A COMPUTATIONAL APPROACH TO STUDY
A LOGISTIC EQUATION

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Abstract

Using a numerical method, we will show the existence of multiple solutions for the well known logistic equation $-\Delta u = \lambda f(x)u(1 - u)$ for $x \in \Omega$, with Dirichlet boundary condition.

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1 Introduction

We study the existence and multiplicity of positive solutions of the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u(x) = \lambda f(x)u(x)(1 - u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded and smooth domain in $\mathbf{R}^N$, $\lambda > 0$ is a real parameter and $f : \Omega \to \mathbf{R}$ is a smooth function which changes sign in Ω.

This problem is a well known model in population genetics, see for example [6]. The case which is well known in the literature is when $f(x) > 0$ on $\bar{\Omega}$.

If u denotes the frequency of allele A_1, it will be natural that we seek solutions $u, 0 \le u \le 1$. The parameter λ corresponds to the reciprocal of the diffusion.

We will work in the Sobolev space $X := H_0^{1,2}$ equipped with the norm

$$\|u\|_X^2 = \int_\Omega |\nabla u(x)|^2 \, dx.$$

where here and henceforth the integrals are taken on the Ω, unless otherwise specified.

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We recall that the linear problem

$$\begin{cases} -\Delta u(x) = \lambda f(x)u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

has positive and negative principal eigenvalues $\lambda_1^+(f)$ and $\lambda_1^-(f)$. It was proved by Ko and Brown [7] using a constrained optimization argument in the case $\lambda < \lambda_1(f)$ and extended to all $\lambda > 0$ by Delgado and Suarez [5] using global bifurcation theory. We will have a glance at these works.

At first consider for fixed λ the eigenvalue problem

$$\begin{cases} -\Delta u(x) - \lambda f(x)u(x) = \mu u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.3}$$

It can be proved that $\mu(\lambda)$ is a concave function that $\lim_{|\lambda|\to\infty} \mu(\lambda) = -\infty$ and also has exactly two zeros $\lambda_1^+(f)$ and $\lambda_1^-(f)$ and that $\mu(\lambda) > 0$ if and only if $\lambda_1^-(f) < \lambda < \lambda_1^+(f)$. Thus, when $\lambda_1^-(f) < \lambda < \lambda_1^+(f)$, all the eigenvalues of the linear operator corresponding to $-\Delta - \lambda f(x)$ with Dirichlet boundary conditions are positive and so $u \equiv 0$ is a stable solution of (1).

In the case $\lambda > \lambda_1^+(f)$ there exists a principal eigenvalue $\mu(\lambda) < 0$ with corresponding positive principal eigenfunction ϕ_1 such that

$$-\Delta\phi_1(x) - \lambda f(x)\phi_1(x) = \mu(\lambda)\phi_1(x) \text{ for } x \in \Omega; \quad u(x) = 0 \text{ for } x \in \partial\Omega.$$

It is easy to see that $\epsilon\phi_1$ is a subsolution provided $\epsilon > 0$ is sufficiently small.

Clearly $u \equiv 1$ is a supersolution and so there exists a minimal positive solution $u_0(\lambda)$ satisfying $0 \le u_0(\lambda)(x) \le 1$ for $x \in \Omega$ whenever $\lambda > \lambda_1^+(f)$.

It is proved in [4] that when $0 < \lambda \le \lambda_1^+(f)$, $u \equiv 0$ is the only non-negative solution satisfying $0 \le u \le 1$ and when $\lambda > \lambda_1^+(f)$, there exists a unique positive solution satisfying $0 \le u \le 1$, i.e., $\lambda_1^+(f)$ is a bifurcation point of (1) that tends to right without any turning and so this branch of solutions lies beneath of the line $||u||_\infty = 1$. Moreover it is proved in [1] that there exists another branch of positive solutions for all $\lambda > 0$ and because of the uniqueness results obtained in Brown and Hess [4] for this solution we have $||u||_\infty > 1$ (see figure 1).

The object of this paper is comparing results obtained by the numerical method presented here and the theoritical results in [1,4] and determining $\lambda_1^+(f)$ with one decimal place.

