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# Convergence of the Cesàro-like Means on $p$-adic Hilbert Spaces

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**Abstract**

Let $T : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ be a nonexpansive mapping (possibly nonlinear) on the $p$-adic Hilbert space $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$. We introduce the Cesàro-like means:

$$F_n x := \frac{1}{P_n}\sum_{k=1}^{n} \alpha_k T^k x, \quad x \in \mathbb{E}_\omega,$$

where $P_n = \alpha_1 + \cdots + \alpha_n$ and $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{K}$ is a sequence of nonzero elements satisfying some additional assumptions. And, we provide results on both the strong and weak convergence of the Cesàro-like means to a fixed point $y$ of $T$.

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## 1 Introduction

In the development of the field of non-Archimedean Functional analysis, there has been particular interest in defining an analogue to the notion of a Hilbert Space. In fact, several definitions have been proposed and an outline of these can be found in [1]. The definition that we consider in our study of nonexpansive mappings is the one given in [7] of a $p$-adic Hilbert space.
Let $\mathbb{K}$ be a complete ultrameteric field and $\omega = (\omega_i)_{i\geq 0} \subset \mathbb{K}$ be a sequence of nonzero terms. Consider

$$\mathbb{E}_\omega := c_0\Big(\mathbb{N}, \mathbb{K}, (|\omega_i|^{\frac{1}{2}})_{i\geq 0}\Big) = \Big\{x = (x_i)_{i\geq 0} \subset \mathbb{K} : \lim_{n\to+\infty} |x_i||\omega_i|^{\frac{1}{2}} = 0\Big\}.$$

We define the basis $(e_i)_{i\geq 0}$ by letting $e_i = (\delta_{i,j})_{j\geq 0}$ where $\delta_{i,j}$ is the Kronecker symbols, $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$. $\mathbb{E}_\omega$ is a free Banach space with $(e_i)_{i\geq 0}$ as an orthogonal basis, that is

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every $x \in \mathbb{E}_\omega$ can be written as, $x = \sum_{i\geq 0} x_i e_i$, where $\|x\| = \sup_{i\geq 0} |x_i||\omega_i|^{\frac{1}{2}}$. The bilinear form $\langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \to \mathbb{K}$ defined as,

$$\langle x, y\rangle = \sum_{i\geq 0} x_i y_i \omega_i,$$

for all $x = \sum_{i\geq 0} x_i e_i \in \mathbb{E}_\omega$, and $y = \sum_{i\geq 0} y_i e_i \in \mathbb{E}_\omega$ is symmetric, non-degenerate, and satisfies the Cauchy-Schwartz inequality,

$$\big|\langle x, y\rangle\big| \leq \|x\|\|y\|.$$

The space $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is called a *p-adic Hilbert space*.

The $p$-adic Hilbert space defined above coincides with the definition given by Khrennikov in [2] in his study of $p$-adic Hilbert spaces and their applications to $p$-adic Quantum Mechanics.

In general, $p$-adic Hilbert spaces do not posses many of the fundamental characteristics of Hilbert spaces. For example, the property that given any Hilbert space $H$, if $D \subset H$, $D = D^{\perp\perp} \iff D \oplus D^{\perp} = H$; does not always hold in the $p$-adic setting. Furthermore, properties such as the Parallelogram Law and Parseval's Inequality fail (see [8]). Moreover, there are also several notable differences between the basic topological notions of $p$-adic Hilbert spaces and their classical counterpart. For example, in $p$-adic Hilbert spaces all balls are clopen (both open and closed), and all points contained in a ball lie at its center. One of the most notable differences between Hilbert spaces and $p$-adic Hilbert spaces is the fact that the norm $\|x\| \neq \sqrt{|\langle x, x\rangle|_{\mathbb{K}}}$ in general. The only connection that we have thus far between the norm and the inner product is given by the Cauchy-Schwartz inequality.

The dissimilarities between the structure of $p$-adic Hilbert spaces and Hilbert spaces give rise to the following questions in the context of nonexpansive mappings. Does the structure of $p$-adic Hilbert spaces affect the behavior of nonexpansive mappings defined on these spaces? And, can we generate results in the $p$-adic setting on the convergence of these mappings and there associated Cesàro means that parallels the results for Hilbert spaces?

Nonexpansive mappings on uniformly convex spaces, particularly Hilbert spaces have been extensively studied. Early contributions to the theory of nonexpansive mappings were made by Baillon [4], Pazy [12], and Reich [13]. The development of the theory of nonexpansive mappings on Hilbert spaces has relied on the geometric structure of the space.Uniform convexity is a geometric property of Banach spaces which was introduced by Clark in 1936 as a result of attempts made to classify Banach spaces according to the convexity of their unit balls, and Hilbert spaces are classic examples of them. We will see shortly that in the non-Archimedean setting the geometric structure of $p$-adic Hilbert spaces does not play as significant a role as it did in the classical setting.

The Cesàro means of nonexpansive mappings on uniformly convex Banach spaces $\mathbb{E}$ are defined as,

$$S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x, \quad \text{for } x \in \mathbb{E}.$$

This definition does not always make sense in our study of nonexpansive mappings over $p$-adic Hilbert spaces. To address this we replace $\frac{1}{n}$ by $\frac{1}{\alpha_1 + \cdots + \alpha_n}$ in the definition of the Cesàro means, where $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence of non-zero elements of $\mathbb{K}$, and we introduce an analogue to the Cesàro means which we refer to as the Cesàro-like means.

