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有关记录次数的计数过程的矩精确完全收敛 *

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摘 要

给出了一个关于i.i.d.绝对连续随机变量列的记录次数的计数过程的矩精确完全收敛性的一般化定理.

关键词: 计数过程, 矩精确完全收敛, 一般化定理.

学科分类号: O211.4.

§1. 引 言

自Hsu和Robbins[1]介绍了完全收敛性的概念以来, 出现了各种形式的这方面的结果. 令$\{X_k, k\geq 1\}$是i.i.d.随机变量序列, $S_n=\sum\limits_{k=1}^{n}X_k$, $n\geq 1$, Heyde在文献[2]中证明了:

$$\lim_{\varepsilon\searrow 0}\varepsilon^2\sum_{n=1}^{\infty}\mathsf{P}(|S_n|\geq\varepsilon n)=\mathsf{E}X_1^2.$$

其中$\mathsf{E}X_1=0$, $\mathsf{E}X_1^2<\infty$. 类似这方面的结果可参见[3–5]. 王岳宝和杨杨[6]把Gut[5]的关于记录次数的计数过程的精确完全收敛性的结果做了一般化的推广. 设$\{X_k, k\geq 1\}$是i.i.d.随机变量序列, 且X_1的分布绝对连续, 令

$$L(1)=1,\qquad L(n)=\min\{k: X_k>X_{L(n-1)}, k>L(n-1)\},\quad n\geq 2.$$

相关的计数过程$\{\mu(n): n\geq 1\}$定义为$\mu(n)=\max\{k: L(k)\leq n\}$. 设$g(x)$和$h(x)$是正可微的函数, 令$\varphi(x)=g'(h(x))h'(x)$,

$$U_0(\varepsilon)=\sqrt{\frac{2}{\pi}}\,\varepsilon\int_{h(n_0)}^{\infty}g(y)e^{-\varepsilon^2y^2/2}\mathrm{d}y,\qquad U(\varepsilon)\sim\frac{1}{U_0(\varepsilon)},\qquad \varepsilon\downarrow a.$$

王岳宝和杨杨[6]获得了如下定理:

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定理 1.1 设$g(x)$和$h(x)$是定义于$[n_0,\infty)$上的严格增可微函数, 且$\lim\limits_{x\to\infty} g(x)=\infty$, $\lim\limits_{x\to\infty} h(x)=\infty$, $\varphi(x)=g'(h(x))h'(x)$, $\varphi(x)$单调, 若单调递增, 则设$\lim\limits_{n\to\infty}\varphi(n+1)/\varphi(n)=1$. 假设存在$a\geq 0$, 使$U_0(\varepsilon)$满足: $U_0(\varepsilon)<\infty$, $\varepsilon>a$, $\lim\limits_{\varepsilon\searrow a} U_0(\varepsilon)=\infty$, 且若下面二个条件之一成立:

$$(1)\quad \sum_{n=n_0}^{\infty}\varphi(n)(\log n)^{-1/2}<\infty; \tag{1.1}$$

$$(2)\quad \lim_{M\to\infty}\lim_{\varepsilon\searrow a}\varepsilon U(\varepsilon)\int_{g^{-1}(U_0(\varepsilon)M)}^{\infty} g(y)e^{-\varepsilon^2y^2/2}\mathrm{d}y=0, \tag{1.2}$$

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow a}\varepsilon^{-q}U(\varepsilon)\int_{g^{-1}(U_0(\varepsilon)M)}^{\infty} y^{-q}\mathrm{d}g(y)=0 \qquad \text{对于某}q\geq 2\text{成立}. \tag{1.3}$$

则有

$$\lim_{\varepsilon\searrow a}U(\varepsilon)\sum_{n\geq n_0}\varphi(n)\mathsf{P}(|\mu(n)-\mathsf{E}\mu(n)|>\varepsilon(\log n)^{1/2}h(n))=1. \tag{1.4}$$

更进一步, 当$a>0$或$a=0$且存在两个常数$b>0$和$M_0>1$, 有对于任意$\varepsilon>0$足够小使得

$$\varepsilon g^{-1}(U_0(\varepsilon)M_0)\geq b$$

成立, 则可用$\log n$代替$\mathsf{E}\mu(n)$.

