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Sequence Inequalities for the Logarithmic Convex (Concave) Function

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Abstract

Let f be a positive strictly increasing logarithmic convex (or logarithmic concave) function on $(0,1]$, then, for k being a nonnegative integer and n a natural number, the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k})$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)dt$. From this, some new inequalities involving $\sqrt[n]{(n+k)!/k!}$ are deduced.

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1. Introduction

In [1], H. Alzer, using the mathematical induction and other techniques, proved that for $r > 0$ and $n \in N$,

$$\frac{n}{n+1} \leq \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1.1}$$

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By the cauchy's mean-value theorem and the mathematical induction, F. Qi in [7] presented that, if n and m are natural numbers, k is a nonnegative integer, $r > 0$, then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}. \tag{1.2}$$

The lower is best possible. From the stirling's formula, for all nonnegative integers k and natural numbers n and m, F.Qi in $[8]$ obtained

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Big/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \leq \sqrt{\frac{n+m}{n+m+k}}. \tag{1.3}$$

Let f be a strictly increasing convex (or concave) function in $(0, 1]$, J.-C. Kuang in [2] verified that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f(x)dx. \tag{1.4}$$

In [10], F. Qi, considering the convexity of a function proved that let f be a strictly increasing convex (or concave) function in $(0, 1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f(\frac{i}{n+k})$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)dt$. That is

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f\left(\frac{i}{n+k+1}\right) > \int_0^1 f(t)dt \tag{1.5}$$

Where k is a nonnegative integer, n a natural number.

The study of Alzer's and Minc-Sathre's inequality has many literature, for example, [1]–[13]. In this article, motivated by [2, 7, 10], i.e. the inequalities in (1.2), (1.3), (1.4) and (1.5), considering the logarithmic convexity of a function, we get

Theorem 1.1: Let f be a positive strictly increasing logarithmic convex (or logarithmic concave) function on $(0, 1]$, then, for k being a nonnegative integer and n a natural number, the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k})$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)dt$, that is

$$\frac{1}{n}\sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k}\right) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} lnf\left(\frac{i}{n+k+1}\right) > \int_0^1 f(t)dt. \tag{1.6}$$

Where k is a nonnegative integer, n a natural number.

If let $f(x) = a^{x^r}$, $r > 0$, or let $k = 0$ in (1.6), then the inequalities in (1.1), (1.2) and (1.4) could be deduced. If let $f(x) = e^{g(x)}$, $g(x)$ be a strictly increasing logarithmic convex (or logarithmic concave) function in $(0, 1]$, the inequalities in (1.5) could be deduced. Therefore, inequality (1.6) generalizes Alzer's and Kuang's inequality in [1, 2] and inequality (1.2) above.

Corollary 1.2 ([10]): For a nonnegative integer k and a natural number $n > 1$, we have

$$\frac{n+k}{n+k+1} < \left[\frac{(2n+2k)!}{(n+2k)!}\right]^{1/n} \Big/ \left[\frac{(2n+2k+2)!}{(n+2k+1)!}\right]^{1/(n+1)} < \left[\frac{(n+k)!}{k!}\right]^{1/n} \Big/ \left[\frac{(n+k+1)!}{k!}\right]^{1/(n+1)}. \tag{1.7}$$

Theorem 1.3: For a natural number $n > 1$, then

$$\left[n^{(n+1)^2} \Big/ (n+1)^{n^2}\right]^{1/(2n+1)} < \left(\prod_{i=1}^{n} i^i\right)^{2/n(n+1)} < \frac{2n+1}{3}. \tag{1.8}$$

2. Proofs of theorems

Proof (Theorem 1.): Let us first assume that f be a positive strictly increasing logarithmic convex function in $(0, 1]$. Taking $x_1 = \frac{i-1}{n+k}$, $x_2 = \frac{i}{n+k}$, $\lambda = \frac{i-k-1}{n}$ and using the logarithmic convexity and monotonicity of f yields

$$\begin{aligned} &\frac{i-k-1}{n} \ln f\left(\frac{i-1}{n+k}\right) + \left(1 - \frac{i-k-1}{n}\right) \ln f\left(\frac{i}{n+k}\right) \\ &\geq \ln f\left(\frac{i-k-1}{n} \cdot \frac{i-1}{n+k} + \frac{n-i+k+1}{n} \cdot \frac{i}{n+k}\right) \\ &= \ln f\left(\frac{ni-i+k+1}{n(n+k)}\right) > \ln f\left(\frac{i}{n+k+1}\right) \end{aligned}$$

for $i = k+1, k+2, \ldots, n+k+1$. Summing up leads to

$$\begin{aligned} &\sum_{i=k+1}^{n+k} \left[\frac{i-k-1}{n} lnf\left(\frac{i-1}{n+k}\right) + \frac{n-i+k+1}{n} lnf\left(\frac{i}{n+k}\right)\right] \\ &\qquad > \sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k+1}\right) \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\sum_{i=k+1}^{n+k} \left[(i-k-1) lnf(\frac{i-1}{n+k}) + (n-i+k+1) lnf(\frac{i}{n+k})\right] \\ &\qquad > n \sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k+1}\right) \end{aligned} \tag{2.2}$$

$$(n+1)\sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k}\right) - nlnf(1) > n\sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k+1}\right) \tag{2.3}$$

$$\begin{aligned}(n+1)\sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k}\right) &> nlnf(1) + n\sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k+1}\right)\\ &= n\sum_{i=k+1}^{n+k+1} lnf\left(\frac{i}{n+k+1}\right)\end{aligned} \tag{2.4}$$

the left inequality in (1.6) is proved.

