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Intuitionistic Fuzzy Euclidean Normed Spaces

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Abstract

In this paper, intuitionistic fuzzy Euclidean normed spaces are considered and compactness in these spaces is studied.

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1 Introduction and Preliminaries

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [7]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology. The concept of fuzzy topology may have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [1]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of intuitionistic fuzzy normed space. This problem has been investigated by Saadati and Park [5]. They have introduced and studied a notion of intuitionistic fuzzy normed spaces. In this section, using the idea of intuitionistic fuzzy metric spaces [3], we define the notion of intuitionistic fuzzy normed spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy normed space due to Saadati and Vaezpour [4].

Definition 1.1. The 5-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an *intuitionistic fuzzy normed space* if V is a vector space, $*$ is a continuous t-norm [6], $\diamond$ is a continuous t-conorm [6], and μ, ν are fuzzy sets on $V \times (0, \infty)$ satisfying the following conditions for every $x, y \in V$ and $t, s > 0$:

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(a) $\mu(x,t)+\nu(x,t)\le 1$,

(b) $\mu(x,t)>0$,

(c) $\mu(x,t)=1$ if and only if $x=0$,

(d) $\mu(\alpha x,t)=\mu(x,\frac{t}{|\alpha|})$ for each $\alpha\ne 0$,

(e) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$,

(f) $\mu(x,\cdot):(0,\infty)\longrightarrow[0,1]$ is continuous,

(g) $\lim_{t\longrightarrow\infty}\mu(x,t)=1$ and $\lim_{t\longrightarrow 0}\mu(x,t)=0$,

(h) $\nu(\alpha x,t)=\nu(x,\frac{t}{|\alpha|})$ for each $\alpha\ne 0$,

(i) $\nu(x,t)\diamond\nu(y,s)\ge\nu(x+y,t+s)$,

(j) $\nu(x,\cdot):(0,\infty)\longrightarrow[0,1]$ is continuous,

(k) $\lim_{t\longrightarrow 0}\nu(x,t)=1$.

In this case (μ,ν) is called an *intuitionistic fuzzy norm* with respect to $*$ and $\diamond$.

Example 1.2. Let $(V,\|\cdot\|)$ be a normed space. Denote $a*b=ab$ and $a\diamond b=\min(a+b,1)$ for all $a,b\in[0,1]$ and let μ_0 and ν_0 be fuzzy sets on $V\times(0,\infty)$ defined as follows:

$$\mu_0(x,t)=\frac{t}{t+m\|x\|},\quad \nu_0(x,t)=\frac{\|x\|}{t+\|x\|},$$

for all $t\in\mathbb{R}^+$ in which $m>1$. Then $(V,\mu_0,\nu_0,*,\diamond)$ is an intuitionistic fuzzy normed space. Here, $\mu_0(x,t)+\nu_0(x,t)=1$ for $x=0$ and $\mu_0(x,t)+\nu_0(x,t)<1$ for $x\ne 0$.

Definition 1.3. A sequence $\{x_n\}$ in an intuitionistic fuzzy normed space $(V,\mu,\nu,*,\diamond)$ is called a *Cauchy sequence* if for each $\varepsilon>0$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that

$$\mu(x_n-x_m,t)>1-\varepsilon\quad\text{and}\quad\nu(x_n-x_m,t)<\varepsilon,$$

for each $n,m\ge n_0$. The sequence $\{x_n\}$ is said to be *convergent* to $x\in V$ in the intuitionistic fuzzy normed space $(V,\mu,\nu,*,\diamond)$ and denoted by $x_n\overset{(\mu,\nu)}{\longrightarrow}x$ if $\mu(x_n-x,t)\longrightarrow 1$ and $\nu(x_n-x,t)\longrightarrow 0$ whenever $n\longrightarrow\infty$ for every $t>0$. An intuitionistic fuzzy normed space is said to be *complete* if and only if every Cauchy sequence is convergent.

Lemma 1.4. *Let $(V,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy normed space. If we define*

$$M(x,y,t)=\mu(x-y,t)\quad\textit{and}\quad N(x,y,t)=\nu(x-y,t),$$

then (M,N) is an intuitionistic fuzzy metric with respect to $*$ and $\diamond$ on V, which is induced by the intuitionistic fuzzy norm (μ,ν) with respect to $*$ and $\diamond$.