For an arbitrary element $x \in \mathbb{E}_\omega$, and a nonexpansive map $T : \mathbb{E}_\omega \to \mathbb{E}_\omega$, the Cesàro-like means are defined as,

$$F_n x := \frac{1}{P_n}\sum_{k=1}^{n} \alpha_k T^k x, \quad x \in \mathbb{E}_\omega,$$

where $P_n = \alpha_1 + \cdots + \alpha_n$ and $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{K}$ is a sequence of nonzero elements satisfying some additional assumptions. The sequence $(\alpha_k)_{k\in\mathbb{N}}$ plays a significant role in our work. Under suitable conditions on the sequence we are able to prove that for $x \in \mathbb{E}_\omega$ if the iterates $T^n x$ converge strongly or weakly to a point $y \in \mathbb{E}_\omega$, then the sequence of Cesàro-like means $F_n x$ also converges to $y$, which is a fixed point of $T$. And, the convergence of the Cesàro-like means is independent of the sequence $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{K}$.

In this paper we study nonexpansive mappings on $p$-adic Hilbert spaces $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$, and provide results on both the strong and weak convergence of their associated Cesàro-like means. We begin with a few definitions and basic notions of nonexpansive mappings, ultrametric Banach spaces, and $p$-adic Hilbert spaces. This is followed by a discussion on the Cesàro-like means in section three. In section four we prove that the Cesàro-like means converge strongly to a fixed point of $T$, and characterize the relationship between $x \in \mathbb{E}_\omega$ and $\lim_{n\to\infty} F_n x$. We also demonstrate that the theorem on the strong convergence of the Cesàro-like means can be applied to the algebraic sum of bounded operators. Section five contains a result on the weak convergence of the Cesàro-like means. The sixth and final section contains several examples which illustrate our work.

## 2 Preliminaries

This section contains a few definitions and basic notions of nonexpansive mappings, ultrametric Banach spaces, and $p$-adic Hilbert spaces.

**Definition 2.1.** Let $(M, d)$ be a metric space. A mapping $T : M \to M$ is a *Lipschitzian* if there exists a number $\lambda > 0$ such that,

$$d(Tx, Ty) \leq \lambda\, d(x, y), \quad \text{for all } x, y \in M.$$

The smallest such $\lambda$ is said to be the *Lipschitz constant* of $T$, and we denote it by $L(T)$.

The map $T : M \to M$ is said to be nonexpansive if $L(T) \leq 1$. It is called a *contraction* if $L(T) < 1$.

The Lipschitz constant of the n-th iterate $T^n$ of a nonexpansive map $T$ is such that

$$L(T^n) \leq [L(T)]^n.$$

Throughout our discussion we let $F_T = \{x \in M : Tx = x\}$ denote the set of fixed points of $T$.

**Definition 2.2.** A field $\mathbb{K}$ is said to be an *ultrametric valued field* if there exists a map $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ such that

(i) $|a| = 0$ if and only if $a = 0$,

(ii) $|ab| = |a||b|, \ \forall\, a, b \in \mathbb{K}$,

(iii) $|a + b| \leq \max\{|a|, |b|\}, \ \forall\, a, b, \in \mathbb{K}$.

The norm $|\cdot|$ induces a metric $d$ on $\mathbb{K}$, which is defined by $d(a, b) = |a - b|$. When $(\mathbb{K}, d)$ is complete with respect to the metric $d$, $\mathbb{K}$ is called a complete ultrametric valued field. The ultrametric norm satisfies:

(i) $\forall\, a \in \mathbb{K}, |-a| = |a|$;

(ii) If $|a| \neq |b|$, then $|a + b| = \max\{|a|, |b|\}$;

(iii) $\forall\, a, b \in \mathbb{K}$, if $|a| < |b|$, then $|b - a| = |b|$;

(iv) A sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ is a Cauchy sequence if and only if $(a_n - a_{n+1})$ is a null sequence in $\mathbb{K}$, i.e. $\lim\limits_{n\to\infty} |a_n - a_{n+1}| = 0$;

A classic example of an ultrametric field is $(\mathbb{Q}_p, |.|_p)$ the field of $p$-adic numbers (see [10]).

**Definition 2.3.** Let $\mathbb{E}$ be a vector space over a complete ultrametric valued field $\mathbb{K}$. The map $\|\cdot\| : \mathbb{E} \mapsto \mathbb{R}^+$ is an ultra (or non-Archimedean) norm on $\mathbb{E}$ if the following conditions are satisfied $\forall\, x, y, \in \mathbb{E}$ and $\forall\, a \in \mathbb{K}$.

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|ax\| = |a|\|x\|$;

(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ (ultrametric inequality).

If $(\mathbb{E}, \|\cdot\|)$ is complete, then $\mathbb{E}$ is said to be an *ultrametric Banach space* over $\mathbb{K}$.

The space

$$c_0(I, \mathbb{K}, \alpha) := \left\{ x = (x_i)_{i\in I} \in \mathbb{K}^I : \lim_{i\in I} |x_i|\, \alpha_i = 0 \right\},$$

where $I$ is the index set, is an example of an ultrametric Banach space. Its norm is given by,

$$\|x\| = \sup_{i\in I} |x_i|\alpha_i.$$

**Definition 2.4.** An ultrametric Banach space $(\mathbb{E}, \|\cdot\|)$ over a non-Archimedean valued field $\mathbb{K}$, is said to be a *free Banach* space if there exists a family $(e_i)_{i\in I} \subset \mathbb{E}$ such that any $x \in \mathbb{E}$ can be written as $x = \sum_{i\in I} x_i e_i$, i.e.

$$\lim_i |x_i|\|e_i\| = 0, \ \text{ and } \ \|x\| = \sup_{i\in I} |x_i|\|e_i\|.$$

We say that $(e_i)_{i\in I}$ is an orthogonal base of $\mathbb{E}$. If in addition $\|e_i\| = 1$, for all $i \in I$ then $(e_i)_{i\in I}$ is said to be an orthonormal base.

