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On an Open Problem for an Algebraic Inequality

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Abstract

In this paper, the open problem published in (Feng Qi, An algebraic inequality, *J. Inequal. Pure Appl. Math.*, **2** (1), Art. 13, 2001) is solved by using analytic arguments. At the same time, the precise scope of r in the open problem is given. The lower bound of the Theorem is refined.

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1. Introduction

In [1], F. Qi, posed the following:

Open problem. Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any positive $r \in \mathbb{R}$, we have

$$\left(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{1.1}$$

The upper bound in (1.1) is best possible.

In [1], F. Qi proved:

Theorem 1.1: Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any positive $r \in \mathbb{R}$, we have

$$\left(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} > \frac{b}{b+\delta} \tag{1.2}$$

The lower bound is best possible.

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The study of Algebraic inequality has many literature, for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The purpose of this paper is to verify the above inequality (1.1) and refines the lower bound of inequality (1.2).

We think that the open problem inequality (1.1) should be decomposed with variable r evaluation. The open problem should be stated as follows.

Theorem 1.2: Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any positive $r \in \mathbb{R}$

(i) If $r > 1$, then

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} \tag{1.3}$$

The upper bound in (1.3) is best possible.

(ii) If $0 < r < \frac{4}{5}$, then

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} > \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} > \frac{b}{b+\delta}. \tag{1.4}$$

2. Proof of theorem 1

Proof: (i) The inequality (1.3) is equivalent to

$$\left[\frac{(b+\delta)^{b+\delta}}{a^a}\right]^{r/(b+\delta-a)} \Big/ \frac{(b+\delta)^{r+1}-a^{r+1}}{b+\delta-a} < \left[\frac{b^b}{a^a}\right]^{r/(b-a)} \Big/ \frac{b^{r+1}-a^{r+1}}{b-a}. \tag{2.1}$$

Therefore, it is sufficient to prove that the function

$$f(s) = \left[\frac{s^s}{a^a}\right]^{r/(s-a)} \Big/ \frac{s^{r+1}-a^{r+1}}{s-a} \tag{2.2}$$

is decreasing for $s > a$.

By direct computation we have

$$f'(s) = \left[\frac{s^s}{a^a}\right]^{r/(s-a)} \cdot g(s) \Big/ (s-a)^6 \cdot \left(\frac{s^{r+1}-a^{r+1}}{s-a}\right)^2, \tag{2.3}$$

where

$$\begin{aligned} g(s) = {} & (1-r)s^{r+2} + (2ar + alna - alns)s^{r+1} - a^2(r+1)s^r - 2a^{r+1}s \\ & + a^{r+2}lns - a^{r+2}lna + 2a^{r+2} \end{aligned} \tag{2.4}$$

and $g(a) = 0$.

Straightforward calculating produces

$$g'(s) = \tau(s)/s, \tag{2.5}$$

where

$$\begin{aligned}\tau(s) = (1-r)(r+2)s^{r+2} + [a(r+1)lna + 2ar(r+1) - a(r+1)lns - a]s^{r+1} \\ - a^2r(r+1)s^r - 2a^{r+1}s + a^{r+2}\end{aligned} \tag{2.6}$$

and $\tau(a) = 0$.

Direct computing gives us

$$\begin{aligned}\tau'(s) = (1-r)(r+2)^2s^{r+1} + [a(r+1)^2lna \\ + 2a(r+1)(r^2+r-1) - a(r+1)^2lns]s^r \\ - a^2r^2(r+1)s^{r-1} - 2a^{r+1}\end{aligned} \tag{2.7}$$

and

$$\tau''(s) = s^{r-1} \cdot h(s), \tag{2.8}$$

where

$$\begin{aligned}h(s) = (1-r^2)(r+2)^2s - ar(r+1)^2lns - a^2r^2(r^2-1)s^{-1} + 2ar^4 \\ + 4ar^3 - ar^2 - 4ar - a + ar(r+1)^2lna\end{aligned} \tag{2.9}$$

and $\tau'(a) = 0$,$h(a) = 3a(1-r^2)$.