Lemma 1.5. *Let (μ,ν) be an intuitionistic fuzzy norm with respect to $*$ and $\diamond$. Then, for any $t>0$, the following hold:*

(1) $\mu(x, t)$ *and* $\nu(x, t)$ *are nondecreasing and nonincreasing with respect to* t*, respectively.*

(2) $\mu(x - y, t) = \mu(y - x, t)$ *and* $\nu(x - y, t) = \nu(y - x, t)$.

Definition 1.6. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. For $t > 0$, we define *open ball* $B(x, r, t)$ with center $x \in V$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in V : \mu(x - y, t) > 1 - r,\ \nu(x - y, t) < r\}.$$

A subset $A \subseteq V$ is called *open* if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subseteq A$. Let $\tau_{(\mu,\nu)}$ denote the family of all open subsets of V. $\tau_{(\mu,\nu)}$ is called the *topology induced by intuitionistic fuzzy norm.*

Note that this topology is the same as the topology induced by intuitionistic fuzzy metric (see, Remark 3.3 of [3]).

Remark 1.7. In an intuitionistic fuzzy normed space $(V, \mu, \nu, *, \diamond)$, if $\mu(x, t) > 1 - r$ and $\nu(x, t) < r$ for $x \in V$, $t > 0$, $0 < r < 1$, we can find a t_0, $0 < t_0 < t$, such that $\mu(x, t_0) > 1 - r$ and $\nu(x, t_0) < r$ (see [2]).

Definition 1.8. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. A subset A of V is said to be *IF-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $\mu(x, t) > 1 - r$ and $\nu(x, t) < r$ for each $x \in A$.

Theorem 1.9. *In an intuitionistic fuzzy normed space every compact set is closed and IF-bounded.*

Proof. By Lemma 1.4 the proof is the same as intuitionistic fuzzy metric space (see, Remark 3.10 of [3]). ■

Lemma 1.10 ([5]). *A subset A of* $\mathbb{R}$ *is IF-bounded in* $(\mathbb{R}, \mu, \nu, *, \diamond)$ *if and only if it is bounded in* $\mathbb{R}$.

Lemma 1.11 ([5]). *A sequence* $\{\beta_n\}$ *is convergent in the intuitionistic fuzzy normed space* $(\mathbb{R}, \mu, \nu, *, \diamond)$ *if and only if it is convergent in* $(\mathbb{R}, |\cdot|)$.

Corollary 1.12. *If the real sequence* $\{\beta_n\}$ *is IF-bounded, then it has at least one limit point.*

Definition 1.13 ([6]). Let $*$ and $*'$ be two continuous t-norms. Then $*'$ dominates $*$, and we write $*' \gg *$, if for all $x_1, x_2, y_1, y_2 \in [0, 1]$,

$$(x_1 *' x_2) * (y_1 *' y_2) \leq (x_1 * y_1) *' (x_2 * y_2).$$

Definition 1.14. A continuous t-norm $*$ is said to be *normal* if $a > 0$ and $b > 0$ then $a * b > 0$ for every $a, b \in]0, 1]$.

For example the product t-norm $a * b = ab$ is normal but t-norm $a * b = \max\{0, a + b - 1\}$ is not normal.

2 Main Results

Definition 2.1. The 5-tuple $(\mathbb{R}^n, \Phi, \Psi, *, \diamond)$ is called an *intuitionistic fuzzy Euclidean normed space* if $*$ is a t-norm, $\diamond$ is a t-conorm and (Φ, Ψ) is an intuitionistic fuzzy Euclidean norm defined by

$$\Phi(x,t) = \prod_{j=1}^{n} \mu(x_j, t) \quad \text{and} \quad \Psi(x,t) = \coprod_{j=1}^{n} \nu(x_j, t),$$

where $x = (x_1, \cdots, x_n)$, $\prod_{j=1}^{n} a_j = a_1 *' \cdots *' a_n$, $*' \gg *$, $\coprod_{j=1}^{n} a_j = a_1 \diamond \cdots \diamond a_n$, $t > 0$, and (μ, ν) is an intuitionistic fuzzy norm with respect to $*$ and $\diamond$.