Although the main approach in this paper is computational and based on finite difference method, we want to have a briefly explain theoretical results that are already known in the literature.

The existence of the positive solutions of the boundary value problem

$$\begin{cases} -\Delta u(x) + q(x)u(x) = \lambda f(x)|u|^{p-1}u & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $p > 1$, is investigated by Afrouzi and Brown in [1], first by showing that the operator $-\Delta + q$ together with Dirichlet boundary conditions generates a self-adjoint operator with lowest eigenvalue ≥ 0, and finding positive solutions of the above problem as critical points of the functional $J : X \to \mathbf{R}$ such that

$$J(u) = \frac{1}{2} \int_\Omega \left[|\nabla u|^2 + q(x)u^2 \right] dx - \frac{1}{p+1} \int_\Omega f(x) |u|^{p+1} dx.$$

They have shown the following version of the Mountain Pass Lemma that was introduced by Ambrosetti and Rabinowitz in 1973, can be applied to the functional J under the standard compactness assumption that $p < \frac{N+2}{N-2}$ (Sobolev critical exponent).

Mountain Pass Lemma [2]: Let E be a Banach space over $\mathfrak{R}$. Let $B_r = \{u \in E : ||u|| < r\}$ and $S_r = \partial B_r$; B_1 and S_1 will be denoted by B and S, respectively. Let $I \in C^1(E, \mathfrak{R})$. If I satisfies $I(0) = 0$ and
(I1) there exist $\rho > 0$ and $\alpha > 0$, such that $I > 0$ in $B_\rho - \{0\}$ and

$$I \geq \alpha > 0$$

on S_ρ.
(I2) there exists $e \in E$, $e \neq 0$ with $I(e) = 0$;
(I3) If $\{u_m\} \subset E$ with the properties that $I(u_m)$ is bounded above, and $I'(u_m) \to 0$ as $m \to \infty$, then $\{u_m\}$ possesses a convergent subsequence. Let

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

Then

$$b :\equiv \inf_{g \in \Gamma} \max_{y \in [0,1]} I(g(y))$$

is a critical value of I with $0 < \alpha \leq b < +\infty$.

Thus by showing that all of the conditions of the Mountain Pass Lemma are satisfied, there exists a critical point $\hat{u}$ of J such that

$$J(\hat{u}) = \inf_{\phi \in \Gamma} \max_{t \in [0,1]} J(\phi(t)) > 0.$$

Since $J(u) = J(|u|)$, we may assume without loss of generality, replacing $\hat{u}$ by $|\hat{u}|$ if necessary, that $\hat{u} \geq 0$ and hence they proved the following theorem in general case:

Theorem 1.1 [1]: *The semilinear boundary value problem*

$$-\Delta u(x) + q(x)u(x) = \lambda f(x) |u|^{p-1} u \text{ for } x \in \Omega; \quad u(x) = 0 \text{ for } x \in \partial\Omega$$

where $p > 1$, $f : \Omega \to \mathbf{R}$ is a smooth function which is somewhere positive and $-\Delta + q$ together with Dirichlet boundary conditions generates a self-adjoint operator has a positive solution if either
(i) the principal eigenvalue of $-\Delta + q > 0$;
(ii) the principal eigenvalue of $-\Delta + q = 0$ and $\int_\Omega f\phi_1^{p+1} dx < 0$ where ϕ_1 denotes the corresponding principal eigenfunction.

Now by using Theorem 1.1 in special case $p = 2$ and $N < 6$ it can be proved the following theorem:

Theorem 1.2: *Consider the semilinear boundary value problem*

$$\begin{cases} -\Delta u(x) = \lambda f(x)u(x)(1 - u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.4}$$

where $N < 6$, $f : \Omega \to \mathbf{R}$ is a smooth function which changes sign in Ω. Then (4) has two different branches of positive solutions with the following properties:

1. *one of them bifurcates from $\lambda_1^+(f)$ and tends to right without any turning back and all of the solutions u on this branch are such that $||u||_\infty < 1$.*

2. *another branch is such that for every $\lambda > 0$ there exists a positive solution with $||u||_\infty > 1$.*

By numerical approach we can show that all of the obtained results by theoretical method are satisfied and moreover we shall find some interesting properties of these branches and the range of that $\lambda_1^+(f)$ belongs to it.

