

Prime Rings with Generalized Derivations

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Abstract The concept of derivations and generalized inner derivations has been generalized as an additive function $\delta : R \rightarrow R$ satisfying $\delta(xy) = \delta(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation on R . Such a function δ is called a generalized derivation. Suppose that U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. In this paper, we prove that $U \subseteq Z(R)$ when one of the following holds: (1) $\delta([u, v]) = u \circ v$ (2) $\delta([u, v]) + u \circ v = 0$ (3) $\delta(u \circ v) = [u, v]$ (4) $\delta(u \circ v) + [u, v] = 0$ for all $u, v \in U$.

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1. Introduction

Daif and Bell^[1] proved that a semiprime ring R must be commutative if it admits a derivation d such that $d([x, y]) = [x, y]$ or $d([x, y]) + [x, y] = 0$. Further, Ashar and Rehman^[2] extended the mentioned result for Lie ideals of R . The purpose of this paper is to generalize these results for generalized derivations and Lie ideals of R .

Throughout this paper, R will always denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and denote by $x \circ y$ the anti-commutator $xy + yx$. Given two subsets A and B of R , then $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[a, b]$ where $a \in A, b \in B$. Recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. An additive map $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the map $I_a : R \rightarrow R$ given $I_a(x) = [a, x]$ is a derivation which is said to be inner derivation. By a generalized inner derivation, one usually means a map of the form $x \mapsto ax + xb$. Hvala^[3] introduced the notions of generalized derivations in rings. An additive function $\delta : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that $\delta(xy) = \delta(x)y + xd(y)$ holds for all $x, y \in R$.

2. Preliminaries

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We begin with some known results which will be used extensively in the sequel to prove our theorems.

Lemma 2.1^[4] *Let R be a 2-torsion free semiprime ring and U a Lie ideal of R . If $[U, U] \subseteq Z(R)$, then $U \subseteq Z(R)$.*

Lemma 2.2 *If U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$, then $2uv \in U$ for all $u, v \in U$.*

Proof For all $w, u, v \in U$,

$$uv + vu = (u + v)^2 - u^2 - v^2 \in U.$$

On the other hand,

$$uv - vu \in U.$$

Adding two expressions, we have $2uv \in U$ for all $u, v \in U$.

Lemma 2.3^[5] *If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.4^[5] *If $d \neq 0$ is a derivation of R , and if U is a Lie ideal of R such that $d(U) \subseteq Z(R)$, then $U \subseteq Z(R)$.*

Lemma 2.5 *A group cannot be a union of two its proper subgroups.*

Proof Suppose on the contrary that $G = M \cup N$, where both M and N are proper subgroups of G . Then there exists $g_1 \in G \setminus M$ (i.e. $g_1 \in N \setminus M$) and $g_2 \in G \setminus N$ (i.e. $g_2 \in M \setminus N$). It is obvious that $g_1g_2 \in G = M \cup N$. Therefore we have either $g_1g_2 \in M$ or $g_1g_2 \in N$. If $g_1g_2 \in M$, we get $g_1 \in M$ since $g_2 \in M$. This is a contradiction. On the other hand, if $g_1g_2 \in N$, we get $g_2 \in N$ since $g_1 \in N$, again a contradiction. This completes the proof of the Lemma.

3. The proof of main theorem

Now we are in a position to prove the following theorem.

Theorem 3.1 *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose R admits a generalized derivation δ with d such that $\delta([u, v]) = u \circ v$ for all $u, v \in U$. If $\delta = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Proof Suppose δ is a generalized derivation of R such that

$$\delta([u, v]) = u \circ v. \quad (1)$$

If $\delta = 0$, then $u \circ v = 0$. By Lemma 2.2, replacing v by $2vw$ in (1) and using the fact $\text{char} R \neq 2$, we have