By similar procedure, if f is a strictly increasing logarithmic concave function in $(0,1]$, then for $i = k+1, k+2, \ldots, n+k+1$, we have

$$\begin{aligned}&\frac{i-k}{n+1}lnf\left(\frac{i+1}{n+k+1}\right) + \left(1-\frac{i-k}{n+1}\right)lnf\left(\frac{i}{n+k+1}\right)\\ &\le lnf\left(\frac{i-k}{n+1}\cdot\frac{i+1}{n+k+1} + \frac{n-i+k+1}{n+1}\cdot\frac{i}{n+k+1}\right)\\ &= lnf\left(\frac{ni+2i-k}{(n+1)(n+k+1)}\right) < lnf\left(\frac{i}{n+k}\right)\end{aligned} \tag{2.5}$$

Summing up leads to

$$\begin{aligned}&\sum_{i=k+1}^{n+k}\left[\frac{i-k}{n+1}lnf\left(\frac{i+1}{n+k+1}\right) + \left(1-\frac{i-k}{n+1}\right)lnf\left(\frac{i}{n+k+1}\right)\right]\\ &= \frac{n}{n+1}\sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k+1}\right) + \frac{n}{n+1}lnf(1)\\ &< \sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k}\right)\end{aligned} \tag{2.6}$$

$$\frac{n}{n+1}\sum_{i=k+1}^{n+k+1} lnf\left(\frac{i}{n+k+1}\right) < \sum_{i=k+1}^{n+k} lnf\left(\frac{i}{n+k}\right). \tag{2.7}$$

The final line in (2.7) implies the left inequality in (1.6).

Finally, by definition of definite integral, the right inequality in (1.6) follows.

The proof is complete. ■

Proof (Corollary 1.): Substituting f by $(x+1)^r$, $r > 0$ or by $\frac{x}{x+1}$ in (1.6) and simplifying yields the first or the second inequality in (1.7), respectively.

Proof (Theorem 2.): Substituting f by x^x and $k = 0$ in Theorem 1, we have

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)ln\left(\frac{i}{n}\right) > \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{i}{n+1}\right)ln\left(\frac{i}{n+1}\right) \tag{2.8}$$

$$\frac{1}{n^2}\sum_{i=1}^{n}\left[i(lni-lnn)\right] > \frac{1}{(n+1)^2}\sum_{i=1}^{n+1}\left[i(lni-ln(n+1))\right] \tag{2.9}$$

$$\begin{aligned}\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]\sum_{i=1}^{n}(ilni) &> \left[\frac{lnn}{n^2}-\frac{ln(n+1)}{(n+1)^2}\right]\sum_{i=1}^{n}i \\ &= \left[\frac{lnn}{n^2}-\frac{ln(n+1)}{(n+1)^2}\right]\frac{n(n+1)}{2}\end{aligned} \tag{2.10}$$

$$(2n+1)ln\left(\prod_{i=1}^{n}i^i\right) > \frac{n(n+1)}{2}ln\left[n^{(n+1)^2}\Big/(n+1)^{n^2}\right]. \tag{2.11}$$

In $[3,\ P.\ 89]$,the following inequalities were given for $n>1$, $n\in N$.

$$\left(\frac{n+1}{2}\right)^{a_n} < \prod_{i=1}^{n}i^i < \left(\frac{2n+1}{3}\right)^{a_n}, \qquad a_n=\frac{n(n+1)}{2}. \tag{2.12}$$

Taking the logarithm yields

$$a_n ln\left(\frac{n+1}{2}\right) < ln\left(\prod_{i=1}^{n}i^i\right) < a_n ln\left(\frac{2n+1}{3}\right). \tag{2.13}$$

By substituting the inequalities in (2.13) into the left term of inequality (2.11), we obtain

$$\begin{aligned}(2n+1)\frac{n(n+1)}{2}ln\left(\frac{2n+1}{3}\right) &> (2n+1)ln\left(\prod_{i=1}^{n}i^i\right) \\ &> \frac{n(n+1)}{2}ln\left[n^{(n+1)^2}\Big/(n+1)^{n^2}\right]\end{aligned} \tag{2.14}$$

$$\left[n^{(n+1)^2}\Big/(n+1)^{n^2}\right]^{1/(2n+1)} < \left(\prod_{i=1}^{n}i^i\right)^{2/n(n+1)} < \frac{2n+1}{3}. \tag{2.15}$$

The proof is complete. ■

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