Lemma 2.2. *Suppose that hypotheses of Definition 2.1 are satisfied. If $*$ is normal and $\diamond = \max$, then $(\mathbb{R}^n, \Phi, \Psi, *, \diamond)$ is an intuitionistic fuzzy normed space.*

Proof. For (a), let $\Psi(x,t) = \max_{j=1}^{n} \nu(x_j, t) = \nu(x_k, t)$ in which $1 \le k \le n$, since $\Phi(x,t) = \prod_{j=1}^{n} \mu(x_j, t) \le \mu(x_k, t)$ then we have $\Phi(x,t) + \Psi(x,t) \le \mu(x_k,t) + \nu(x_k,t) \le 1$. The properties of (b)-(d), (f)-(h) and (j), (k) are immediate from the definition. For triangle inequalities (e) and (i), suppose that $x, y \in V$ and $t, s > 0$.

$$\begin{aligned}
\Phi(x,t) * \Phi(y,s) &= \prod_{j=1}^{n} \mu(x_j, t) * \prod_{j=1}^{n} \mu(y_j, s) \\
&= (\mu(x_1,t) *' \cdots *' \mu(x_n,t)) * (\mu(y_1,t) *' \cdots *' \mu(y_n,t)) \\
&\le (\mu(x_1,t) * \mu(y_1,t)) *' \cdots *' (\mu(x_n,t) * \mu(y_n,t)) \\
&\le \mu(x_1+y_1, t+s) *' \cdots *' \mu(x_n+y_n, t+s) \\
&= \prod_{j=1}^{n} \mu(x_j + y_j, t+s) = \Phi(x+y, t+s).
\end{aligned}$$

The proof for (i) is similar to (e). ■

Example 2.3. Let (Φ, Ψ) be the standard intuitionistic fuzzy Euclidean norm on $\mathbb{R}^n$, i.e. $\Phi(x,t) = \frac{t}{t+\|x\|}$, $\Psi(x,t) = \frac{\|x\|}{t+\|x\|}$ and (μ, ν) be the standard intuitionistic fuzzy norm on $\mathbb{R}$, i.e. $\mu(x_j, t) = \frac{t}{t+|x_j|}$, $\nu(x_j,t) = \frac{|x_j|}{t+|x_j|}$, $* = \min$ and $\diamond = \max$. Then we have $\Phi(x,t) = \min_j \mu(x_j,t)$ and $\Psi(x,t) = \max_j \nu(x_j,t)$ or equivalently $\|x\| = \max_j |x_j|$.

Lemma 2.4. *Suppose that hypotheses of Lemma 2.2 are satisfied. If A is an infinite and IF-bounded subset of $\mathbb{R}^n$, then A has at least one limit point.*

Proof. Let $\{x_m\} \subseteq A$ be an infinite sequence. Since A is IF-bounded, so is $\{x_m\}$. Then there exist $0 < r_0 < 1$ and $t_0 > 0$ such that for each m,

$$r_0 \;>\; \Psi(x_m, t_0) = \max_j \nu\left(x_j^{(m)}, t_0\right) = \nu\left(1, \frac{t_0}{\max_j |x_j^{(m)}|}\right).$$

Therefore there exists a $k \in \mathbb{R}^+$ such that $\max_j |x_j^{(m)}| \le k$, for every m, i.e. the real sequences $\{x_j^{(m)}\}$, $j = 1, \ldots, n$, are bounded. Hence there are subsequences $\{x_j^{(m_k)}\}$, $j = 1, \ldots, n$, converging to x_j in A in the intuitionistic fuzzy norm (μ, ν). Hence the subsequence $\{x_{m_k}\}$ tends to $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. ■

Lemma 2.5. *Let $\{Q_1, Q_2, \ldots\}$ be a countable collection of nonempty subsets in $\mathbb{R}^n$ such that $Q_{k+1} \subseteq Q_k$, each Q_k is closed and Q_1 is IF-bounded. Then $\bigcap_{k=1}^{\infty} Q_k$ is nonempty and closed.*

Proof. Using the above lemma, the proof proceeds as in the classical case. ■

We call an n-dimensional ball $B(x, r, t)$ a rational ball if $x \in \mathbb{Q}^n$, $r \in \mathbb{Q} \cap (0, 1)$ and $t \in \mathbb{Q}^+$.

Theorem 2.6. *Suppose that hypotheses of Lemma 2.2 are satisfied. Let $G = \{A_1, A_2, \ldots\}$ be the countable collection of n-dimensional rational open balls. If $x \in \mathbb{R}^n$ and S is an open subset of $\mathbb{R}^n$ containing x, then there exists $A_k \in G$ such that $x \in A_k \subseteq S$, for some $k \geq 1$.*

