equation (20) is the first step to relate the depth of the wind-driven motion at the eastern boundary, that is, $D_I(x_E, y)$, to the wind-stress field evaluated at the latitude of vanishing curl [Eq. (3.16) of the original paper].

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