For a free Banach space $\mathbb{E}$, let $\mathbb{E}^*$denote its topological dual and $B(\mathbb{E})$ the set of all bounded linear operators on $\mathbb{E}$. Both $\mathbb{E}^*$ and $B(\mathbb{E})$ are endowed with their respective usual norms.

Let $(e_i)_{i\in\mathbb{N}}$ be an orthogonal base for $\mathbb{E}$, then one can define $e_i' \in \mathbb{E}^*$ by:

$$e_i'(x) = x_i,\ x = \sum_{i\in\mathbb{N}} x_i e_i.$$

Let $\mathbb{E}$ and $\mathbb{F}$ be free Banach spaces over $\mathbb{K}$ with respective bases $(e_j)_{j\in J}$ and $(f_i)_{i\in I}$. For $x' \in \mathbb{E}^*$(the dual of $\mathbb{E}$) and $y \in \mathbb{F}$, define the operator $x' \otimes y : \mathbb{E} \to \mathbb{F}$ by,

$$x' \otimes y(x) = \langle x', x\rangle y$$

where $\|x' \otimes y\| = \|x'\|\|y\|$.

Each operator $A$ on $\mathbb{E}$ can be expressed as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$, such that for any $j \in \mathbb{N}$, $\lim\limits_{i\to\infty} |a_{ij}|\, \|e_i\| = 0$, and

$$Ax = \sum_{ij} a_{ij}(e_j' \otimes e_i)x.$$

Moreover, for any $s \in \mathbb{N}$,

$$Ae_s = \sum_{i\in\mathbb{N}} a_{is} e_i$$

and

$$\|A\| = \sup_{i,j} \frac{|a_{ij}|\, \|e_i\|}{\|e_j\|}.$$

If $\|A\|$ is finite then $A$ is said to be bounded.

**Definition 2.5.** Let $\omega = (\omega_i)_{i\ge 0} \subset \mathbb{K}$ be a sequence of nonzero terms. Consider

$$\mathbb{E}_\omega := c_0\Big(\mathbb{N}, \mathbb{K}, (|\omega_i|^{\frac{1}{2}})_{i\ge 0}\Big) = \Big\{x = (x_i)_{i\ge 0} \subset \mathbb{K} : \lim_{n\to+\infty} |x_i||\omega_i|^{\frac{1}{2}} = 0\Big\}.$$

We define the basis $(e_i)_{i\ge 0}$ by letting $e_i = (\delta_{i,j})_{j\ge 0}$ where $\delta_{i,j}$ is the Kronecker symbols, $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \ne j$. $\mathbb{E}_\omega$ is a free Banach space with $(e_i)_{i\ge 0}$ as an orthogonal basis. (The orthogonality described here is with respect to the norm (see [1])). Indeed, every $x \in \mathbb{E}_\omega$ can be written as, $x = \sum\limits_{i\ge 0} x_i e_i$, where $\|x\| = \sup\limits_{i\ge 0} |x_i||\omega_i|^{\frac{1}{2}}$. From this, we conclude that $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

Let $\langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \to \mathbb{K}$ be the bilinear form defined by,

$$\langle x, y\rangle = \sum_{i\ge 0} x_i y_i \omega_i,$$

for all $x = \sum\limits_{i\ge 0} x_i e_i \in \mathbb{E}_\omega$, and $y = \sum\limits_{i\ge 0} y_i e_i \in \mathbb{E}_\omega$. The bilinear form $\langle\cdot,\cdot\rangle$ is a symmetric and non-degenerate bilinear form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, and it is satisfies the Cauchy-Schwartz inequality,

$$|\langle x, y\rangle| \le \|x\|\|y\|.$$

The space $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is called a *p-adic Hilbert space*.

Further discussion on Free Banach spaces and $p$-adic Hilbert spaces can be found in [5] and [7].

**Definition 2.6.** Let $u, v \in B(\mathbb{E}_\omega)$, say, $u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i$ and $v = \sum_{i,j} \beta_{ij} e'_j \otimes e_i$, where $\lim_{i\to+\infty} |\alpha_{ij}||\omega_i|^{\frac{1}{2}} = 0$, and $\lim_{i\to+\infty} |\beta_{ij}||\omega_i|^{\frac{1}{2}} = 0$ for all $j \geq 0$.

The operator $v \in B(\mathbb{E}_\omega)$, is the *adjoint* of $u$ with respect to $\langle\cdot,\cdot\rangle$ if $\langle u(x),\ y\rangle = \langle x,\ v(y)\rangle$, for all $x, y \in \mathbb{E}_\omega$.

Notice that $v$ is also the adjoint of $u$ since $\langle\cdot,\cdot\rangle$ is symmetric. Furthermore, the adjoint of an operator $u$, if it exists, is unique, since $\langle\cdot,\cdot\rangle$ is nondegenerate.

**Theorem 2.7.** *Let $(\omega_i)_{i\geq 0} \subset \mathbb{K}$ be a sequence of nonzero terms and let $u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i \in B(\mathbb{E}_\omega)$. Then $u$ has an adjoint $v = u^* \in B(\mathbb{E}_\omega)$ if and only if*

$$\lim_{j\to+\infty} |\omega_j|^{-\frac{1}{2}} |\alpha_{ij}| = 0,\ \forall\, i \geq 0.$$

*In this event, $u^* = \sum_{ij} \omega_i^{-1} \omega_j \alpha_{ij} e'_j \otimes e_i$.*

*Proof.* See [7]. ∎

Theorem 2.7 demonstrates that contrary to classical Hilbert spaces, not all continuous linear operators on $p$-adic Hilbert spaces have an adjoint. Let

$$B_0(\mathbb{E}_\omega) = \left\{ u = \sum_{i,j} \alpha_{ij} e'_j \otimes e_i \in B(\mathbb{E}_\omega) : \lim_{j\to+\infty} |\omega_j|^{-\frac{1}{2}} |\alpha_{ij}| = 0,\ \forall\, i \geq 0 \right\}$$

denote the space of continuous operators $u \in B(\mathbb{E}_\omega)$ which have an adjoint.