Proof. Since $x \in S$ and S is open, there exist $0 < r < 1$ and $t > 0$ such that $B(x, r, t) \subseteq S$. We find rational number $\eta \in (0, 1)$ such that $(1 - \eta) * (1 - \eta) > 1 - r$ and $\eta < r$. Let $\{\xi_k\}_{k=1}^{n}$ be a finite sequence such that $1 - \eta < \prod_{k=1}^{n}(1 - \xi_k)$, $\eta > \xi_k$ and $x = (x_1, \ldots, x_n)$. By density property of $\mathbb{Q}^n$ we can find $y = (y_1, \ldots, y_n) \in \mathbb{Q}^n$ such that $\mu(x_k - y_k, t/2) > 1 - \xi_k$ and $\nu(x_k - y_k, t/2) < \xi_k$. Therefore

$$\Phi(x - y, t/2) = \prod_{k=1}^{n} \mu(x_k - y_k, t/2) \geq \prod_{k=1}^{n}(1 - \xi_k) > 1 - \eta$$

and

$$\Psi(x - y, t/2) = \max_{k=1}^{n} \nu(x_k - y_k, t/2) \leq \max_{k=1}^{n}(\xi_k) < \eta,$$

and so $x \in B(y, \eta, t/2)$. Now we prove that $B(y, \eta, t/2) \subseteq B(x, r, t)$. Let $z \in B(y, \eta, t/2)$. Then $\Phi(y - z, t/2) > 1 - \eta$ and $\Psi(y - z, t/2) < \eta$, and hence

$$\Phi(x - z, t) \geq \Phi(x - y, t/2) * \Phi(y - z, t/2) \geq (1 - \eta) * (1 - \eta) > 1 - r$$

and

$$\Psi(x - z, t) \leq \Psi(x - y, t/2) \diamond \Phi(y - z, t/2) \leq \eta \diamond \eta < r.$$

On the other hand, by Remark 1.7, there exists $t_0 \in \mathbb{Q}$ such that $t_0 < t/2$ and $x \in B(y, \eta, t_0) \subseteq B(y, \eta, t/2) \subseteq S$. Now $B(y, \eta, t_0) \in G$ the proof is complete. ■

Corollary 2.7. *Suppose that hypotheses of Lemma 2.2 are satisfied. In an intuitionistic fuzzy Euclidean normed space $(\mathbb{R}^n, \Phi, \Psi, *, \diamond)$ every closed IF-bounded set is compact.*

Theorem 2.8. *Suppose that hypotheses of Lemma 2.2 are satisfied and $S \subseteq \mathbb{R}^n$. If S is a compact set then it is IF-bounded and closed.*

Proof. S is IF-bounded (see Theorem 3.9 of [3]). We prove that S is closed. If not, then S has a limit point, say y, which is not in S. For each $x \in S$ we put

$$\Phi(x - y, 2t_x) = (1 - r_x) * (1 - r_x) \quad \text{and} \quad \Psi(x - y, 2t_x) = r_x \diamond r_x$$

in which $r_x \in]0, 1[$ and $t_x > 0$ and $(\mathbb{R}^n, \Phi, \Psi, *, \diamond)$ is an intuitionistic fuzzy normed space defined in Lemma 2.2. Since $y \neq x$ and $r_x \neq 0$, the collection $\{B(x, r_x, t_x) : x \in S\}$ is an open cover for S. Since S is compact, a finite number of members of above collection cover S, say

$S \subseteq \bigcup_{k=1}^{p} B(x_k, r_k, t_k)$. Let $r = \min\{r_k : 1 \le k \le p\}$ and $t = \min\{t_k : 1 \le k \le p\}$. We show that for every $k = 1, \ldots, p$,

$$B(y, r, t) \cap B(x_k, r_k, t_k) = \emptyset.$$

In fact, if $x \in B(y, r, t)$ we have

$$\Phi(x - y, t) > 1 - r \ge 1 - r_k \quad \text{and} \quad \Psi(x - y, t) < r \le r_k.$$

On the other hand, we have

$$\Phi(y - x_k, 2t_k) \ge \Phi(y - x, t_k) * \Phi(x - x_k, t_k)$$

and

$$\Psi(y - x_k, 2t_k) \le \Psi(y - x, t_k) \diamond \Psi(x - x_k, t_k).$$

Therefore

$$(1 - r_k) * (1 - r_k) > (1 - r_k) * \Phi(x - x_k, t_k)$$

and

$$r_k \diamond r_k < r_k \diamond \Psi(x - x_k, t_k).$$

By properties of the t-norm and t-conorm, we have $\Phi(x - x_k, t_k) \le 1 - r_k$ and $\Psi(x - x_k, t_k) \ge r_k$ which means that x is not in $B(x_k, r_k, t_k)$, for each $k = 1, \ldots, p$, and so $B(y, r, t) \cap S = \emptyset$, which is a contradiction. ■

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