文献[7]考虑了矩精确完全收敛性(有关矩精确完全收敛性的极限定理可参考[7]), 于是自然要问: 是否有类似于定理1.1的矩精确完全收敛性成立?

在这篇文章中, 我们考虑了这个问题, 并得到了类似的结果, 且去掉了(1.2).

§2. 主要结果

下面我们仍旧设$\{X_k,k\geq 1\}$是i.i.d.随机变量序列, 且X_1的分布绝对连续, $\mu(n)$与节1中的相同. 设$g(x)$和$h(x)$是可微的正函数, 令

$$\varphi(x)\sqrt{\log x}=g'(h(x))h'(x), \qquad G_0(\varepsilon)=\int_{h(n_0)}^{\infty}\varepsilon g(x)\mathsf{P}(|N|>\varepsilon x)\mathrm{d}x,$$
$$G(\varepsilon)\sim\frac{1}{G_0(\varepsilon)}, \qquad \varepsilon\downarrow a. \tag{2.1}$$

定理 2.1 设$g(x)$和$h(x)$是定义于$[n_0,\infty)$上的严格增的可微函数, 且$\lim\limits_{x\to\infty} g(x)=\infty$, $\lim\limits_{x\to\infty} h(x)=\infty$, $\varphi(x)\sqrt{\log x}$单调, 若单调递增, 则设$\lim\limits_{n\to\infty}\varphi(n+1)/\varphi(n)=1$. 假设存在$a\geq 0$, 使$G_0(\varepsilon)$满足: $G_0(\varepsilon)<\infty$, $\varepsilon>a$, $\lim\limits_{\varepsilon\searrow a} G_0(\varepsilon)=\infty$, 且若下面二个条件之一成立:

$$(1)\quad \sum_{n=n_0}^{\infty}\varphi(n)(\log n)^{\alpha}<\infty \qquad \text{对于某}\alpha>0\text{成立}; \tag{2.2}$$

$$(2)\quad \lim_{M\to\infty}\lim_{\varepsilon\searrow a}\varepsilon^{-q}G(\varepsilon)\int_{g^{-1}(G_0(\varepsilon)M)}^{\infty} y^{-q}\mathrm{d}g(y)=0 \qquad \text{对于某}q\geq 2\text{成立}. \tag{2.3}$$

则有

$$\lim_{\varepsilon\searrow a} G(\varepsilon)\sum_{n\geq n_0}\varphi(n)\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+=1. \tag{2.4}$$

类似于[6], 我们能得到下面几个推论:

令$g(x)=x^r l(x)$, $r\geq 0$, 其中$l(x)$是一缓变函数. 由(2.1), 令

$$G(\varepsilon)=\frac{\varepsilon^r(l(\varepsilon^{-1}))^{-1}}{(r+1)^{-1}\mathsf{E}|N|^{r+1}},\qquad \varepsilon>0=a,$$

于是成立:

推论 2.1　令$h(x)$是定义于$[n_0,\infty)$的正且可微的函数, 严格增于无穷, $\varphi(x)=(r(h(x))^{r-1}l(h(x))+(h(x))^r l'(h(x)))h'(x)/\sqrt{\log x}$单调, 若单调递增, 则假设$\lim\limits_{n\to\infty}[\varphi(n+1)/\varphi(n)]=1$. 此外, $l(x)$在$[n_0,\infty)$的每个紧子集上有界. 则成立

$$\lim_{\varepsilon\searrow a}\varepsilon^r(l(\varepsilon^{-1}))^{-1}\sum_{n=n_0}^{\infty}\varphi(n)\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+=\frac{1}{r+1}\mathsf{E}|N|^{r+1}. \tag{2.5}$$

令$g(x)=e^{rx^2}$, $r>0$, 由(2.1), 令

$$G(\varepsilon)=\sqrt{\frac{\pi}{2}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}},\qquad \varepsilon>a=\sqrt{2r},$$

于是成立:

推论 2.2　令$h(x)$是定义于$[n_0,\infty)$的正且可微的函数, 严格增于无穷, $\varphi(x)=2re^{rh^2(x)}h(x)h'(x)/\sqrt{\log x}$单调, 若单调递增, 则假设$\lim\limits_{n\to\infty}\varphi(n+1)/\varphi(n)=1$. 最后假设$h(x)$满足(2.2), 则成立

$$\lim_{\varepsilon\searrow\sqrt{2r}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}}\sum_{n=n_0}^{\infty}\varphi(n)\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+=\sqrt{\frac{2}{\pi}}. \tag{2.6}$$