The definition for the weak convergence is similar to what is used in the classical setting. It is stated here for $p$-adic Hilbert spaces.

**Definition 2.8.** Let $\mathbb{E}$ be a $p$-adic Hilbert space. The sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{E}$ converges *weakly* to $x \in \mathbb{E}$ if for all $y \in \mathbb{E}$,

$$\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle.$$

This is denoted by $x_n \rightharpoonup x$.

**Definition 2.9.** A mapping $T$ on a $p$-adic Hilbert space is *asymptotically regular* if $T^n x - T^{n+1} x \to 0$ as $n \to \infty$. It is said to be *weakly asymptotically regular*, if $T^n x - T^{n+1} x \rightharpoonup 0$ as $n \to \infty$.

Note that for a nonexpansive map $T : \mathbb{E}_\omega \to \mathbb{E}_\omega$, asymptotically regularity implies weak asymptotic regularity.

**Lemma 2.10.** *Let $x$ be an arbitrary element of $\mathbb{E}_\omega$ and let $T$ be a strict contractive linear mapping on $\mathbb{E}_\omega$. Then, $T$ is weakly asymptotically regular.*

*Proof.* See [8]. ∎

# 3 The Cesàro - like Means

In the classical setting(i.e. Banach spaces over $\mathbb{R}$ or $\mathbb{C}$), the Cesàro means of a nonexpansive map $T$ are defined by

$$S_n = \frac{1}{n}\sum_{k=1}^{n} T^k x, \quad \text{for } x \in \mathbb{E}, n \in \mathbb{N}$$

where $T : \mathbb{E} \mapsto \mathbb{E}$ is a nonexpansive map on a uniformly convex Banach space $\mathbb{E}$. However, when attempting to study the convergence of these maps in the ultrametric setting a few problems arise. First, we find that this definition does not always make sense, because, the asymptotic behavior of $\frac{1}{n}$ may vary depending on the valuation. Take for instance the ultrametric field $(\mathbb{Q}_p, |\cdot|_p)$ for a fixed prime $p \geq 2$. We have that $\left|\frac{1}{n}\right|_p = p^{ord_p(n)} \to \infty$ as $n \to \infty$.

To address these difficulties we will replace $\frac{1}{n}$ by $\frac{1}{\alpha_1 + \cdots + \alpha_n}$ in the definition of the Cesàro mean, where $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{K}$ is a sequence of nonzero terms satisfying additional assumptions. We can now introduce an analogue to the Cesàro means which we will refer to as the Cesàro-like means. Let $(\mathbb{E}, \|\cdot\|)$ be an ultrametric Banach space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$, and let $T : \mathbb{E} \mapsto \mathbb{E}$ be a nonexpansive mapping. The Cesàro-like means are defined as,

$$F_n x := \frac{1}{P_n}\sum_{k=1}^{n} \alpha_k T^k x, \quad x \in \mathbb{E},$$

where $P_n = \alpha_1 + \cdots + \alpha_n$ with $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{K}$ being a sequence of nonzero elements. We suppose

(H.1) $|P_n| := |\alpha_n|$, where $P_n = \alpha_1 + \cdots + \alpha_n$;

(H.2) $\lim_{n\to\infty}\left(\frac{|\alpha_n|}{|\alpha_{n+1}|}\right) = 0.$

The sequence $(\alpha_k)_{k\in\mathbb{N}}$ plays a significant role in our work.

Notice that condition (H.1) implies that $(|P_n|)_{n\in\mathbb{N}}$ is a strictly increasing sequence. Indeed, by definition $|P_n| = |\alpha_n| = \max_{1\leq k\leq n} |\alpha_k|$ implies that for all $i, j = 1, \ldots, n$, $|\alpha_i| \neq |\alpha_j|$ for all $i \neq j$. So we have,

$$\begin{aligned} |\alpha_{n+1}| = |P_{n+1}| &= \max_{1\leq k\leq n+1} |\alpha_k| \\ &= \max\left\{\max_{1\leq k\leq n} |\alpha_k|,\ |\alpha_{n+1}|\right\} \\ &= \max\left\{|P_n|,\ |\alpha_{n+1}|\right\}, \end{aligned}$$

by (H.1). And therefore, $|P_n| < |\alpha_{n+1}| = |P_{n+1}|$.

For the consistency of our work, we give examples of sequences $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{K}$ satisfying both (H.1) and (H.2).

**Example 3.1.** Take $\mathbb{K} = \mathbb{Q}_p$, the $p$-adic number field. An example of such a sequence $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{Q}_p$ is given by

$$\alpha_k = p^{-p^k}, \quad k \in \mathbb{N}.$$

Let us check that conditions (H.1) and (H.2) are satisfied. Indeed,

$$|P_n|_p = \max_{1\le k\le n} |\alpha_k|_p = \max_{1\le k\le n} p^{p^k} = p^{p^n} = |\alpha_n|_p.$$

And,

$$\lim_{n\to\infty} \frac{|\alpha_n|_p}{|\alpha_{n+1}|_p} = \lim_{n\to\infty} \frac{|p^{-p^n}|_p}{|p^{-p^{n+1}}|_p} = \lim_{n\to\infty} \frac{p^{p^n}}{p^{p^{n+1}}} = 0.$$

A more generalized family of sequences satisfying the conditions above is given by $\alpha_k = p^{-p^{f(k)}}$, where $f : \mathbb{N} \mapsto \mathbb{N}$ is a strictly increasing function.

