Communications in Mathematical Analysis
ISSN 0973-3841 Vol.1 No.1 (2006), pp. 46-51

http://www.ripublication.com/cma.htm

On a summability factor theorem

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(Communicated by Toka Diagana)

Abstract

In this paper a main theorem on $\mid N, p_n \mid_k$ summability factors, which generalizes a known result on $\mid N, p_n \mid$ summability factors, has been proved.

AMS Subject Classification: Primary 40D15, 40F05; Secondary 40G05
Keywords: Nörlund summability, summability factors.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (w_n), respectively.

The series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^\alpha \mid^k < \infty. \tag{1.1}$$

If we take $\alpha = 1$, then $\mid C, \alpha \mid_k$ summability reduces to $\mid C, 1 \mid_k$ summability.

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \ (n \geq 0). \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1.3}$$

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defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $\mid N, p_n \mid$, if (see [5])

$$\sum_{n=1}^{\infty} \mid \sigma_n - \sigma_{n-1} \mid < \infty, \tag{1.4}$$

and it is said to be summable $\mid N, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1} \mid \sigma_n - \sigma_{n-1} \mid^k < \infty. \tag{1.5}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \ \alpha \geq 0 \tag{1.6}$$

the Nörlund mean reduces to the (C, α) mean and $\mid N, p_n \mid_k$ summability becomes $\mid C, \alpha \mid_k$ summability. For $p_n = 1$, we get the $(C, 1)$ mean and then $\mid N, p_n \mid_k$ summability becomes $\mid C, 1 \mid_k$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

2. The Known Results

Concerning the $\mid C, 1 \mid$ and $\mid N, p_n \mid$ summabilities Kishore [4] has proved the following theorem.

Theorem 2.1: Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $\mid C, 1 \mid$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $\mid N, p_n \mid$.

Later on Ram [6] has proved the following theorem related to the absolute Nörlund summability factors of infinite series.

Theorem 2.2: Let (p_n) be as in Theorem 2.1. If

$$\sum_{v=1}^{n} \frac{1}{v} \mid s_v \mid = O(X_n) \ as \ n \to \infty, \tag{2.1}$$

where (X_n) is a positive non-decreasing sequence and (λ_n) is a sequence such that

$$\sum_{n=1}^{\infty} n \mid \Delta^2 \lambda_n \mid X_n < \infty, \tag{2.2}$$

$$\mid \lambda_n \mid X_n = O(1) \ as \ n \to \infty, \tag{2.3}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N, p_n \mid$.

Bor [1] has proved Theorem 2.2 under weaker conditions in the following form.

Theorem 2.3: Let (p_n) be a sequence as in Theorem 2.1. If

$$\sum_{v=1}^{n} \frac{1}{v} \mid t_v \mid = O(X_n) \; as \; n \to \infty, \tag{2.4}$$

where (t_n) is the n-th (C,1)mean of the sequence (na_n), and the sequences (λ_n), (X_n) are such that conditions (2.2)-(2.3) of Theorem 2.2 are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N, p_n \mid$.

Remark 2.4: It should be noted that condition (2.1) implies the condition (2.4), but the converse need not be true (see [1] for details).

Also Varma [7] has proved the following summability factor theorem.

Theorem 2.5: Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $\mid C, 1 \mid_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $\mid N, p_n \mid_k$, $k \geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.3 for $\mid N, p_n \mid_k$ summability. Now we shall prove the following theorem.

Theorem 3.1: Let (p_n) be as in Theorem 2.1. If

$$\sum_{v=1}^{n} \frac{1}{v} \mid t_v \mid^k = O(X_n) \; as \; n \to \infty, \tag{3.1}$$

and the sequences (λ_n) and (X_n) satisfy the conditions (2.2) and (2.3) of Theorem 2.2, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N, p_n \mid_k$, $k \geq 1$.

Remark 3.2: It should be noted that if we take $k = 1$, then we get Theorem 2.3.

We need the following lemma for the proof of our theorem.