By direct computation we obtain

$$h'(s) = p(s)/s^2, \tag{2.10}$$

where

$$\begin{aligned}p(s) &= (1-r^2)(r+2)^2s^2 - ar(r+1)^2s + a^2r^2(r^2-1), \\ p(a) &= a^2(r+1)^2(4-5r), \\ p'(s) &= 2(1-r^2)(r+2)^2s - ar(r+1)^2, \\ p'(a) &= a(r+1)(-2r^3-7r^2-r+8),\end{aligned} \tag{2.11}$$

and

$$p''(s) = 2(1-r^2)(r+2)^2. \tag{2.12}$$

Then, when $r > 1$ and $s > a$, $p''(s) < 0$, $p'(s) \searrow$, $p'(s) < p'(a) < 0$; $p(s) \searrow$, $p(s) < p(a) < 0$, and thus $h'(s) < 0$, $h(s) \searrow$, $h(s) < h(a) < 0$; therefore $\tau''(s) < 0$, $\tau'(s) \searrow$, $\tau'(s) < \tau'(a) = 0$, and then $\tau(s) \searrow$, $\tau(s) < \tau(a) = 0$; $g'(s) < 0$, $g(s) \searrow$, $g(s) < g(a) = 0$; hence $f'(s) < 0$. The inequality (1.3) follows.

(ii). The left inequality in (1.4) is equivalent to

$$\left[\frac{(b+\delta)^{b+\delta}}{a^a}\right]^{r/(b+\delta-a)} \bigg/ \frac{(b+\delta)^{r+1}-a^{r+1}}{b+\delta-a} > \left[\frac{b^b}{a^a}\right]^{r/(b-a)} \bigg/ \frac{b^{r+1}-a^{r+1}}{b-a}. \tag{2.13}$$

Therefore, it is sufficient to prove that the function

$$f(s) = \left[\frac{s^s}{a^a}\right]^{r/(s-a)} \bigg/ \frac{s^{r+1}-a^{r+1}}{s-a} \tag{2.14}$$

is increasing for $s > a$.

In a similar way, when $0 < r < \frac{4}{5}$, and $s > a$, we have $p''(s) > 0$, $p'(s) \nearrow$, $p'(s) > p'(a) > 0$; $p(s) \nearrow$, $p(s) > p(a) > 0$, and thus $h'(s) > 0$, $h(s) \nearrow$, $h(s) > h(a) > 0$; therefore $\tau''(s) > 0$, $\tau'(s) \nearrow$, $\tau'(s) > \tau'(a) = 0$, and then $\tau(s) \nearrow$, $\tau(s) > \tau(a) = 0$; $g'(s) > 0$, $g(s) \nearrow$, $g(s) > g(a) = 0$; hence $f'(s) > 0$. The left inequality in (1.4) holds.

The right inequality in (1.4) is equivalent to

$$\left[\frac{b^b}{a^a}\right]^{1/(b-a)} \bigg/ b > \left[\frac{(b+\delta)^{b+\delta}}{a^a}\right]^{1/(b+\delta-a)} \bigg/ (b+\delta). \tag{2.15}$$

Therefore, it is sufficient to prove that the function

$$L(s) = \left[\frac{s^s}{a^a}\right]^{1/(s-a)} \bigg/ s \tag{2.16}$$

is decreasing for $s > a$.

By direct computation, we obtain

$$L'(s) = \left[\frac{s^s}{a^a}\right]^{1/(s-a)} \cdot M(S) \bigg/ s^2(s-a)^2, \tag{2.17}$$

where

$$M(s) = as - aslns + aslna - a^2 \tag{2.18}$$

and $M'(s) = alna - alns$.

Therefore $M''(S) = -\frac{a}{s} < 0$, $M'(s) \searrow$, $M'(s) < M'(a) = 0$; $M(s) \searrow$, $M(s) < M(a) = 0$, and then $L'(s) < 0$. The right inequality in (1.4) follows.

Remark 2.1: Let $r \in \mathbb{R}$ and $0 < r < \frac{4}{5}$, inequality (1.4) refines the lower bound of inequality (1.2).

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