**Example 3.2.** Let $\mathbb{K} = \mathbb{Q}_p$. The sequence $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{Q}_p$ given by

$$\alpha_k = p^{-k!}, \quad k \in \mathbb{N},$$

satifies both (H.1) and (H.2).

## 4 Strong Convergence of the Cesàro – like Means on $p$-adic Hilbert Spaces

Before formally stating the theorem on the strong convergence of Cesàro-like means $(F_n x)_{n\in\mathbb{N}}$, let us first look to following lemmas and theorem.

**Lemma 4.1.** *The map $F_n : \mathbb{E}_\omega \to \mathbb{E}_\omega$ is nonexpansive.*

*Proof.* For arbitrary $x, y \in \mathbb{E}_\omega$,

$$\begin{aligned}
\|F_n x - F_n y\| &= \left\| \frac{1}{\alpha_1 + \cdots + \alpha_n} \sum_{k=1}^{n} \alpha_k \Big(T^k x - T^k y\Big) \right\| \\
&\le \frac{1}{|P_n|} \max_{1\le k\le n} \left\| \alpha_k \Big(T^k x - T^k y\Big) \right\| \\
&\le \frac{1}{|P_n|} \cdot |P_n| \max_{1\le k\le n} \|T^k x - T^k y\| \\
&\le \|x - y\|.
\end{aligned}$$

■

**Lemma 4.2.** *Let $x$ be an arbitrary element of a $p$-adic Hilbert space $\mathbb{E}_\omega$ and let $T$ be a contractive mapping (possibly nonlinear) on $\mathbb{E}_\omega$. Then $F_n x - T^n x \to 0$ as $n \to \infty$.*

*Proof.* Let $x$ be an arbitrary element of $\mathbb{E}_\omega$,

$$\begin{aligned}
\|F_nx - T^nx\| &= \left\|\frac{1}{P_n}\sum_{k=1}^{n}\alpha_kT^kx - T^nx\right\| \\
&= \frac{1}{|P_n|}\left\|\sum_{k=1}^{n}\alpha_kT^kx - P_nT^nx\right\| \\
&= \frac{1}{|P_n|}\left\|\sum_{k=1}^{n-1}\alpha_kT^kx + (\alpha_n - P_n)T^nx\right\| \\
&= \frac{1}{|P_n|}\left\|\sum_{k=1}^{n-1}\alpha_kT^kx - P_{n-1}T^nx\right\| \\
&\le \frac{1}{|P_n|}\max\left\{\left\|\sum_{k=1}^{n-1}\alpha_kT^kx\right\|, \|P_{n-1}T^nx\|\right\} \\
&\le \frac{1}{|P_n|}\max\left\{\max_{1\le k\le n-1}\|\alpha_kT^kx\|, |P_{n-1}|\|T^nx\|\right\} \\
&\le \frac{1}{|P_n|}\max\left\{|P_{n-1}|\max_{1\le k\le n-1}\|T^kx\|, |P_{n-1}|\|T^nx\|\right\} \\
&\le \frac{|P_{n-1}|}{|P_n|}\|x\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{|\alpha_n|}{|\alpha_{n+1}|} = 0$ and $\|x\| < \infty$, $\forall\, x \in \mathbb{E}_\omega$, as $n \to \infty$ we have $\|F_nx - T^nx\| \to 0$. ■

The following theorem is due to Banach (see [9]).

**Theorem 4.3.** *Let $T$ be a self-mapping of a complete metric space $(M, d)$. If $T$ is a contraction, then it has a unique fixed point $y$, and*

$$\lim_{n\to\infty} T^nx = y, \ \forall x \in M.$$

*We now state and prove the theorem on the strong convergence of the Cesàro-like means.*

**Theorem 4.4.** *Let $x$ be an arbitrary element of a $p$-adic Hilbert space $\mathbb{E}_\omega$ and let $T$ be a contractive mapping (possibly nonlinear) on $\mathbb{E}_\omega$. If $F_T \neq \phi$, then the ultrametric Cesàro-like means $(F_nx)_{n\in\mathbb{N}}$ converge strongly to a fixed point $y \in F_T$.*

*Proof.* To prove that $(F_nx)_{n\in\mathbb{N}}$ is Cauchy it suffices to show that

$$\lim_{n\to\infty}\|F_nx - F_{n+1}x\| = 0.$$

For an arbitrary $x \in E_\omega$,

$$\begin{aligned}
\|F_nx - F_{n+1}x\| &= \|F_nx - T^nx + T^nx - T^{n+1}x + T^{n+1}x - F_{n+1}x\| \\
&\le \|F_nx - T^nx\| + \|T^nx - T^{n+1}x\| + \|T^{n+1}x - F_{n+1}x\|.
\end{aligned}$$

By Lemma 4.2: $\|F_n x - T^n x\| \to 0$ as $n \to \infty$, and by Theorem 4.3: $\|T^n x - T^{n+1} x\| \to 0$ as $n \to \infty$, therefore

$$\lim_{n\to\infty} \|F_n x - F_{n+1} x\| = 0.$$

Hence $(F_n x)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{E}_\omega$ and therefore converges to a point $g(x) \in \mathbb{E}_\omega$. Since $\|F_n x - T^n x\| \to 0$ as $n \to \infty$, both $(F_n x)_{n\in\mathbb{N}}$ and $(T^n x)_{n\in\mathbb{N}}$ converge to the same limit; and, by Theorem 4.3, $g(x)$ is a fixed point of $T$. ■