In the next section we state basic concepts of the numerical method that we want to apply it for solving our problem and at the end of this section we provide some examples and obtain their numerical solutions and produce the graph of bifurcation diagram.

2 Numerical Results

In this section we present a numerical results is based on "finite difference method".

The essence of the method of differences for the solution of differential equations is that instead of solving a differential equation one solves a corresponding finite difference equation that is obtained by substituting differences expressions with higher or lower level of accuracy for the derivatives.

At first we state the basic concepts of this approach. For simplicity we consider Ω be a region like a cube in $\mathfrak{R}^N$, and $f(x) : \Omega \to \mathfrak{R}$ be a sign changing smooth function in Ω.

Our main purpose is replacing the approximation by the derivative. By using Taylor expansion for the function $u(x_1, x_2, \cdots, x_i + \triangle x_i, \cdots, x_N)$ around $(x_1, x_2, \cdots, x_N)$ we have:

$$u(x_1, x_2, \cdots, x_i + \triangle x_i, \cdots, x_N) = u(x_1, x_2, \cdots, x_N) + \triangle x_i \frac{\partial u}{\partial x_i}(x_1, x_2, \cdots, x_N) +$$

$$\frac{(\triangle x_i)^2}{2!} \frac{\partial^2 u}{\partial x_i^2}(x_1, x_2, \cdots, x_N) + \cdots$$

After dividing by $\triangle x_i$ we can obtain forward difference for the partial derivative of i-th variable:

$$\frac{\partial u}{\partial x_i}(x_1, \cdots, x_N) \cong \frac{1}{\triangle x_i}[u(x_1, \cdots, x_i + \triangle x_i, \cdots, x_N) - u(x_1, \cdots, x_N)].$$

The backward difference is gained by using the same approach i.e.

$$\frac{\partial u}{\partial x_i}(x_1, \cdots, x_N) \cong \frac{1}{\triangle x_i}[u(x_1, \cdots, x_N) - u(x_1, \cdots, x_i - \triangle x_i, \cdots, x_N)].$$

By putting together these formula we have the "central difference" that is the best approximation for $\frac{\partial u}{\partial x_i}$:

$$\frac{\partial u}{\partial x_i}(x_1, \cdots, x_N) \cong \frac{1}{2\triangle x_i}[u(x_1, \cdots, x_i + \triangle x_i, \cdots, x_N) - u(x_1, \cdots, x_i - \triangle x_i, \cdots, x_N)].$$

We can continue this procedure to gain higher derivative such as:

$$\frac{\partial^2 u}{\partial x_i^2}(x_1, \cdots, x_N) \cong \frac{1}{(\triangle x_i)^2}[u(x_1, \cdots, x_i + \triangle x_i, \cdots, x_N)$$
$$- 2u(x_1, \cdots, x_N) + u(x_1, \cdots, x_i - \triangle x_i, \cdots, x_N)].$$

We substitute these approximations for the first and second derivatives in ordinary and partial differential equation for $n = 1$ and $n > 1$ respectively. It is important to note, moreover, that on substitution of the differential equation, a difference equation is obtained that combines the value of the required function only in individual, discretely distributed points. The points are usually chosen so to form a quadrate network, i.e., we find an array **u** of real numbers agreeing with solution u on a grid $\Omega \subset \Omega$ and then one can study behavior of solution by considering this numerical solution.

The method of differences is especially suitable for the solution of boundary value problems, for instance, the problem of determining a function that satisfies the Laplace equation in the interior of a given field Ω and possesses given values at the boundary of the field; such problems arise in the exploration of stationary temperature distribution when the temperature at the boundary of the field is known, in investigating the tension in a twisted rod of prismatic section, etc. In this cases the procedure is as above.

In the following subsections we look for the solutions that the existence and behavior of them is proved in section 1.

