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On Wallis' inequality

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Abstract

For all positive integers n,

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}.$$

Both bounds are the best possible.

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The Wallis formula follows from the infinite product representation of the sine (see [11, 15])

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right). \tag{1}$$

Taking in (1) $x=\dfrac{\pi}{2}$ gives

$$\frac{\pi}{2} = \lim_{n\to\infty}\frac{[(2n)!!]^2}{[(2n-1)!!]^2(2n+1)} = \prod_{n=1}^{\infty}\left[\frac{(2n)^2}{(2n-1)(2n+1)}\right]. \tag{2}$$

The Wallis formula can also be expressed as

$$\frac{\pi}{2} = \left[4^{\zeta(0)}e^{-\zeta'(0)}\right]^2, \tag{3}$$

see [11], where ζ is the Riemann zeta function [10].

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A derivation of the Wallis formula from $\zeta'(0)$ using the Hadamard product [9] for the Riemann zeta function $\zeta(s)$ due to Y. L. Yung can be found in [11]. The Wallis formula can also be reversed to derive $\zeta'(0)$ from the Wallis formula without using the Hadamard product [14].

It is noted that Wallis sine (cosine) formula [12, 13] is as follows

$$\int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{n\Gamma(\frac{n}{2})} = \begin{cases} \dfrac{\pi}{2} \cdot \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ even}, \\ \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ odd}, \end{cases} \tag{4}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d}t \quad (x > 0)$$

is the Euler's gamma function.

For more information on Wallis formula, please refer to [1, p. 258], [3, pp. 17–28], [4, p. 468], [6, pp. 63–64], and references therein.

Motivated by (2), D. K. Kazarinoff [5] proved that, for all positive integers n,

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{5}$$

The inequality (5) is called Wallis' inequality in [7, p. 103], see also [2, p. 259]. We here improve the lower bound and confirm the upper in (5).

Theorem 1: For all positive integers n,

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{6}$$

The constants $\dfrac{4}{\pi} - 1$ and $\dfrac{1}{4}$ are the best possible.

Proof: Since

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad 2^n n! = (2n)!!,$$

the double inequality (6) is equivalent to

$$\frac{1}{4} < \left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - n \le \frac{4}{\pi} - 1. \tag{7}$$

Define for $x > -\dfrac{1}{2}$,

$$\theta(x) = \left[\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\right]^2 - x,$$

clearly, $\theta(1) = \frac{4}{\pi} - 1$. In order to prove (7), it is sufficient to show that the function θ is strictly decreasing on $[1, \infty)$ and with $\lim_{x\to\infty} = \frac{1}{4}$. The following proof shows that in fact $\theta'(x) < 0$ holds on $\left(-\frac{1}{2}, \infty\right)$. Easy computation yields

$$\theta'(x) = 2(\theta(x) + x)(\psi(x+1) - \psi(x+1/2)) - 1,$$
$$\frac{\theta''(x)}{2(\theta(x)+x)} = \psi'(x+1) - \psi'(x+1/2) + 2(\psi(x+1) - \psi(x+1/2))^2,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$, the logarithmic derivative of the gamma function, is psi function or digamma function.

Using the representations [8, p. 16]

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\,\mathrm{d}t,$$
$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt}\,\mathrm{d}t$$

for $x > 0$ and $n = 1, 2, \ldots$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant, it follows that

$$\frac{\theta''(x)}{2(\theta(x)+x)} = -\int_0^\infty t\delta(t) e^{-(x+1/2)t}\,\mathrm{d}t + 2\left(\int_0^\infty \delta(t) e^{-(x+1/2)t}\,\mathrm{d}t\right)^2,$$

where

$$\delta(t) = \left(1 + e^{-t/2}\right)^{-1}.$$

By using the convolution theorem for Laplace transformas, we have

$$\begin{aligned}\frac{\theta''(x)}{2(\theta(x)+x)} &= -\int_0^\infty t\delta(t) e^{-(x+1/2)t}\,\mathrm{d}t \\ &\qquad + 2\int_0^\infty \left[\int_0^t \delta(s)\delta(t-s)\,\mathrm{d}s\right] e^{-(x+1/2)t}\,\mathrm{d}t \\ &= \int_0^\infty e^{-(x+1/2)t}\omega(t)\,\mathrm{d}t,\end{aligned} \tag{8}$$

where

$$\omega(t) = \int_0^t \left[2\delta(s)\delta(t-s) - \delta(t)\right]\mathrm{d}s.$$

Set $P_a(y) = \delta(a(1-y))\delta(a(1+y))$, and let $s = \frac{t}{2}(1+y)$, and take into account that $y \mapsto P_{t/2}(y)$ is an even function. Then we get

$$\omega(t) = \int_0^1 \left[2P_{t/2}(y) - \delta(t)\right] t\,\mathrm{d}y. \tag{9}$$

Easy computation yields

$$\frac{P_a'(y)}{P_a(y)} = -\frac{1}{1+e^{a(1-y)/2}} + \frac{1}{1+e^{a(1+y)/2}} < 0,$$

which implies for $0 < y < 1$,

$$2P_{t/2}(y) > 2P_{t/2}(1) = 2\delta^2(t/2) = 2(1+e^{-t/4})^{-2} > (1+e^{-t/2})^{-1} = \delta(t),$$

and thus $\omega(t) > 0$ by (9). Combining (8) and (9) leads to $\theta''(x) > 0$ and $\theta'(x)$ is strictly increasing on $(-1/2, \infty)$.

From the representations [1, p. 257 and p. 259]

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\big(x^{-2}\big) \quad (x \to \infty), \tag{10}$$

$$\psi(x) = \ln x - \frac{1}{2x} + O\big(x^{-2}\big) \quad (x \to \infty), \tag{11}$$

we conclude that

$$\lim_{x\to\infty} x^{-\frac{1}{2}}\frac{\Gamma(x+1)}{\Gamma\big(x+\frac{1}{2}\big)} = 1, \tag{12}$$

$$\lim_{x\to\infty} x\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right] = \frac{1}{2}. \tag{13}$$

From (12), (13) and the monotonicity of the function θ', we imply

$$\begin{aligned}\theta'(x) &< \lim_{x\to\infty}\theta'(x)\\ &= \lim_{x\to\infty} 2\left[x^{-1/2}\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\right]^2 x(\psi(x+1) - \psi(x+1/2)) - 1 = 0.\end{aligned}$$

Using the asymptotic formula (10) we conclude from

$$\theta(x) = x\left[x^{-1/2}\frac{\Gamma(x+1)}{\Gamma(x+1/2)} + 1\right]\left[x^{-1/2}\frac{\Gamma(x+1)}{\Gamma(x+1/2)} - 1\right]$$

that

$$\lim_{x\to\infty}\theta(x) = 1/4.$$

The proof is complete. ■

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