The following theorem characterizes the map $g$ which takes

$$x \mapsto g(x) = \lim_{n\to\infty} F_n x.$$

**Theorem 4.5.** *The mapping $g$ on $\mathbb{E}_\omega$ which takes $x \mapsto g(x)$ is uniformly continuous.*

*Proof.* For $x \in \mathbb{E}_\omega$, $\lim_{n\to\infty} F_n x = g(x) \in E_\omega$. Thus the operator $g$ maps $\mathbb{E}_\omega$ into $\mathbb{E}_\omega$. For $x, y \in \mathbb{E}_\omega$, and $\forall\, n \in \mathbb{N}$,

$$\begin{aligned} \|g(x) - g(y)\| &\leq \|g(x) - F_n x\| + \|F_n x - F_n y\| + \|F_n y - g(y)\| \\ &\leq \|g(x) - F_n x\| + \|x - y\| + \|F_n y - g(y)\| + \|F_n y - g(y)\|, \end{aligned}$$

Since $F_n$ is nonexpansive $\|F_n x - F_n y\| \leq \|x - y\|$. Now if we take the limit as $n \to \infty$, we have that $\forall\, x, y \in \mathbb{E}$, $\|g(x) - g(y)\| \leq \|x - y\|$. ■

We can apply the results on the strong convergence of the Cesàro-like means to the algebraic sum of bounded operators. Namely, we perturb a contraction by a bounded linear operator in an ultrametric Banach space $\mathbb{E}$.

Consider the operator, $\Lambda_{\alpha,\beta} : \mathbb{E} \to \mathbb{E}$,

$$\Lambda_{\alpha,\beta} = \alpha T + \beta B, \tag{4.1}$$

where $\mathbb{E}$ is an ultrametric Banach space over a complete ultrametric field $\mathbb{K}$, $T : \mathbb{E} \to \mathbb{E}$ is a linear contraction, $B : \mathbb{E} \to \mathbb{E}$ is a bounded linear operator, and $\alpha, \beta \in \mathbb{K}$ are nonzero elements.

The following assumptions will be made:

$(H.3)$: $\|\alpha T + \beta B\| < \|B\|$;

$(H.4)$: $\|B\| < \dfrac{1}{\max(1, |\beta|)}$.

**Theorem 4.6.** *Under assumptions $(H.3) - (H.4)$, $\Lambda_{\alpha,\beta}$ is a contraction on $\mathbb{E}$.*

*Proof.* We have

$$\begin{aligned} \|\alpha T\| = \|\alpha T + \beta B - \beta B\| &\leq \max\Big\{\|\alpha T + \beta B\|, |\beta|\|B\|\Big\} \\ &\leq \max(1, |\beta|).\ \max\Big\{\|\alpha T + \beta B\|, \|B\|\Big\} \\ &= \max\Big\{1, |\beta|\Big\}\|B\|, \end{aligned} \tag{4.2}$$

by $(H.3)$.

$$\begin{aligned}
\|\alpha T + \beta B\| &\leq \max\Big\{\|\alpha T\|, |\beta|\|B\|\Big\} \\
&\leq \max\Big\{\max\{1, |\beta|\}\|B\|, |\beta|\|B\|\Big\} \text{ (by (4.2))} \\
&= \max\Big\{1, |\beta|\Big\}\|B\| \\
&< 1,
\end{aligned}$$

by $(H.4)$.

So the iterates $\left(\Lambda^n_{\alpha,\beta}(x)\right)_{n\in\mathbb{N}}$ as well as the corresponding Cesàro-like means $(F_n x)_{n\in\mathbb{N}}$ for $x \in \mathbb{E}$ converge strongly to the unique fixed-point of $\Lambda_{\alpha,\beta}$. Notice that $F_{\Lambda_{\alpha,\beta}} = N(I - \Lambda_{\alpha,\beta})$.

## 5 Weak Convergence of the Cesàro-like means on $p$-adic Hilbert Spaces

We begin with the following lemmas.

**Lemma 5.1.** *Let $\mathbb{E}_\omega$ be a $p$-adic Hilbert space over a complete ultrametric valued field $\mathbb{K}$, and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{E}_\omega$. If $x_n \rightharpoonup y$ in $\mathbb{E}_\omega$, then*

$$\frac{1}{P_n}\sum_{k=1}^{n}\alpha_k x_k \rightharpoonup y.$$

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{E}_\omega$ which is weakly convergent to $y \in \mathbb{E}_\omega$. Here $M$ is the bound on the sequence of real numbers $\{|\langle x_n, z\rangle|\}_{n\in\mathbb{N}}$ in $\mathbb{R}$. $\forall\, z \in \mathbb{E}_\omega$,

$$\begin{aligned}
\left|\left\langle \frac{1}{P_n}\sum_{k=1}^{n}\alpha_k x_k - y,\ z\right\rangle\right| &= \left|\left\langle \frac{1}{P_n}\left(\sum_{k=1}^{n}\alpha_k x_k -\ P_n y\right),\ z\right\rangle\right| \\
&= \left|\left\langle \frac{1}{P_n}\left(\sum_{k=1}^{n-1}\alpha_k x_k + \alpha_n x_n -\ P_n y\right),\ z\right\rangle\right| \\
&= \left|\left\langle \frac{1}{P_n}\left(\sum_{k=1}^{n-1}\alpha_k x_k + (P_n - P_{n-1})x_n -\ P_n y\right),\ z\right\rangle\right| \\
&= \frac{1}{|P_n|}\left|\left\langle \sum_{k=1}^{n-1}\alpha_k x_k - P_{n-1}x_n + P_n(x_n - y),\ z\right\rangle\right| \\
&\leq \frac{1}{|P_n|}\left|\left\langle \sum_{k=1}^{n-1}\alpha_k x_k,\ z\right\rangle\right| + \left|\left\langle \frac{P_{n-1}}{P_n}\, x_n,\ z\right\rangle\right| \\
&\quad + |\langle x_n - y,\ z\rangle|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|P_n|}\left|\sum_{k=1}^{n-1}\alpha_k\langle x_k,\ z\rangle\right| + \frac{|P_{n-1}|}{|P_n|}|\langle x_n,\ z\rangle| \\
&\quad + |\langle x_n - y,\ z\rangle| \\
&\leq \frac{|P_{n-1}|}{|P_n|}\max_{1\leq k\leq n-1}|\langle x_k,\ z\rangle| + \frac{|P_{n-1}|}{|P_n|}|\langle x_n,\ z\rangle| \\
&\quad + |\langle x_n - y,\ z\rangle| \\
&\leq 2M\frac{|P_{n-1}|}{|P_n|} + |\langle x_n - y,\ z\rangle|.
\end{aligned}
$$