特别地, 令$h(x)=(\log\log\log x)^{1/2}$, 我们有

$$\lim_{\varepsilon\searrow\sqrt{2r}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}}\sum_{n=n_0}^{\infty}\frac{(\log\log n)^{r-1}}{n(\log n)^{3/2}}\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n\log\log\log n)^{1/2}\}_+=\sqrt{\frac{2}{\pi}}\frac{1}{r}.$$

令$g(x)=e^{rx}$, $r>0$, 由(2.1), 令$G(\varepsilon)=2^{-1}\varepsilon^{-1}re^{-r^2/(2\varepsilon^2)}$, $\varepsilon>0=a$, 于是成立:

推论 2.3　令$h(x)$是定义于$[n_0,\infty)$的正且可微的函数, 严格增于无穷, $\varphi(x)=re^{rh(x)}h'(x)/\sqrt{\log x}$单调, 若单调递增, 则假设$\lim\limits_{n\to\infty}\varphi(n+1)/\varphi(n)=1$. 最后假设$h(x)$满足(2.2), 则成立

$$\lim_{\varepsilon\searrow 0}\varepsilon^{-1}e^{-r^2/(2\varepsilon^2)}\sum_{n=n_0}^{\infty}\varphi(n)\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+=\frac{2}{r}. \tag{2.7}$$

特别地, 令$h(x)=\log\log x$, $0<r<1/2$, 成立

$$\lim_{\varepsilon\searrow 0}\varepsilon^{-1}e^{-r^2/(2\varepsilon^2)}\sum_{n=n_0}^{\infty}n^{-1}(\log n)^{r-3/2}\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}\log\log n\}_+=\frac{2}{r^2}.$$

§3. 有关$\mu(n)$的一些性质及定理的证明

下面的引理3.1和引理3.2分别来自文献[8]的定理1和定理3.

引理 3.1 (1) $(\mu(n)-\log n)/\sqrt{\log n}\xrightarrow{d}N(0,1)$; (2) 序列$\{|(\mu(n)-\log n)/\sqrt{\log n}|^r, n\geq 3\}$一致可积,

$$\mathsf{E}\Big|\frac{\mu(n)-\log n}{\sqrt{\log n}}\Big|^r\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|^re^{-x^2/2}\mathrm{d}x,\qquad \text{任意}r>0.$$

引理 3.2 对于$k\geq 2$我们有

$$\sup_n\Big|\mathsf{P}(\mu(k)\geq n)-\Phi\Big(\frac{n-\log k}{\sqrt{\log k}}\Big)\Big|\geq\frac{1.9}{\sqrt{\log k}}.$$

定理2.1的证明: 由题设知$\exists n_1(\delta)$, 当$n>n_1$时,

$$\frac{\varphi(n+1)\sqrt{\log(n+1)}}{\varphi(n)\sqrt{\log(n)}}<1+\delta,\qquad \frac{\varphi(n)\sqrt{\log(n)}}{\varphi(n+1)\sqrt{\log(n+1)}}>1-\delta,$$

其中任意$0<\delta<1$. 而

$$\begin{aligned}\sum_{n\geq n_1}\varphi(n)\sqrt{\log n}\mathsf{E}\{|N|-\varepsilon h(n)\}_+ &= \sum_{n\geq n_1}\varphi(n)\sqrt{\log n}\mathsf{E}\Big\{\int_{\varepsilon h(n)}^{\infty}I\{|N|>t\}\mathrm{d}t\Big\}\\ &= \sum_{n\geq n_1}\varphi(n)\sqrt{\log n}\int_{\varepsilon h(n)}^{\infty}\mathsf{P}(|N|>t)\mathrm{d}t\\ &:= \Delta.\end{aligned}\tag{3.1}$$

于是

$$\begin{aligned}&\frac{1}{1+\delta}\int_{n_1+1}^{\infty}\varphi(x)\sqrt{\log x}\int_{\varepsilon h(x)}^{\infty}\mathsf{P}(|N|>t)\mathrm{d}t\mathrm{d}x\leq\Delta\\ \leq&\frac{1}{1-\delta}\int_{n_1}^{\infty}\varphi(x)\sqrt{\log x}\int_{\varepsilon h(x)}^{\infty}\mathsf{P}(|N|>t)\mathrm{d}t\mathrm{d}x.\end{aligned}$$