$$uvw + vwu = 0. \quad (2)$$

Combining (1) with (2), we have

$$-vuw + vwu = v[w, u] = 0 = v[u, w]. \quad (3)$$

Again replacing v by $[u, r]$ gives

$$[u, r][u, w] = 0. \quad (4)$$

Replacing r by sr in (4), we have

$$0 = [u, sr][u, w] = s[u, r][u, w] + [u, r]r[u, w] = [u, s]R[u, w]. \quad (5)$$

In particular, we have $[u, w]R[u, w] = 0$. The primeness of R forces that $[u, v] = 0$ (i.e. $[U, U] = 0$), and hence by Lemma 2.1 we get the required result. Therefore, from now on we shall assume that $\delta \neq 0$. Suppose on the contrary that $U \not\subseteq Z(R)$. For any $u, v \in U$, we have $\delta([u, v]) = u \circ v$. In other words, we have obtained

$$\delta(u)v + ud(v) - \delta(v)u - vd(u) = u \circ v. \quad (6)$$

Replacing v by $2vu$ in (6) and using the fact $\text{char}R \neq 2$, we find that

$$\delta(u)vu + ud(v)u + [u, v]d(u) - \delta(v)u^2 - vd(u)u = u \circ (vu). \quad (7)$$

Combining (6) with (7), we have $[u, v]d(u) = 0$ for all $u, v \in U$. Again replacing v by $2wv$ in the above gives $0 = [u, wv]d(u) = [u, w]vd(u) + w[u, v]d(u) = [u, w]vd(v)$. That is,

$$[u, w]Ud(u) = 0. \quad (8)$$

Thus we have either $[u, w] = 0$ or $d(u) = 0$ by Lemma 2.3. Now let $U_1 = \{u \in U \mid [u, w] = 0\}$ and $U_2 = \{u \in U \mid d(u) = 0\}$. Then U_1, U_2 are both additive subgroups of U and $U_1 \cup U_2 = U$. Thus $U = U_1$ or $U = U_2$ by Lemma 2.5. If $U = U_1$, then $[u, w] = 0$ (i.e. $[U, U] = 0$) for all $u, w \in U$. Hence we get $U \subseteq Z(R)$ by Lemma 2.1, a contradiction. On the other hand, if $U = U_2$ then $d(U) = 0$ for all $u \in U$. Thus by Lemma 2.4, we get $U \subseteq Z(R)$, again a contradiction.

Using the same technique with necessary variations we get the following.

Theorem 3.2 *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose R admits a generalized derivation δ with d such that $\delta([u, v]) + u \circ v = 0$ for all $u, v \in U$. If $\delta = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Theorem 3.3 *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose R admits a generalized derivation δ with d such that $\delta(u \circ v) = [u, v]$ for all $u, v \in U$. If $\delta = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Proof If $\delta = 0$, then we have $[u, v] = 0$ (i.e. $[U, U] = 0$). By Lemma 2.1, we have $U \subseteq Z(R)$. Now, we assume that $\delta \neq 0$. Suppose on the contrary that $U \not\subseteq Z(R)$. For any $u, v \in U$, we have

$$\delta(u \circ v) = [u, v]. \quad (9)$$

Replacing v by $2vu$ in (9), and using the fact $\text{char}R \neq 2$, we have

$$\delta(u)vu + ud(v)u + uvd(u) + \delta(v)u^2 + vd(u)u + vud(u) = [u, v]u. \quad (10)$$

Combining (9) with (10), we have

$$(u \circ v)d(u) = 0. \quad (11)$$

Again replacing v by $2wv$ in (11) and using $\text{char}R \neq 2$ yields

$$(u \circ (wv))d(u) = 0. \quad (12)$$

Using the identity $x \circ (yz) = y(x \circ z) + [x, y]z$ to expand (12) leads to

$$w(u \circ v)d(u) + [u, w]vd(u) = 0. \quad (13)$$

Combining (11) with (13), we have $[u, w]vd(u) = 0$. That is, $[u, w]Ud(u) = 0$. Note that the argument given in the proof of Theorem 3.1 is still valid in the present situation and hence repeating the same process we get the required result.

Using the same technique with necessary variations, one can prove the following.

Theorem 3.4 *Let R be a 2-torsion free prime ring and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose R admits a generalized derivation δ with d such that $\delta(u \circ v) + [u, v] = 0$ for all $u, v \in U$. If $\delta = 0$ or $d \neq 0$, then $U \subseteq Z(R)$.*

Remark In view of the above results, it is natural to ask if these also hold to left multiplier (i.e., a generalized derivation with $d = 0$). We leave this as an open question.

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