2.1 Numerical Results: ODE case

Let $\Omega = [0, 1]$ and $f(x) = \frac{1}{2} - x$ a sign changing function in Ω, we want to obtain a numerical solution for the problem

$$-u'' = \lambda f(x)u(1 - u) \quad \text{for } x \in \Omega, \qquad u(0) = u(1) = 0$$

By choosing $h = \triangle x = \frac{1}{10}$ we divide Ω into 10 section that according to the Dirichlet condition we have $u_0 = u(0) = 0$ and $u_{10} = u(1) = 0$. Suppose that $x_i = ih$ and $u_i = u(x_i)$. By using above discussion we have the following system of equations:

$$-\frac{1}{h^2}[u_{i+1} - 2u_i + u_{i-1}] = \lambda f(x_i)u_i(1 - u_i) \quad i = 1, 2, \cdots, 9 \tag{2.1}$$

Our main purpose is obtaining variables $u_1, u_2, \cdots, u_9$. For solving this system of equations we have used "Matlab toolbox" and a useful algorithm that can solve any $(n - 1) \times (n - 1)$ system of equation and find the position of the bifurcation point "$\lambda_1^+(f)$". The obtained results shows there are two array of solutions that before $\lambda_1^+(f)$ one of them is identically zero (in Mathlab toolbox we assume that the numbers less than $d_1.d_2d_3 \cdots \times 10^-6$ is identical zero) and another has the norm above the horizontal asymptote 1 when we define

$$||u|| = ||u||_\infty = \sup_{x \in [0,1]} u(x),$$

and after $\lambda_1^+(f)$ one of the array of solutions has values less than 1 and again another greater than 1 (see the first and second tables) for brevity we express just some of those numerical results.

Table 1.

| λ	**u** in				
	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
1	1.88×10^{-16}	1.70×10^{-15}	4.62×10^{-15}	7.23×10^{-15}	4.33×10^{-15}
10	2.27×10^{-10}	7.35×10^{-10}	1.40×10^{-9}	1.98×10^{-9}	1.09×10^{-9}
100	1.00×10^{-7}	4.53×10^{-7}	1.13×10^{-6}	1.43×10^{-6}	1.01×10^{-6}
100.1	0.000027	0.000123	0.000308	0.000474	0.000275
100.2	0.000090	0.000407	0.00102	0.00156	0.00091
105	0.00288824	0.0132691	0.0336236	0.0517604	0.0303053
110	0.00537165	0.0250811	0.0642849	0.0993022	0.0585243
200	0.017342	0.106881	0.332809	0.541458	0.352052
500	0.00715019	0.0913763	0.473675	0.843527	0.649998
10^5	1.47×10^{-6}	0.00197915	0.499998	0.997986	0.975604
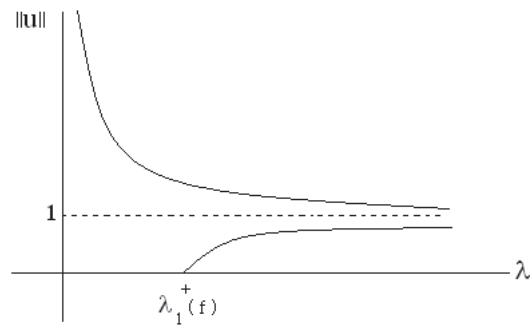
Figure 1. Bifurcation diagram

For any value of λ we can draw the interpolation diagram of numerical values of the solution but it is not important that for any λ what is the diagram of solution. In fact we are looking for the behavior of bifurcation diagram (a diagram in $(\lambda, ||u||)$- plane).

According to the above tables we can determine the bifurcation point with a good accuracy, i.e., $\lambda^+(f)$ is around 100.1 that before it there is not any nonzero solution with $||u||_\infty < 1$ and for small λ the values of solutions tends to ∞, moreover when we consider large λ the solutions with $||u||_\infty < 1$ and other solution achieve same values in each case (see figure 1).

Table 2.