而

$$\begin{aligned}\int_{n_1+1}^{\infty}\varphi(x)\sqrt{\log x}\int_{\varepsilon h(x)}^{\infty}\mathsf{P}(|N|>t)\mathrm{d}t\mathrm{d}x &= \int_{n_1+1}^{\infty}\int_{\varepsilon h(x)}^{\infty}\mathsf{P}(|N|>t)\mathrm{d}t\mathrm{d}g(h(x))\\ &= \int_{h(n_1+1)}^{\infty}\int_{\varepsilon y}^{\infty}\mathsf{P}(|N|>t)\mathrm{d}t\mathrm{d}g(y)\\ &= \int_{h(n_1+1)}^{\infty}\varepsilon g(y)\mathsf{P}(|N|>\varepsilon y)\mathrm{d}y+C.\end{aligned}\tag{3.2}$$

其中C为一无关紧要的常数(下同). 又因为

$$G(\varepsilon)\sim\frac{1}{G_0(\varepsilon)}=\frac{1}{\displaystyle\int_{h(n_0)}^{\infty}\varepsilon g(x)\mathsf{P}(|N|>\varepsilon x)\mathrm{d}x}.\tag{3.3}$$

由(3.1)-(3.3)以及$G_0(\varepsilon)$的性质可得

$$\begin{aligned}\frac{1}{1+\delta} &\le \liminf_{\varepsilon\searrow a} G(\varepsilon)\sum_{n\ge n_0}\varphi(n)\sqrt{\log n}\,\mathsf{E}\{|N|-\varepsilon h(n)\}_+\\ &\le \limsup_{\varepsilon\searrow a} G(\varepsilon)\sum_{n\ge n_0}\varphi(n)\sqrt{\log n}\,\mathsf{E}\{|N|-\varepsilon h(n)\}_+\\ &\le \frac{1}{1-\delta}.\end{aligned}$$

令$\delta\to 0$, 得

$$\lim_{\varepsilon\searrow a} G(\varepsilon)\sum_{n\ge n_0}\varphi(n)\sqrt{\log n}\,\mathsf{E}\{|N|-\varepsilon h(n)\}_+ = 1. \tag{3.4}$$

现证

$$\lim_{\varepsilon\searrow a} G(\varepsilon)\sum_{n\ge n_0}\varphi(n)|\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+-\sqrt{\log n}\,\mathsf{E}\{|N|-\varepsilon h(n)\}_+| = 0. \tag{3.5}$$

若(2.2)成立:

$$\begin{aligned}&|\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+ - \sqrt{\log n}\,\mathsf{E}\{|N|-\varepsilon h(n)\}_+|\\ =&\ \Big|\mathsf{E}\int_0^\infty I\Big\{\frac{|\mu(n)-\log n|}{\sqrt{\log n}} > \varepsilon h(n)+\frac{x}{\sqrt{\log n}}\Big\}\mathrm{d}x - \sqrt{\log n}\,\mathsf{E}\int_0^\infty I\{|N|>\varepsilon h(n)+x\}\mathrm{d}x\Big|\\ =&\ \Big|\int_0^\infty \mathsf{P}\Big(\frac{|\mu(n)-\log n|}{\sqrt{\log n}} > \varepsilon h(n)+\frac{x}{\sqrt{\log n}}\Big)\mathrm{d}x - \sqrt{\log n}\int_0^\infty \mathsf{P}(|N|>\varepsilon h(n)+x)\mathrm{d}x\Big|\\ =&\ \Big|\sqrt{\log n}\int_0^\infty \mathsf{P}\Big(\frac{|\mu(n)-\log n|}{\sqrt{\log n}} > \varepsilon h(n)+x\Big)\mathrm{d}x - \sqrt{\log n}\int_0^\infty \mathsf{P}(|N|>\varepsilon h(n)+x)\mathrm{d}x\Big|\\ \le&\ \sqrt{\log n}\int_0^\infty\Big|\mathsf{P}\Big(\frac{|\mu(n)-\log n|}{\sqrt{\log n}} > \varepsilon h(n)+x\Big) - \mathsf{P}(|N|>\varepsilon h(n)+x)\Big|\mathrm{d}x\\ :=&\ \sqrt{\log n}\int_0^{(\log n)^a}\Delta_1\mathrm{d}x + \sqrt{\log n}\int_{(\log n)^a}^\infty \Delta_1\mathrm{d}x.\end{aligned} \tag{3.6}$$