Lemma 3.3 ([1]): Under the conditions on (X_n) and (λ_n), as taken in the statement of Theorem 2.2, the following conditions hold,

$$nX_n \Delta\lambda_n = O(1) \; as \; n \to \infty, \tag{3.2}$$

$$\sum_{n=1}^{\infty} \Delta\lambda_n X_n < \infty. \tag{3.3}$$

4. Proof of the Theorem

In order to prove the theorem, we need consider only the special case in which (N, p_n) is $(C, 1)$, that is, we shall prove that $\sum a_n \lambda_n$ is summable $\mid C, 1 \mid_k$. Our theorem will then follow by means of Theorem 2.5. Let T_n be the n-th $(C,1)$ mean of the sequence $(na_n\lambda_n)$, that is,

$$T_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v\lambda_v. \tag{4.1}$$

Using Abel's transformation, we have

$$\begin{aligned} T_n &= \frac{1}{n+1}\sum_{v=1}^{n} va_v\lambda_v = \frac{1}{n+1}\sum_{v=1}^{n-1}\Delta\lambda_v(v+1)t_v + \lambda_n t_n \\ &= T_{n,1} + T_{n,2}\ ,\ say. \end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n} \mid T_{n,r} \mid^k < \infty \ \ for\ r = 1, 2,\ by\ (1.1). \tag{4.2}$$

Now, we have that

$$\begin{aligned} \sum_{n=2}^{m+1}\frac{1}{n} \mid T_{n,1} \mid^k &\le \sum_{n=2}^{m+1}\frac{1}{n(n+1)^k}\left\{\sum_{v=1}^{n-1}\frac{v+1}{v} v \mid \Delta\lambda_v \mid\mid t_v \mid\right\}^k \\ &= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{k+1}}\left\{\sum_{v=1}^{n-1} v \mid \Delta\lambda_v \mid\mid t_v \mid\right\}^k \\ &= O(1)\sum_{n=2}^{m+1}\frac{1}{n^2}\sum_{v=1}^{n-1}(v \mid \Delta\lambda_v \mid)^k \mid t_v \mid^k \times \left\{\frac{1}{n}\sum_{v=1}^{n-1} 1\right\}^{k-1} \\ &= O(1)\sum_{n=2}^{m+1}\frac{1}{n^2}\sum_{v=1}^{n-1}(v \mid \Delta\lambda_v \mid)^k \mid t_v \mid^k \\ &= O(1)\sum_{v=1}^{m}(v \mid \Delta\lambda_v \mid)^k \mid t_v \mid^k \sum_{n=v+1}^{m+1}\frac{1}{n^2} \\ &= O(1)\sum_{v=1}^{m} v \mid \Delta\lambda_v \mid (v \mid \Delta\lambda_v \mid)^{k-1}\frac{\mid t_v \mid^k}{v} \\ &= O(1)\sum_{v=1}^{m} v \mid \Delta\lambda_v \mid \frac{\mid t_v \mid^k}{v} \\ &= O(1)\sum_{v=1}^{m-1}\Delta(v \mid \Delta\lambda_v \mid)\sum_{r=1}^{v}\frac{\mid t_r \mid^k}{r} + O(1)m \mid \Delta\lambda_m \mid \sum_{v=1}^{m}\frac{\mid t_v \mid^k}{v} \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} \mid \Delta(v \mid \Delta\lambda_v \mid) \mid X_v + O(1)m \mid \Delta\lambda_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1} \mid (v+1) \mid \Delta^2\lambda_v \mid - \mid \Delta\lambda_v \mid\mid X_v + O(1)m \mid \Delta\lambda_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta^2\lambda_v \mid X_v \\
&\quad + O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1)m \mid \Delta\lambda_m \mid X_m \\
&= O(1) \; as \; m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the Theorem 3.1 and Lemma 3.3.

Again

$$
\begin{aligned}
\sum_{n=1}^{m} \frac{1}{n} \mid T_{n,2} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^k \frac{\mid t_n \mid^k}{n} \\
&= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid \frac{\mid t_n \mid^k}{n} = O(1)\sum_{n=1}^{m} \mid \lambda_n \mid \frac{\mid t_n \mid^k}{n} \\
&= O(1)\sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \frac{\mid t_v \mid^k}{v} + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \frac{\mid t_n \mid^k}{n} \\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \; as \; m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the Theorem 3.1 and Lemma 3.3.

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