λ	**u** *in*				
	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
0.001	55308.	112417.	79703.2	40143.	12336.1
1	55.7605	113.566	80.5946	40.6831	12.5421
10	5.97384	12.3945	8.87244	4.56917	1.44939
100	4.29464	2.59059	1.68119	1.2703	0.538632
105	0.862761	2.25422	1.76872	1.08568	0.448388
110	0.83322	2.20715	1.73967	1.07533	0.449291
200	0.511211	1.77038	1.50136	1.0148	0.502894
500	1.71038	0.477379	0.656528	0.890166	0.658966
10^5	1.02565	1.00005	0.999995	0.999965	0.975606

Moreover we find another numerical behavior of the bifurcation point 100.1. By using "Matlab toolbox" we find the eigenvalues of the following problem

$$AU = \lambda BU \qquad (2.2)$$

where A is tridiagonal matrix with elements determined by the right hand side of the system (5), and B is a diagonal matrix with elements $f(x_i)$ on the principal diagonal.

As such as theoretical arguments the first eigenvalue of (6) is exactly the bifurcation point of the problem (5), i.e. 100.1! In this case our main idea is finding the numerical positive solutions and drawing the bifurcation diagram, in this way we can approximate the value of $\lambda_1^+(f)$. If we only want to find position of $\lambda_1^+(f)$, the easier way is investigating of (6).

2.2 Numerical Results: PDE case

Let $\Omega = [0, 1] \times [0, 1]$ and $f(x) = 0.1 - xy$ that changes sign in Ω. The grid $\Omega \subseteq \Omega$ be a division of Ω and $h = \triangle x = \triangle y = \frac{1}{10}$ and we solve numerically the problem (1). Drichlet boundary condition leads us to have $u_{0,j} = u_{i,0} = u_{i,10} = u_{10,j} = 0$ for $i, j = 0, 1, 2, \cdots, 10$.

Table 3.

λ	u		
10^{-4}	$u_{11} = 3.06 \times 10^{-21}$ $u_{44} = 5.30 \times 10^{-20}$ $u_{77} = 1.29 \times 10^{-19}$	$u_{22} = 1.23 \times 10^{-20}$ $u_{55} = 8.69 \times 10^{-20}$ $u_{88} = 9.30 \times 10^{-20}$	$u_{33} = 2.83 \times 10^{-20}$ $u_{66} = 1.20 \times 10^{-19}$ $u_{99} = 3.01 \times 10^{-20}$
0.01	$u_{11} = 3.06 \times 10^{-15}$ $u_{44} = 5.29 \times 10^{-14}$ $u_{77} = 1.29 \times 10^{-13}$	$u_{22} = 1.23 \times 10^{-14}$ $u_{55} = 8.67 \times 10^{-14}$ $u_{88} = 9.31 \times 10^{-14}$	$u_{33} = 2.83 \times 10^{-14}$ $u_{66} = 1.20 \times 10^{-13}$ $u_{99} = 3.01 \times 10^{-14}$
1000	$u_{11} = 1.32 \times 10^{-10}$ $u_{44} = 4.18 \times 10^{-11}$ $u_{77} = 7.60 times 10^{-14}$	$u_{22} = 2.47 \times 10^{-10}$ $u_{55} = 7.82 \times 10^{-12}$ $u_{88} = 4.13 \times 10^{-15}$	$u_{33} = 1.39 \times 10^{-10}$ $u_{66} = 9.47 \times 10^{-13}$ $u_{99} = 1.46 \times 10^{-16}$
1400	$u_{11} = 4.33 \times 10^{-10}$ $u_{44} = 1.20 \times 10^{-10}$ $u_{77} = 8.91 \times 10^{-14}$	$u_{22} = 7.59 \times 10^{-10}$ $u_{55} = 1.81 \times 10^{-11}$ $u_{88} = 3.20 \times 10^{-15}$	$u_{33} = 4.39 \times 10^{-10}$ $u_{66} = 1.61 \times 10^{-12}$ $u_{99} = 7.40 \times 10^{-17}$
1409.2	$u_{11} = 5.29 \times 10^{-5}$ $u_{44} = 1.47 \times 10^{-5}$ $u_{77} = 1.06 \times 10^{-8}$	$u_{22} = 9.29 \times 10^{-5}$ $u_{55} = 2.20 \times 10^{-6}$ $u_{88} = 3.79 \times 10^{-10}$	$u_{33} = 5.35 \times 10^{-5}$ $u_{66} = 1.94 \times 10^{-7}$ $u_{99} = 8.70 \times 10^{-12}$
1409.3	$u_{11} = 0.000065$ $u_{44} = 0.000018$ $u_{77} = 1.31 \times 10^{-8}$	$u_{22} = 0.00011$ $u_{55} = 2.72 \times 10^{-6}$ $u_{88} = 4.69 \times 10^{-10}$	$u_{33} = 0.000066$ $u_{66} = 2.40 \times 10^{-7}$ $u_{99} = 1.07 \times 10^{-11}$
1500	$u_{11} = 0.04593$ $u_{44} = 0.01245$ $u_{77} = 7.51 \times 10^{-6}$	$u_{22} = 0.07930$ $u_{55} = 0.00178$ $u_{88} = 2.47 \times 10^{-7}$	$u_{33} = 0.04609$ $u_{66} = 0.000147$ $u_{99} = 5.20 \times 10^{-9}$
10000	$u_{11} = 0.7985$ $u_{44} = 0.05842$ $u_{77} = 3.6 \times 10^{-7}$	$u_{22} = 0.95044$ $u_{55} = 0.00112$ $u_{88} = 9.63 \times 10^{-11}$	$u_{33} = 0.63679$ $u_{66} = 8.76 \times 10^{-6}$ $u_{99} = 1.63 \times 10^{-13}$