由引理3.1和引理3.2知

$$\Delta_1 \le \frac{3.8}{\sqrt{\log n}},\qquad \Delta_1\le\frac{C_r}{(\varepsilon h(n)+x)^{r+1}},\quad \forall r>0.$$

令r足够大, 我们有

$$(3.6)\text{式} \le 3.8(\log n)^\alpha + \sqrt{\log n}\frac{C_r}{(\log n)^{\alpha r}} \le C(\log n^\alpha). \tag{3.7}$$

由(2.1)-(2.2)、(3.6)-(3.7)即可知(3.5)成立.

若(2.3)成立:

令$b(\varepsilon)=h^{-1}\circ g^{-1}(G_0(\varepsilon)M)$, 则当$\varepsilon\searrow a$时, $b(\varepsilon)\to\infty$. 和[6]中的证明一样, 若$\varphi(x)\sqrt{\log x}$非降, 由[6]的引理3知存在ε_0, 当$a<\varepsilon<\varepsilon_0$时, 有

$$\begin{aligned}4G_0(\varepsilon)M &= 4g(h(b(\varepsilon)))\geq g\circ h(b(\varepsilon)+1)\geq\int_{n_0}^{b(\varepsilon)+1}\mathrm{d}g(h(x))\\ &= \int_{n_0}^{b(\varepsilon)+1}\varphi(x)\sqrt{\log x}\mathrm{d}x\geq\sum_{n=n_0}^{b(\varepsilon)}\int_n^{n+1}\varphi(x)\sqrt{\log x}\mathrm{d}x\\ &\geq \sum_{n=n_0}^{b(\varepsilon)}\varphi(n)\sqrt{\log n}\geq(1-\delta)\Big(\sum_{n=n_0}^{b(\varepsilon)+1}\varphi(n)\sqrt{\log n}-\varphi(n_1)\sqrt{\log n_1}\Big). \quad (3.8)\end{aligned}$$

若$\varphi(x)\sqrt{\log x}$非增, 类似的有

$$4G_0(\varepsilon)M\geq\sum_{n=n_0}^{b(\varepsilon)+1}\varphi(n)\sqrt{\log n}-\varphi(n_0)\sqrt{\log n_0}. \quad (3.9)$$

而

$$\begin{aligned}&\sum_{n=n_0}^{b(\varepsilon)+1}\varphi(n)|\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+-\sqrt{\log n}\mathsf{E}\{|N|-\varepsilon h(n)\}_+|\\ \leq&\sum_{n=n_0}^{b(\varepsilon)+1}\varphi(n)\sqrt{\log n}\int_0^\infty\Big|\mathsf{P}\Big(\frac{|\mu(n)-\log n|}{\sqrt{\log n}}>\varepsilon h(n)+x\Big)-\mathsf{P}(|N|>\varepsilon h(n)+x)\Big|\mathrm{d}x\\ :=&\sum_{n=n_0}^{b(\varepsilon)+1}\varphi(n)\sqrt{\log n}\Delta_{n\varepsilon}. \quad (3.10)\end{aligned}$$

由引理3.1和引理3.2知, 对于任意$r>0$, 有

$$\begin{aligned}\Delta_{n\varepsilon} &= \int_0^{(\log n)^{1/4}}\Delta_1\mathrm{d}x+\int_{(\log n)^{1/4}}^\infty\Delta_1\mathrm{d}x\\ &\leq \frac{3.8}{(\log n)^{1/4}}+C\int_{(\log n)^{1/4}}^\infty\frac{1}{(\varepsilon h(n)+x)^{r+1}}\mathrm{d}x\\ &\leq \frac{3.8}{(\log n)^{1/4}}+\frac{C}{(\log n)^{r/4}}\to 0. \quad (3.11)\end{aligned}$$