Taking the limit as $n \to \infty$ we observe

$$\left|\left\langle \frac{1}{P_n}\sum_{k=1}^{n}\alpha_k x_k - y,\ z\right\rangle\right| \to 0,$$

and therefore

$$\frac{1}{P_n}\sum_{k=1}^{n}\alpha_k x_k \rightharpoonup y.$$

■

**Lemma 5.2.** *If $T : \mathbb{E}_\omega \to \mathbb{E}_\omega$ is a bounded linear mapping and the adjoint $T^*$ of $T$ exists, then $T$ is weakly continuous.*

*Proof.* Assume that for some sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{E}_\omega$, $x_n \rightharpoonup x$. We want to show that $Tx_n \rightharpoonup Tx$ as $n \to \infty$. In fact,

$$\left|\left\langle Tx_n - Tx, y\right\rangle\right| = \left|\left\langle x_n - x,\ T^*y\right\rangle\right| \longrightarrow 0, \quad \text{as } n \to \infty, \tag{5.1}$$

since $\langle x_n - x,\ z\rangle \to 0$, as $n \to \infty$ for all $z \in \mathbb{E}_\omega$. Therefore, $Tx_n$ converges weakly to $Tx$. This completes the proof. ■

We now state and prove the main result on the weak convergence of the Cesàro-like means on $p$-adic Hilbert spaces.

**Theorem 5.3.** *Let $T$ be a contractive linear mapping on $\mathbb{E}_\omega$ whose adjoint $T^*$ exists. If $T^n x \rightharpoonup y$, then $F_n x = \frac{1}{P_n}\sum_{k=1}^{n}\alpha_k x_k \rightharpoonup y$ and $y \in F_T$.*

*Proof.* By Lemma 5.1,

$$T^n x \rightharpoonup y \implies F_n x = \frac{1}{P_n}\sum_{k=1}^{n}\alpha_k x_k \rightharpoonup y.$$

Now we prove that $y$ is a fixed point of $T$.

$$
\begin{aligned}
\left|\left\langle y - Ty,\ z\right\rangle\right| &= \left|\left\langle y - T^n x + T^n x - T^{n+1}x + T^{n+1}x - Ty,\ z\right\rangle\right| && (5.2)\\
&\leq \left|\left\langle y - T^n x,\ z\right\rangle\right| + \left|\left\langle T^n x - T^{n+1}x,\ z\right\rangle\right| \\
&\quad + \left|\left\langle T^{n+1}x - Ty,\ z\right\rangle\right| && (5.3)\\
&= \left|\left\langle y - T^n x,\ z\right\rangle\right| + \left|\left\langle T^n x - T^{n+1}x,\ z\right\rangle\right| \\
&\quad + \left|\left\langle T(T^n x) - Ty,\ z\right\rangle\right|. && (5.4)
\end{aligned}
$$

By Lemma 5.1:

$$\left|\left\langle T^n x - T^{n+1}x,\ z\right\rangle\right| \to 0 \quad \text{as } n \to \infty,$$

and by Lemma 5.2,

$$\left|\left\langle T(T^n x) - Ty,\ z\right\rangle\right| \to 0 \quad \text{as } n \to \infty.$$

Passing to the limit in (5.4) we have

$$\left|\left\langle y - Ty,\ z\right\rangle\right| \to 0 \quad \text{as } n \to \infty,$$

so $y$ is a fixed point of $T$. ■

# 6 Examples

In this section we provide examples for contractions on $p$-adic Hilbert spaces. We also provide examples of contractions defined on clopen, bounded, and convex subsets of ultrametric Banach spaces.

**Example 6.1.** Consider the $p$-adic Hilbert space $\mathbb{E}_\omega$, where $\omega = (\omega_i)_{i\in\mathbb{N}}$ and $\omega_i = p^i$, $\forall i \geq 0$.

Define $T : \mathbb{E}_\omega \to \mathbb{E}_\omega$ by,

$$Tx = \sum_{k=0}^{\infty} \lambda_k x_k e_{k+1},$$

where $x = (x_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$, and the sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $\mathbb{Q}_p$ is defined as $\lambda_k = p^{p^k}$.

First we check that $T$ is well defined, i.e. $\lim\limits_{k\to\infty} |\lambda_k|_p |x_k|_p |\omega_{k+1}|_p^{\frac{1}{2}} = 0$.

$$\begin{aligned}\lim_{k\to\infty} |\lambda_k|_p\ |x_k|_p\ |\omega_{k+1}|_p^{1/2} &= \lim_{k\to\infty} p^{-p^k}\ |x_k|_p\ p^{-k/2} p^{-1/2}\\ &\leq \|x\|\ \lim_{k\to\infty} p^{-p^k}\ p^{-1/2}\\ &= 0.\end{aligned}$$

Clearly, $T$ is a linear contraction on $\mathbb{E}_\omega$, and $F_T = \{\vec{0}\}$.