By using the approximation of u_{xx} and u_{yy} we have again a linear system of equations of this type

$$100(u_{21} + u_{12} - 4u_{11}) + \lambda(0.1 - 0.1 * 0.1)(u_{11} - u_{11}^2) = 0 \qquad \text{for } i = j = 1 \qquad (2.3)$$

Table 4.

λ	u		
10^{-4}	$u_{11} = 28888.8$	$u_{22} = 116450.$	$u_{33} = 269273.$
	$506199.$	$u_{55} = 846495.$	$u_{66} = 1.22 \times 10^6$
	$u_{77} = 1.30 \times 10^6$	$u_{88} = 805470.$	$u_{99} = 213902.$
0.01	$u_{11} = 124048$	$u_{22} = 263286.$	$u_{33} = 1.26 \times 10^6$
	$u_{44} = 1.00 \times 10^6$	$u_{55} = 25727.6$	$u_{66} = 168078$
	$u_{77} = 178700$	$u_{88} = 178337$	$u_{99} = 87317.7$
10	$u_{11} = 0.32021$	$u_{22} = 1.28483$	$u_{33} = 2.95689$
	$u_{44} = 5.5528$	$u_{55} = 9.34157$	$u_{66} = 13.643$
	$u_{77} = 14.6001$	$u_{88} = 8.79291$	$u_{99} = 2.25532$
100	$u_{11} = 0.204221$	$u_{22} = 0.750349$	$u_{33} = 1.58515$
	$u_{44} = 2.66565$	$u_{55} = 3.14669$	$u_{66} = 2.50618$
	$u_{77} = 1.7146$	$u_{88} = 2.85901$	$u_{99} = 1.84918$
10000	$u_{11} = 0.799268$	$u_{22} = 0.969581$	$u_{33} = 0.990234$
	$u_{44} = 1.00707$	$u_{55} = 1.00159$	$u_{66} = 1.00001$
	$u_{77} = 1.00002$	$u_{88} = 0.999371$	$u_{99} = 1.02851$

$$100(u_{22} + u_{13} + u_{11} - 4u_{12}) + \lambda(0.1 - 0.1 * 0.2)(u_{12} - u_{12}^2) = 0 \quad \text{for } i = 1, j = 2 \quad (2.4)$$

$$\vdots$$

$$100(u_{89} + u_{98} - 4u_{99}) + \lambda(0.1 - 0.9 * 0.9)(u_{99} - u_{99}^2) = 0 \quad \text{for } i = 9, j = 9 \quad (2.5)$$

After solving this system that has 81 equation and 81 by "Matlab toolbox" we obtain **u** in grid Ω that leads us to understand the behavior of solution branches. We express just some of values of u_{ij}s in the following tables. It is easy to see that $\lambda_1^+(f)$ in this case is 1409.3 with decimal accuracy.

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