结合(3.8)-(3.11)即可得

$$\lim_{\varepsilon\searrow a}G(\varepsilon)\sum_{n_0\leq n\leq b(\varepsilon)+1}\varphi(n)|\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+-\sqrt{\log n}\mathsf{E}\{|N|-\varepsilon h(n)\}_+|=0. \quad (3.12)$$

而

$$\begin{aligned}&\sum_{n>b(\varepsilon)+1}\varphi(n)|\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n^{1/2})h(n)\}_+-\sqrt{\log n}\mathsf{E}\{|N|-\varepsilon h(n)\}_+|\\ \leq&\sum_{n>b(\varepsilon)+1}\varphi(n)\sqrt{\log n}\Delta_{n\varepsilon}. \quad (3.13)\end{aligned}$$

因为

$$\Delta_{n\varepsilon} \le C_q \int_0^\infty \frac{1}{(\varepsilon h(n)+x)^{q+1}}\mathrm{d}x = C_q \frac{1}{\varepsilon^q h^q(n)},$$

所以

$$\begin{aligned}\sum_{n>b(\varepsilon)+1}\varphi(n)\sqrt{\log n}\Delta_{n\varepsilon} &\le C_q \sum_{n>b(\varepsilon)+1}\varphi(n)\sqrt{\log n}\frac{1}{\varepsilon^q h^q(n)}\\ &\le C\int_{b(\varepsilon)}^\infty \frac{\varphi(x)\sqrt{\log x}}{\varepsilon^q h^q(x)}\mathrm{d}x\\ &= C\int_{b(\varepsilon)}^\infty \frac{1}{\varepsilon^q h^q(x)}\mathrm{d}g(h(x))\\ &= C\int_{h(b(\varepsilon))}^\infty \frac{1}{\varepsilon^q y^q}\mathrm{d}g(y)\\ &= C\int_{g^{-1}(G_0(\varepsilon)M)}^\infty \frac{1}{\varepsilon^q y^q}\mathrm{d}g(y). \end{aligned} \tag{3.14}$$

由(2.3)、(3.13)-(3.14)得

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow a} G(\varepsilon)\sum_{n>b(\varepsilon)+1}\varphi(n)|\mathsf{E}\{|\mu(n)-\log n|-\varepsilon(\log n)^{1/2}h(n)\}_+ - \sqrt{\log n}\mathsf{E}\{|N|-\varepsilon h(n)\}_+| = 0. \tag{3.15}$$

由(3.12)和(3.15)即得(3.5).

结合(3.4)、(3.5)即得定理2.1. □

推论2.1的证明: 易知

$$G_0(\varepsilon) = \int_{h(n_0)}^\infty \varepsilon x^r l(x)\mathsf{P}(|N|>\varepsilon t)\mathrm{d}t = \int_{\varepsilon h(n_0)}^\infty \epsilon^{-r} t^r l(\varepsilon^{-1}t)\mathsf{P}(|N|>t)\mathrm{d}t.$$

令

$$G(\varepsilon) = \frac{\varepsilon^r(l(\varepsilon^{-1}))^{-1}}{(r+1)^{-1}\mathsf{E}|N|^{r+1}},$$

类似[6]推论2.1的证明, 易知$\lim_{\varepsilon\searrow 0} G(\varepsilon)G_0(\varepsilon)=1$.

而(2.3)在[6]的推论2.1的证明中也有验算, 从略. □

推论2.2的证明: 易知

$$G_0(\varepsilon) = \int_{h(n_0)}^\infty \varepsilon e^{rx^2}\mathsf{P}(|N|>\varepsilon x)\mathrm{d}x.$$

由正态分布的尾概率估计, 易知对$\forall\,\delta>0$, 成立

$$C_2 + \frac{1}{1+\delta}\int_{h(n_0)}^\infty \varepsilon e^{rx^2}\frac{2}{\sqrt{2\pi}\varepsilon x}e^{-\varepsilon^2x^2/2}\mathrm{d}x \le G_0(\varepsilon) \le \frac{1}{1-\delta}\int_{h(n_0)}^\infty \varepsilon e^{rx^2}\frac{2}{\sqrt{2\pi}\varepsilon x}e^{-\varepsilon^2x^2/2}\mathrm{d}x + C_1,$$

其中C_1和C_2是两个只和δ有关的常数.