**Example 6.2.** Consider $\mathbb{E}_\omega$ where $\omega_i = p^i$. Its canonical orthogonal base $(e_s)_{s\in\mathbb{N}}$ has the property that $\|e_s\| = |\omega_i|_p^{\frac{1}{2}}$. (We can instead take any $\omega_i$ such that $\{|\omega_i|_p\}_{i\geq 0}$ is a decreasing sequence of real numbers.)

Define the operator $T : \mathbb{E}_\omega \to \mathbb{E}_\omega$ as follows:

$$Te_s = p^s e_s + (1 - p^s)e_{s+1}, \quad \forall s \in \mathbb{N}. \tag{6.1}$$

For every $x = \sum_s x_s e_s \in \mathbb{E}_\omega$,

$$Tx = \sum_{s=0}^{+\infty} x_s Te_s. \tag{6.2}$$

One can easily check that,

$$\|T\| = \sup_{s\in\mathbb{N}} \frac{\|Te_s\|}{\|e_s\|} = p^{-\frac{1}{2}} < 1, \tag{6.3}$$

hence $T$ is a contraction of $\mathbb{E}_\omega$.

Now we calculate the fixed points of $T$.

$$\begin{aligned} Tx - x &= \sum_{s=1}^{+\infty} \left(x_s p^s + (1 - p^{s-1})x_{s-1}\right) e_s - x_s e_s && (6.4)\\ &= \sum_{s=1}^{+\infty} \left(x_s(p^s - 1) + (1 - p^{s-1})x_{s-1}\right) e_s. && (6.5)\end{aligned}$$

The equation in (6.4) equals zero when,

$$(p^s - 1)x_s + (1 - p^{s-1})x_{s-1} = 0 \quad \forall s \geq 1. \tag{6.6}$$

The fixed points of $T$ are given by

$$F_T = \Big\{x = (x_0, 0, 0, \ldots) \in \mathbb{E}_\omega, \text{ where } x_0 \in \mathbb{K}\Big\}.$$

The following are examples of a nonlinear, contractive operator with a nontrivial fixed point set $F_T$.

**Example 6.3.** Consider the $p$-adic Hilbert space $\mathbb{E}_\omega$ with the complete ultrametric field $\mathbb{K} = \mathbb{Q}_p$ defined above and the sequence $\omega = (\omega_s)_{s\in\mathbb{N}}$ and $\omega_s = p^s$, $\forall s \geq 0$. Let $\lambda \in \mathbb{K}$, where $|\lambda| > 1$ and fix $v_0 \in \mathbb{E}_\omega$, such that $v_0 \not\equiv 0$. For $x \in \mathbb{E}_\omega$, let

$$T_{v_0}(x) = \frac{x - v_0}{\lambda}.$$

Clearly, $T_{v_0}$ is a non-linear self-mapping on $\mathbb{E}_\omega$. And,

$$\begin{aligned} \left\|T_{v_0}(x) - T_{v_0}(y)\right\| &= \left\|\frac{x-y}{\lambda}\right\| \\ &= \frac{1}{|\lambda|} \cdot \|x - y\| \\ &< \|x - y\| \end{aligned}$$

implies that $T_{v_0}$ is a strict contraction, with fixed point set

$$F_{T_{v_0}} = \left\{\frac{v_0}{1-\lambda}\right\}.$$

The iterates of $T_{v_0}$ are given by $T_{v_0}^n(x) = \dfrac{x + v_0 \cdot \sum_{i=0}^{n-1} \lambda^i}{\lambda^n}$. By Theorem 4.3 and Theorem 4.4, the iterates and the corresponding Cesáro-like means converge to the unique fixed point of $T_{v_0}$.

Now we look to an example of a linear contraction on a $p$-adic Hilbert space, with adjoint that satisfies the assumptions of the theorem on the weak convergence of the Cesàro-like means.

**Example 6.4.** Consider $\mathbb{E}_\omega$ where $\omega_s = p^s$ for each $s \in \mathbb{N}$. Define the $p$-adic unilateral shift by: $Ue_s = e_{s+1}$ for each $s \in \mathbb{N}$.

Clearly, $\dfrac{\|Ue_s\|}{\|e_s\|} = \dfrac{1}{\sqrt{p}} < 1$. So $U$ is a contraction on $\mathbb{E}_\omega$. It is also clear that the adjoint $U^*$ of $U$ does exist and that $U^*e_{s+1} = e_s$ for each $s \in \mathbb{N}$. In addition to that, the $U$ iterates are given by:

$$U^n x = \sum_{s\geq 0} x_s e_{s+n},$$

for each $n \in \mathbb{N}$, and each $x = (x_s)_{s\in\mathbb{N}} \in \mathbb{E}_\omega$. Fix $x = (x_s)_{s\in\mathbb{N}} \in \mathbb{E}_\omega$. If $y = (y_s)_{s\in\mathbb{N}} \in \mathbb{E}_\omega$, then

$$\langle U^n x, y\rangle = \sum_{s\geq 0} \omega_{n+s} x_s y_{s+n}.$$

In view of the above,

$$|\langle U^n x, y\rangle| \leq p^{-n/2}\|x\| \cdot \|y\|,$$

hence, $\langle U^n x, y\rangle \to 0$ as $n \to \infty$. In other words, $U^n x \rightharpoonup 0$ in $\mathbb{E}_\omega$. Using Theorem 5.3 it follows that the corresponding Cesàro-like means $(F_n x)_{n\in\mathbb{N}}$ defined by

$$F_n x = \frac{1}{P_n}\sum_{k=1}^{n} \alpha_k \left(\sum_{s\geq 0} x_s e_{k+s}\right)$$

converges weakly to 0.

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