而

$$\begin{aligned}\int_{h(n_0)}^{\infty}\varepsilon e^{rx^2}\frac{2}{\sqrt{2\pi}\varepsilon x}e^{-\varepsilon^2x^2/2}\mathrm{d}x &= \int_{h(n_0)}^{\infty}\frac{2}{\sqrt{2\pi}x}e^{-x^2(\varepsilon^2-2r)/2}\mathrm{d}x\\ &= \int_{\sqrt{\varepsilon^2-2r}h(n_0)}^{\infty}\frac{2}{\sqrt{2\pi}t}e^{-t^2/2}\mathrm{d}t.\end{aligned}$$

所以

$$\begin{aligned}\lim_{\varepsilon\searrow\sqrt{2r}}\sqrt{\frac{\pi}{2}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}}G_0(\varepsilon) &\le \frac{1}{1-\delta}\lim_{\varepsilon\searrow\sqrt{2r}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}}\int_{\sqrt{\varepsilon^2-2r}h(n_0)}^{\infty}\frac{1}{t}e^{-t^2/2}\mathrm{d}t\\ &= \frac{1}{1-\delta},\\ \lim_{\varepsilon\searrow\sqrt{2r}}\sqrt{\frac{\pi}{2}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}}G_0(\varepsilon) &\ge \frac{1}{1+\delta}\lim_{\varepsilon\searrow\sqrt{2r}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}}\int_{\sqrt{\varepsilon^2-2r}h(n_0)}^{\infty}\frac{1}{t}e^{-t^2/2}\mathrm{d}t\\ &= \frac{1}{1+\delta}.\end{aligned}$$

故

$$\lim_{\varepsilon\searrow\sqrt{2r}}\sqrt{\frac{\pi}{2}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}}G_0(\varepsilon)=1,$$

于是可取

$$G(\varepsilon)=\sqrt{\frac{\pi}{2}}\frac{1}{-\log\sqrt{\varepsilon^2-2r}},\qquad a=\sqrt{2r}.$$

□

推论2.3的证明:

$$\begin{aligned}G_0(\varepsilon) &= \int_{h(n_0)}^{\infty}\varepsilon e^{rx}\mathsf{P}(|N|>\varepsilon x)\mathrm{d}x\\ &= \int_{\varepsilon h(n_0)}^{\infty}\varepsilon e^{rt/\varepsilon}\mathsf{P}(|N|>t)\mathrm{d}t\\ &= \frac{\varepsilon}{r}e^{rh(n_0)}\mathsf{P}(|N|>\varepsilon h(n_0))+\frac{2\varepsilon}{\sqrt{2\pi}r}\int_{\varepsilon h(n_0)}^{\infty}e^{rt/\varepsilon-t^2/2}\mathrm{d}t\\ &= \frac{\varepsilon}{r}e^{rh(n_0)}\mathsf{P}(|N|>\varepsilon h(n_0))+\frac{2\varepsilon}{\sqrt{2\pi}r}e^{r^2/(2\varepsilon^2)}\int_{\varepsilon h(n_0)}^{\infty}e^{-(t-r/\varepsilon)^2/2}\mathrm{d}t\\ &= \frac{\varepsilon}{r}e^{rh(n_0)}\mathsf{P}(|N|>\varepsilon h(n_0))+\frac{2\varepsilon}{\sqrt{2\pi}r}e^{r^2/(2\varepsilon^2)}\int_{\varepsilon h(n_0)-r/\varepsilon}^{\infty}e^{-t^2/2}\mathrm{d}t.\end{aligned}$$

所以

$$\lim_{\varepsilon\searrow0}2^{-1}\varepsilon^{-1}re^{-r^2/(2\varepsilon^2)}G_0(\varepsilon)=\lim_{\varepsilon\searrow0}\frac{1}{\sqrt{2\pi}}\int_{\varepsilon h(n_0)-r/\varepsilon}^{\infty}e^{-t^2/2}\mathrm{d}t=1,$$

于是可取

$$G(\varepsilon)=2^{-1}\varepsilon^{-1}re^{-r^2/(2\varepsilon^2)},\qquad a=0.$$

□

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The Quadrature Complete Convergence for Record Time and the Associated Counting Process

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The paper gives the quadrature's precision and complete convergence general theorem, which is about the i.i.d. absolute continue variable arrange for the record time and the associated counting process.

Keywords: The counting process, the quadrature's precision and complete convergence, general theorem.

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